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ON CLASSES OF RELATIONS AND GRAPHS DETERMINED BY SUBOBJECTS AND FACTOROBJECTS

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The classes of relations and graphs determined by subobjects and factorobjects are studied. We investigate whether such classes are closed under products, whether they are finitely generated by products and subobjects and whether a class can be described alternatively by subobjects and factorobjects. This is related to good characterizations.

0. Introduction

Most graph-theoretical notions are defined by means of the existence (or non-existence) of certain special homomorphisms. Typical examples include: chromatic number $\chi(G)$ (= existence of homomorphisms into complete graphs); independence number $\alpha(G)$ (= existence of embeddings of discrete graphs); clique number $\omega(G)$ (= existence of embeddings of complete graphs); connectivity of a graph (= non-existence of a homomorphism onto $(\{1,2\},\{\{1\},\{2\}\})$); hamiltonian graphs (= existence of monomorphisms of a circuit onto a graph).

The ways one can describe classes of graphs by means of homomorphisms or particular type homomorphisms vary, and consequently also the properties of the classes thus obtained are in general very diverse. There are, however, some questions one can answer in a satisfactory generality.

In particular, we are interested in the questions as to whether a class can be determined (using existence of homorphisms) in substantially different ways, whether it is closed under products and whether it is generated (forming products and full subobjects) by a finite number of objects.

In the last question there is a surprising difference between the classes of graphs determined by factorobjects (i.e. homomorphisms *from* the graphs; a typical example is the class of all graphs with chromatic number $\leq n$), and classes of graphs determined by subobjects (i.e. homomorphisms *into* the graphs; a typical example is the class of all graphs with clique number $\leq n$), see Theorems 2.6 and 3.2 below. In consequence, the classes which can be determined both by factorobjects and subobjects are extremely rare. (This question is discussed in Section 3.)

There may be more in the last question than meets the eye: The classes $\mathscr{C}$ which can be defined both by existence of homomorphisms into some of $B_1, \ldots,$ or B_n and by non-existence of homomorphisms from $A_1, \ldots,$ and A_m are the simplest ones having the *good characterization* in the sense of Edmonds ([2]; the search for a mapping $G \to B_i$ proving G is in $\mathscr{C}$, and the search for one $A_i \to G$ disproving it, are both non-deterministic polynomial. In the more complex cases the good characterization can be often viewed, roughly speaking, as defining the class both by the existence of homomorphisms into a $B \in \mathscr{B}$, and by the non-existence of homomorphisms from an $A \in \mathscr{A}$—or dually non-existence in the former and existence in the latter case—where $\mathscr{A}$, $\mathscr{B}$ are defined by reasonable formulas.) Thus, Theorem 4.1 may be interpreted as a list of all trivial good characterizations.

Let us remark here that of the four possible combinations of descriptions, the two cases mentioned, namely those with the non-existence of homomorphisms on the one hand and the existence on the other hand, are those which make sense. (If $\mathscr{C}$ is determined by the existence of a homomorphism from an A_i, $i = 1, \ldots, n$, and by the existence of one into a B_i, $i = 1, \ldots, m$, then necessarily $\mathscr{C}$ is the class of all graphs. Indeed, consider an arbitrary X and take $X \times A_1$; since $A_1 \in \mathscr{C}$ we have a homomorphism $A_1 \to B_i$, hence $X \times A_1 \to B_i$, hence there is a homomorphism $A_j \to X \times A_1 \to X$, so that $X \in \mathscr{C}$.)

The paper is divided into five parts. Section 1 is concerned with the necessary definitions, and summarizes known results. In Section 2 we prove that all classes determined by the existence of factorobjects are finitely generated. In Section 3, on the other hand, we show that this is the case with only a few classes determined by forbidden subobjects. In Section 4, classes determined both by subobjects and factorobjects are discussed. While the classes determined by the existence of homomorphisms into a given object, or non-existence of homomorphisms from given ones, or also by existence of homomorphisms from one given object, are always productive, in the case of non-existence of homomorphisms into a given object one obtains both productive and non-productive classes. Section 5 contains some results on productivity concerning particular types of targets.

1. Preliminary results

1.1. Conventions. We will be concerned with classes $\mathscr{C}$ of digraphs (≡sets with binary relations; or objects which may be viewed as such, e.g. undirected graphs). A *homomorphism* between two digraphs (X, R), (Y, S) is a mapping $f: X \to Y$ such that $(f(x), f(y)) \in S$ whenever $(x, y) \in R$. In such a case we write $f: (X, R) \to (Y, S)$. If $f: (X, R) \to (Y, S)$ is a homomorphism with $f: X \to Y$ invertible and $f^{-1}: (Y, S) \to (X, R)$ a homomorphism, we speak about an *isomorphism*.

Let (X, R) be a digraph, $Y \subseteq X$. The digraph $(Y, R \setminus Y) = (Y, R \cap Y \times Y)$ is called *subobject* of (X, R) (carried by Y).

Given a class $\mathscr{C}$ of digraphs and a set X, denote by $\mathscr{C}X$ the set of all $(X, R) \in \mathscr{C}$ partially ordered by $(X, R) < (X, S)$ iff $R \subseteq S$. The symbols $\bigwedge$, $\wedge$ for meet (≡ infimum) are used in accordance with the ordering $<$.

An equivalence relation E on X is said to be a $\mathscr{C}$-congruence (or, simply, congruence, if there is no danger of confusion) on $A=(X,R)\in\mathscr{C}$, if $A/_E=(X/_E, R_E)$, where $(Ex, Ey)\in R_E$ iff there exist $x'\in Ex$, $y'\in Ey$ such that $(x', y')\in R$, is in $\mathscr{C}$. The trivial congruence $\{(x,x)\colon x\in X\}$ will be denoted by Δ. A non-trivial congruence E on A is said to be *critical* if for every $B>_{\neq}A$ on which E is still a congruence, $B/_E>_{\neq}A/_E$.

A *product* of digraphs is defined by

$$\mathop{\mathsf{X}}_{i=1}^{n}(X_i, R_i)=\left(\mathop{\mathsf{X}}_{i=1}^{n} X_i, R\right),$$

where $((x_i),(y_i))\in R$ iff $(x_i, y_i)\in R_i$ for all i. For products of two objects we use the symbol $(X,R)\times(Y,S)$. The homomorphisms $\mathsf{X}(X_i,R_i)\to(X_j,R_j)$ sending $(x_i: i=1,\dots,n)$ to x_j are called *projections*. The product of n copies of the same digraph A will be denoted by A^n.

All the classes $\mathscr{C}$ we will deal with will be supposed *closed under isomorphisms*.

1.2. Definition. A class $\mathscr{C}$ of finite digraphs is said to be *hereditary* if it is closed under subobjects, *productive* if it is closed under products. It is said to have *multiplication of points* if for every onto mapping $f\colon X\to Y$ and every $(Y,S)\in\mathscr{C}$, $(X,(f\times f)^{-1}(S))$ is in $\mathscr{C}$.

1.3. Remark. The converse property that if $(X,(f\times f)^{-1}(S))$ is in $\mathscr{C}$, also (Y,S) is in $\mathscr{C}$ is available in every hereditary $\mathscr{C}$. Indeed, (Y,S) is isomorphic to a subobject of $(X,(f\times f)^{-1}(S))$.

1.4. Definition. Let $A_1,\dots,A_n$ be objects of $\mathscr{C}$. We put:

$(A_1,\dots,A_n)\not\to\mathscr{C}=\{A\in\mathscr{C}\colon \text{for no } i \text{ there is an } f\colon A_i\to A\}$,

$\mathscr{C}\to(A_1,\dots,A_n)=\{A\in\mathscr{C}\colon \text{there is an } i \text{ and an } f\colon A\to A_i\}$,

$(A_1,\dots,A_n)\neg\mathscr{C}=\{A\in\mathscr{C}\colon A \text{ has no subobject isomorphic to an } A_i\}$,

$SP\mathscr{C}(A_1,\dots,A_n)=\{A\in\mathscr{C}\colon A$ is isomorphic to a subobject of a nontrivial product of copies of $A_i\}$.

1.5. Conventions. (1) To avoid the trivial case, in $\mathscr{C}\to(A_1,\dots,A_n)$ we will always assume that no A_i has a loop (i.e. no (x,x) is in its relation).

(2) Write $A\sim B$ if there are homomorphisms $A\to B$ and $B\to A$. Obviously the first two classes from Definition 1.4 do not change after replacing A_i by B_i such that $A_i\sim B_i$.

1.6. We obtain easily

Proposition. *Let $\mathscr{C}$ be hereditary productive. Then every $(A_1,\dots,A_n)\not\to\mathscr{C}$, $\mathscr{C}\to(A_1,\dots,A_n)$ and $SP\mathscr{C}(A_1,\dots,A_n)$ is hereditary productive, and every $(A_1,\dots,A_n)^-\,\mathscr{C}$ is hereditary.*

1.7. Proposition. *For every system $A_1, \ldots, A_n$ in a hereditary $\mathscr{C}$ there are $B_1, \ldots, B_m$ such that*

$$(A_1, \ldots, A_n) \not\to \mathscr{C} = (B_1, \ldots, B_m) \neg \mathscr{C}$$

Proof. It suffices to take all the B such that there is a homomorphic image C of an A_i in $\mathscr{C}$ with $C < B$.

1.8. Definition. An object A of a hereditary productive $\mathscr{C}$ is said to be *subdirectly irreducible* if, whenever $f: A \to \times B_i$ is an isomorphism onto a subobject in $\mathscr{C}$ such that $p_i \circ f$ is onto for every projection p_i, then at least one of the $p_i \circ f$ is an isomorphism.

1.9. By [6, 2.6] we have

Proposition. *For every A in a productive and hereditary $\mathscr{C}$ there is an isomorphism f of A onto a subobject of $\times B_i$ such that the B_i are subdirectly irreducible and $p_i \circ f$ is onto for all the projections p_i.*

1.10. Definition. A hereditary productive class $\mathscr{C}$ is said to be *finitely generated* if there are $A_1, \ldots, A_n$ such that

$$\mathscr{C} = SP\mathscr{C}(A_1, \ldots, A_n).$$

1.11. By 1.9, we obtain easily

Proposition. *$\mathscr{C}$ is finitely generated iff it has only finitely many subdirectly irreducibles.*

1.12. Let us reformulate here in a more handy form the main theorem of [6] applied to classes of digraphs, which we will use in the sequel. In the formulation, the expressions "maximal" and "meet irreducible" are meant with respect to the ordering $\prec$ from 1.1. (Thus, $(X, R) \in \mathscr{C}$ is meet irreducible iff, whenever $R = \bigcap R_i$ with $(X, R_i) \leftarrow \mathscr{C}$, at least one of the R_i coincides with R.) We have

Theorem. *In a hereditary productive $\mathscr{C}$, A is subdirectly irreducible iff either A is maximal and $\bigcap_{i=1}^{n} E_i \neq \Delta$ for every family of congruences E_i on A unless some E_i is Δ, or A is not maximal, A is meet irreducible, and there is no critical congruence on A.*

2. Existence of factorobjects

2.1. Definition. A digraph $A = (X, R)$ is said to be *reduced* if every homomorphism $A \to A$ is an isomorphism.

2.2. Lemma. *For every A there is (up to isomorphism) exactly one reduced B such that $A \sim B$.*

Proof. Take a homomorphism $f: A \to A$ with the least possible cardinality of the image $B = f(A)$. Now, necessarily every homomorphism $g: B \to B$ is onto and hence (since the object B is finite) an isomorphism. If also $A \sim C$ with a reduced C, we obtain $B \to C$, $C \to B$ which compose to isomorphisms, and hence are isomorphisms themselves.

2.3. Lemma. *Let $A = (X_0, R_0)$ be reduced, let $\mathscr{C}$ have the multiplication of points. Then (X, R) is maximal in $\mathscr{C} \to A$ iff there is a mapping $f: X \to X_0$ such that $R = (f \times f)^{-1}(R_0)$.*

Proof. Let (X, R) be maximal, let $f: (X, R) \to (X_0, R_0)$ be a homomorphism. By the multiplication of points, $(X, (f \times f)^{-1}(R_0))$ is in $\mathscr{C}$. Since it follows (X, R) in $<$, we obtain $R = (f \times f)^{-1}(R_0)$. On the other hand, let $S \supsetneq (f \times f)^{-1}(R_0)$, $(x, y) \in S \setminus (f \times f)^{-1}(R_0)$, and let there be a homomorphism $g: (X, S) \to (X_0, R_0)$. Put $h(f(x)) = x$, $h(f(y)) = y$ (since we assume A antireflexive—see 1.5—$f(x) \neq f(y)$) and choose an arbitrary point $h(z) \in f^{-1}(z)$ otherwise. Thus a homomorphism $h: (X_0, R_0) \to (X, S)$ is obtained (actually, h is already a homomorphism into $(X, (f \times f)^{-1}(R_0))$. We have $\varphi = gh$ an isomorphism. Consequently,

$$(f(x), f(y)) = (\varphi^{-1} ghf(x), \varphi^{-1} ghf(y)) = (\varphi^{-1} g(x), \varphi^{-1} g(y)) \in R$$

in contradiction with the assumption.

2.4. Lemma. *Let $\mathscr{C} \to (A_1, \ldots, A_m) = SP\mathscr{C}(B_1, \ldots, B_k)$ and*

$$\mathscr{C} \to (A_{1+m}, \ldots, A_n) = SP\mathscr{C}(B_{1+k}, \ldots, B_s).$$

Then

$$\mathscr{C} \to (A_1, \ldots, A_n) = SP\mathscr{C}(B_1, \ldots, B_s).$$

Proof. The inclusion $\mathscr{C} \to (A_1, \ldots, A_n) \subset SP\mathscr{C}(B_1, \ldots, B_s)$ is obvious. On the other hand, let there be an $f: A \to \mathsf{X} B_i^{n_i}$, which is an isomorphism onto a subobject of a non-trivial product. Choose an i with $n_i \neq 0$. We have an isomorphism $A \to B_i$ obtained by composing f with one of the projections onto B_i.

2.5. Notation. (1) Let M be a subset of X. We denote by $\mathscr{E}(M)$ the equivalence on X defined by $(x, y) \in \mathscr{E}(M)$ iff $x, y \in M$ or $x = y$.

(2) The reduced B from Lemma 2.2 will be denoted by $\tilde{A}$.

(3) For an $A = (X, R)$ and for an $(x, y) \in R$ define digraphs $2xyA$ and $\overline{2xyA}$ as follows: The set of vertices is $X \cup \{\bar{x}, \bar{y}\}$ where $\bar{x}, \bar{y} \notin X$, $\bar{x} \neq \bar{y}$. If we denote by q the mapping of $X \cup \{\bar{x}, \bar{y}\}$ onto X sending z to z for $z \in X$, $\bar{x}$ to x and $\bar{y}$ to y, the

relation of $2xyA$ is $(q\times q)^{-1}(R)\setminus\{(\bar{x},\bar{y})\}$, that of $\overline{2xy}A$ is $(q\times q)^{-1}(R)\setminus\{(\bar{x},\bar{y}), (\bar{y},\bar{x})\}$.

2.6. Theorem. *Let $\mathscr{C}$ be hereditary productive with multiplication of points. Suppose one of the following conditions is satisfied:*

(a) *if (X, R) is in $\mathscr{C}$, then every $(X, R\setminus\{(x, y)\})$ is in $\mathscr{C}$;*

(b) *all the (X, R) in $\mathscr{C}$ are symmetric, and for each of them, every $(X, R\setminus\{(x, y), (y, x)\}$ is in $\mathscr{C}$.*

Then $\mathscr{C}\to(A_1, \ldots, A_n)$ is generated by all $2xyA_i$ in case (a), *by all the $\overline{2xy}A_i$ in case* (b).

Proof. By Lemmas 2.4 and 2.2 it suffices to prove that the $\mathscr{C}\to A$ with reduced A are generated by the $2xyA$ (respectively $\overline{2xy}A$). We will do this by looking at the subdirectly irreducibiles in $\mathscr{C}\to A$ (see Proposition 1.9) by means of Theorem 1.12. By Lemma 2.3, for a maximal object (X, R) we have a mapping $f: X\to X_0$ such that $R=(f\times f)^{-1}(R_0)$. Suppose $f^{-1}(x_0)$ contains three distinct points x_1, x_2, x_3. Then (see Remark 1.3), $\mathscr{C}(\{x_i, x_j\})$ are congruences and $\bigcap_{i\neq j}\mathscr{C}(\{x_i, x_j\})$ is trivial. Similarly, if $x_0\neq y_0$ and $f^{-1}(x_0)$ and $f^{-1}(y_0)$ contain two distinct points each, say x_1, x_2 and y_1, y_2, we have trivial $\mathscr{C}(\{x_1, x_2\})\cap\mathscr{C}(\{y_1, y_2\})$. Thus the maximal subdirectly irreducibles are evidently subobjects of $2xyA$ (respectively $\overline{2xy}A$).

Let (X, R) be non-maximal meet irreducible. By the condition (a) (respectively (b)), there is an $f:(X, R)\to(X_0, R_0)$ such that $(f\times f)^{-1}(R_0)\setminus R=\{(u, v)\}$ (respectively $(f\times f)^{-1}(R_0)\setminus R=\{(u, v), (v, u)\}$). If for an $x\in X_0$, $f^{-1}(x)\setminus\{u, v\}$ has two distinct elements x_1, x_2, $\mathscr{C}(\{x_1, x_2\})$ is a critical congruence. This leaves as subdirectly irreducibles only subobjects of $2xyA$ in the former case, of $\overline{2xy}A$ in the later one. Finally, by the multiplication of points and by the condition (a) (respectively (b)), $2xyA$ (respectively $\overline{2xy}A$) really is in $\mathscr{C}$.

2.9. Examples. (1) The class of all digraphs such that there is a mapping into $\to$ is generated by the digraph of Fig. 1.

Fig. 1.

(2) The class of all digraphs such that there is a mapping into the 3-cycle is generated by the digraph of Fig. 2.

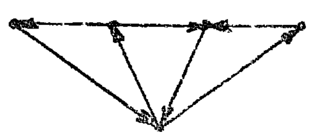

Fig. 2.

(3) More generally, the class of digraphs such that there is a homomorphism into an n-cycle is generated by the digraph of Fig. 3.

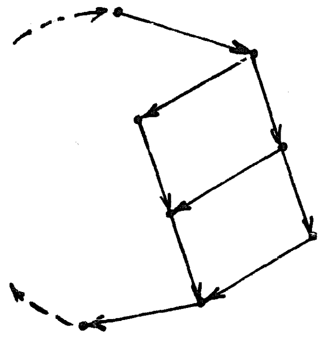

Fig. 3.

(4) The class of all n-chromatic undirected graphs is generated by $K_{n-2} \oplus P_3$, where $X \oplus Y$ is the graph obtained from the disjoint union of X and Y by joining all $x \in X$ with all $y \in Y$, K_k is the complete graph with k points, $P_n = (\{0, 1, \ldots, n\}, \{\{i, i+1\}: i = 0, \ldots, n-1\})$.

(5) Similarly as in (3), the class of all undirected graphs such there is a homomorphism into the n-circuit, n odd, is generated by the graph of Fig. 4.

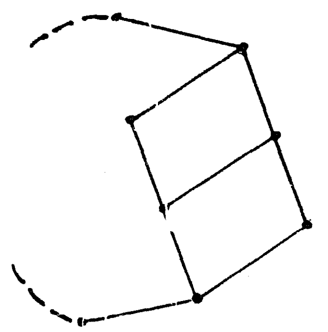

Fig. 4.

2.10. Remark. Of course, once we know or guess the generators in a concrete instance, there is often an easy direct proof of the fact that they really generate the given class. This is the case with the examples given, in particular with that under (4).

3. Forbidden subobjects

3.1. Notation. Denote by $\mathscr{D}$ the class of all discrete graphs (i.e. those with void relation); by $\mathscr{O}$ its subclass consisting of the one-point graphs; by $\mathscr{B}$ the class of disjoint sums of complete bipartite graphs; by $\mathscr{S}$ the subclass of $\mathscr{B}$ consisting of the disjoint sums of copies of K_2 and K_1.

3.2. Theorem. *Let $\mathscr{C}$ be the class of undirected graphs (without loops). Then the only finitely generated productive classes of the form $(A_1, \ldots, A_n)\neg\mathscr{C}$ are $\{\emptyset\}$, $\mathscr{D}$, $\mathscr{O}$, $\mathscr{B}$ and $\mathscr{P}$. Consequently, the only finitely generated $(A_1, \ldots, A_n)\rightarrow\mathscr{C}$ are $\mathscr{P}$ and $\{\emptyset\}$.*

Proof. Let $\mathscr{A}=(A_1, \ldots, A_n)\neg\mathscr{C}=SP\mathscr{C}(A_1', \ldots, A_m')$. Thus, in particular, the chromatic number $\chi(A)$ of no $A\in\mathscr{A}$ exceeds $c=\max(\chi(A_i);\ i=1, \ldots, m)$. Suppose every A_i contains a non-trivial circuit, let d be the maximum of the lengths of circuits in A_i. By [3] there exists a graph C with $\chi(C)>c$ which does not contain circuits of lengths $\leq d$. This is a contradiction, since $C\in\mathscr{A}$ and $C\notin SP\mathscr{C}(A_1', \ldots, A_m')$. Hence, at least one A_i is a forest. Because of the productivity and the fact that P_3 generates the system of all bipartite graphs (cf. 2.9(4)), we have $P_3\notin\mathscr{A}$ and hence one of its full subgraphs has to be among the A_i. We will assume that it is A_1. Thus, A_1 is one of the graphs of Fig. 5

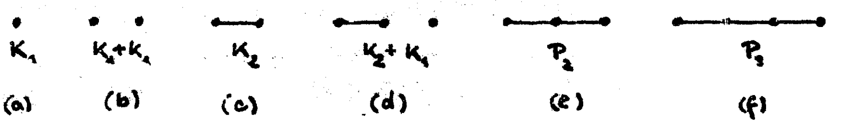

Fig. 5.

In case (a) we see that $\mathscr{A}=\{\emptyset\}$.

In case (b) we see that $\mathscr{A}$ is a subsystem of the system of all complete graphs. Such an $\mathscr{A}$, however, is productive only if equal to $\mathscr{C}$ (and this, on the other hand, is of the required form, namely $(K_1+K_1, K_2)\neg\mathscr{C}$).

In case (c) we obtain a subsystem of $\mathscr{D}$. There are just two productive hereditary ones: $\mathscr{D}$ and $\mathscr{O}$.

Case (d) does not bring anything new: namely, the graph in question is a subobject of $K_2\times K_2$ and hence P_2 has to be forbidden, too.

For cases (e) and (f) note first that the graphs in question are subobjects of $K_3\times K_3$. Thus, we either return to case (a) or (c), or, say, $A_2=K_3$. In case (e) this leaves only productive subclasses of $\mathscr{P}$, which are, as one easily sees, exactly $\{\emptyset\}$, $\mathscr{O}$, $\mathscr{D}$ and $\mathscr{P}$. Finally in case (f), we have $\mathscr{A}'=(P_3, K_3)\neg\mathscr{C}=\mathscr{B}$. (Indeed, let us show that a connected $(X, R)\in\mathscr{A}'$ is complete bipartite. Suppose, by way of contradiction, that there are others and let (X, R) be a counterexample with the smallest possible cardinality. Obviously, there is an x such that, for $Y=X\setminus\{x\}$, $(Y, R|_Y)$ is still connected. Thus, $(Y, R|_Y)$ is a complete bipartite with, say, the classes Y_1, Y_2. Now, since (X, R) is connected, x has to be joined with some y, say in Y_1. Then, since K_3 is forbidden, x is joined with no element of Y_2. If z is any other element of Y_1, take a $u\in Y_2$ and consider the path $xyuz$. Since P_3 is forbidden, we have necessarily x joined with z.) Now, obviously, the only productive subsystems of $\mathscr{B}$ are $\{\emptyset\}$, $\mathscr{O}$, $\mathscr{B}$, $\mathscr{P}$ and $\mathscr{B}$.

3.3. Remark. If we consider the class of all undirected graphs (loops are admitted), the list of the finitely generated $(A_1, \ldots, A_n)\neg\mathscr{C}$ given in Theorem 3.2 is not complete. We have to add, then, the void class, the class consisting of one-point graphs with loop, the class of all discrete graphs with all loops added, the class of all complete graphs with all loops, the class of all disjoint sums of complete graphs with all loops, and the class of all graphs with all loops, see [6, 4.2].

4. Classes determined by both factorobjects and forbidden subobjects

4.1. By Theorem 3.2 we obtain immediately

Corollary. *Let $\mathscr{C}$ be the class of all undirected graphs or the class of all undirected graphs without loops. Except for $K_1 \not\to \mathscr{C} = \mathscr{C} \to \emptyset$ and $K_2 \not\to \mathscr{C} = \mathscr{C} \to K_1$ there is no $(A_1, \ldots, A_n) \not\to \mathscr{C}$ equal to a $\mathscr{C} \to (A'_1, \ldots, A'_m)$.*

4.2. Proposition. *Put* $L_n = (\{0, 1, \ldots, n\}, \{(i, i+1): i = 0, 1, \ldots, n-1\})$, $\bar{L}_n = (\{0, 1, \ldots, n\}, \{(i, j): i < j\})$. *Let $\mathscr{C}$ be the class of all digraphs or of all digraphs without loops. Then $L_n \not\to \mathscr{C} = \mathscr{C} \to \bar{L}_{n-1}$.*

Proof. Since there is no homomorphism $L_n \to \bar{L}_{n-1}$, we have $\mathscr{C} \to \bar{L}_{n-1} \subseteq L_n \not\to \mathscr{C}$. On the other hand, suppose there is no homomorphism $\varphi : L_n \to (X, R)$. Put

$$P(x) = \{i: \text{there exist } x_0 R x_1 R x_2 \cdots x_{i-1} R x_i = x\},$$
$$\psi(x) = \max P(x).$$

We have $k \notin P(x)$ for all $k \geq n$, and $\psi(x) < \psi(y)$ whenever xRy. Thus, ψ is a homomorphism $(X, R) \to \bar{L}_{n-1}$.

4.3. Proposition. *Let $\mathscr{C}$ be the class of all digraphs or of all digraphs without loops. Assume that for some $A \in \mathscr{C}$ there is an A' such that $A \not\to \mathscr{C} = \mathscr{C} \to A'$. Then, for a suitably large n, there is a homomorphism $A \to L_n$.*

Proof. The sequences $x_0, \ldots, x_n$ of elements of digraphs (X, R) such that always $(x_i, x_{i+1}) \in R$ or $(x_{i-1}, x_i) \in R$ will be referred to as *quasipaths* of length n. For a quasipath q put $P(q)$ (respectively $N(q)$) the number of occurrences of the former (respectively latter) of the two cases. If $x_0 = x_n$ we speak about *quasicycles*. Suppose that in the symmetrization of an antisymmetric (Y, S) there is no proper cycle of length $\leq d$. Then for every quasicycle q of length $\leq d$ in any (X, R) with a homomorphism $\varphi : (X, R) \to (Y, S)$ we have $P(q) = N(q)$. (Indeed, let n be the smallest natural number such that there is a counterexample $x_0, \ldots, x_n$. Since there are no short cycles in (Y, S), there are i, j, $i < j \leq n$, such that $\varphi(x_i) = \varphi(x_j)$. Thus we have homomorphisms $\alpha : P \to (Y, S)$, $\beta : Q \to (Y, S)$ where P (respectively Q) is obtained from the full subgraph spanned by $x_i, x_{i+1}, \ldots, x_j$ (respectively

$x_j, x_{j+1}, \ldots, x_n = x_0, x_1, \ldots, x_i$) by identifying x_i with x_j. This contradicts the choice of n since at least in one of P, Q there is a quasipath q with $P(q) \neq N(q)$.

Now, consider $(X, R) = A$. To prove our statement, it obviously suffices to prove that $P(q) = N(q)$ for every quasicycle $q = (x_0, \ldots, x_n)$ in A.

Take a symmetric $(Y, \bar{S})$ with chromatic number $\chi(Y, \bar{S}) > \chi(\bar{A}')$ where $\bar{A}'$ is the symmetrization of A', and such that every proper cycle in $(Y, \bar{S})$ is longer than n. (Such a $(Y, \bar{S})$ exists by [3].) Take an arbitrary orientation (Y, S) of $(Y, \bar{S})$. Since there is no homomorphism $(Y, S) \to A'$, there has to be a $\varphi : (X, R) \to (Y, S)$, so that $P(q) = N(q)$.

4.4. Remarks. In contrast with the symmetric case (see Remark 3.3) we are so far unable to prove much more than what is stated in Section 3. One can show very easily that if $A \not\to \mathscr{C} = \mathscr{C} \to A'$, there is a connected A_1 with $A_1 \not\to \mathscr{C} = A \not\to \mathscr{C}$. Also among the A such that there is no homomorphism $L_3 \to A$, $L_2 \not\to \mathscr{C}$ is the only $A \not\to \mathscr{C}$ equal to a $\mathscr{C} \to A'$.

4.5. We suspect that Proposition 4.2 covers all the possibilities. Let us formulate this as

Problem 1. Let $A \not\to \mathscr{C} = \mathscr{C} \to A'$ for $\mathscr{C}$ the class of all digraphs. Is then necessarily $A \sim L_n$ (and hence also $A' \sim \bar{L}_{n-1}$) for some n?

5. Forbidden factorobjects

5.1. Definition. Define the classes

$$(A_1, \ldots, A_n) \to \mathscr{C} = \{A \in \mathscr{C} : \text{there is an } i \text{ and an } f : A_i \to A\},$$

$$\mathscr{C} \not\to (A_1, \ldots, A_n) = \{A \in \mathscr{C} : \text{for no } i \text{ there is an } f : A \to A_i\},$$

(compare Definition 1.4.).

5.2. Interesting problems arise here in the question of productivity. We have:

Theorem. *$(A_1, \ldots, A_n) \to \mathscr{C}$ is productive iff it is equal to $A_i \to \mathscr{C}$ for some i.*

Proof. Obviously each class $A \to \mathscr{C}$ is productive. Consider a class $\mathscr{K} = (A_1, \ldots, A_n) \to \mathscr{C}$, $n > 1$. Clearly we may write $\mathscr{K} = (A_1', \ldots, A_m') \to \mathscr{C}$ such that there is no homomorphism $A_i' \to A_j'$ for every $i \neq j$, $i, j \in \{1, 2, \ldots, m\}$. If $m = 1$ then $\mathscr{K} = A_1' \to \mathscr{C}$ and $\mathscr{K}$ is productive. If $m > 1$ then $A_i \in \mathscr{K}$ but $A_1 \times A_2 \notin \mathscr{K}$ as $f : A_i \to A_1 \times A_2$ implies the existence of morphisms $A_i \to A_1$ and $A_i \to A_2$.

5.3. The situation with productivity of classes $\mathscr{C} \not\to (A_1, \ldots, A_n)$ is less transparent. Take, e.g. $A, B \in \mathscr{C}$ such that there is neither a homomorphism $A \to B$ nor

$B \to A$. (This occurs very often in reasonable classes $\mathscr{C}$, e.g. in the class of digraphs, undirected graphs, graphs containing a given one, n-chromatic graphs with $n \geq 3$, digraphs without cycles, etc.—see [4, 5].) Then A, B are in $\mathscr{C} \not\to (A \times B)$ while $A \times B$ is not. Thus, e.g. in digraphs consider A the 2-cycle and B the 3-cycle, $A \times B$ being the 6-cycle. In undirected graphs the smallest $A \times B$ thus obtained seems to be unnecessarily large. There arises

Problem 2. Find the smallest A in the class $\mathscr{C}$ of all undirected graphs (respectively digraphs) such that $\mathscr{C} \not\to A$ is not productive.

5.4. Positive problems of this kind are perhaps even more interesting. A trivial example is obtained as an immediate Corollary 4.2: If $\mathscr{C}$ is the class of all digraphs, every $\mathscr{C} \not\to \bar{L}_n (= L_{n+1} \to \mathscr{C})$ is productive. The following two statements are less trivial:

5.5. Theorem. *If $\mathscr{C}$ is the class of all digraphs then $\mathscr{C} \not\to L_n$ is a productive class for each $n \geq 1$.*

5.6. Theorem. *If $\mathscr{C}$ is the class of all digraphs and p is a prime number, then $\mathscr{C} \not\to C_p$ is a productive class (C_p is the directed cycle of length p).*

Remark. Thus, after proving the theorem we will see that, in contrast with Theorem 5.2, one has productive classes $\mathscr{C} \not\to (A_1, \ldots, A_n)$ with any number of A_i such that there are no homomorphisms $A_i \to A_j$. One can take e.g. $A_i = C_{p_i}$, where p_i are distinct primes, or $A_1 = \bar{L}_k$ and $A_i = C_{p_i}$ for $i \geq 2$. (Indeed, since $\mathscr{C} \not\to (A_1, \ldots, A_n) = \bigcap (\mathscr{C} \not\to A_i)$, it is productive whenever all the $\mathscr{C} \not\to A_i$ are.)

The proofs of Theorems 5.5 and 5.6 will be given after proving first two lemmas.

5.7. Observations. By Fig. 6, the products of quasipaths (see Proposition 4.3) formed by two arrows are indicated. The paths produced are assumed to be oriented from left to right, and from below upwards. (Thus, the products of the quasipath formed by two consecutive reversed arrows are not depicted explicitly; the reader can check easily that the following observations are true for them, too.)

(1) First, we see that in all the cases, an even number of arrows meet in the middle point.

(2) Second, we observe that if we orient paths in the product as indicated by dotted oriented courves, the position of the arrows in the product follows the rule given in Table 1 where the first two lines indicate the position of the arrows

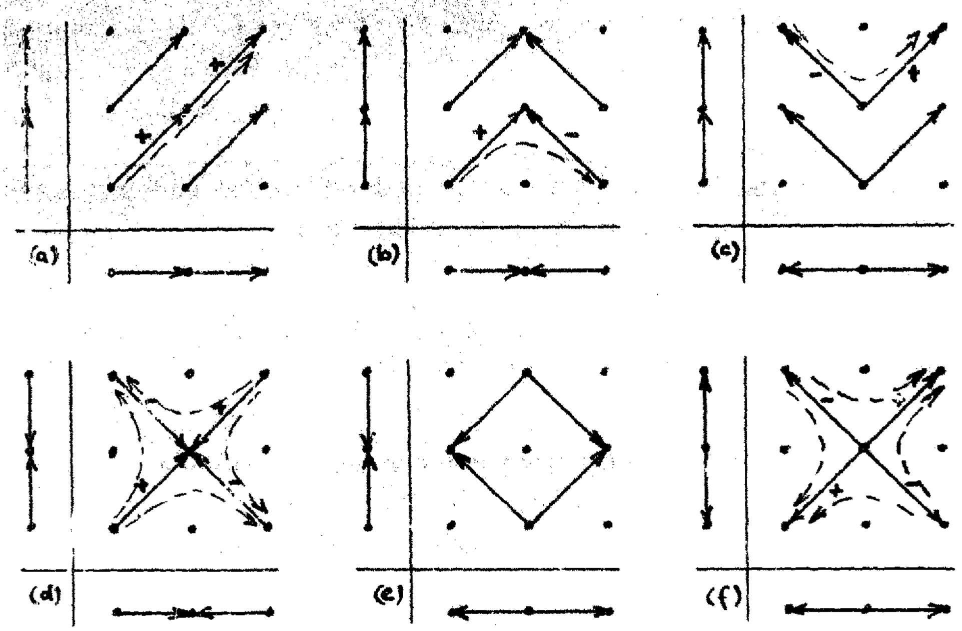

Fig. 6.

coming into the product, the third line indicates the position of the result:

Table 1

+	+	−	−
+	−	+	−
+	−	−	+

5.8. Lemma. *A product of two quasipaths of length n contains a quasipath of the same length n.*

Proof. Let $(\{0, \ldots, r\}, R), (\{0, \ldots, s\}, S)$ be the two quasipaths. Consider their product $(\{(i, j) : 0 \leq i \leq r, 0 \leq j \leq s\}, T)$. Put $\bar{T} = \{\{u, v\} : (u, v) \in T\}$ (the symmetrization of T). By 5.7(1), for $0 < i < r$ and $0 < j < s$, $\bar{T}\{i, j\}$ has even cardinality.

We can assume that no proper subquasipath of $(\{0, \ldots, r\}, R)$ or $(\{0, \ldots, s\}, S)$ has length n (otherwise choose, instead of the original quasipaths, a shortest subquasipath of the length n), and that $(0, 1) \in R$ and $(0, 1) \in S$. Then there is just one edge in $\bar{T}\{0, 0\}$. Now consider the longest quasipath q in T starting from $(0, 0)$: $(0, 0), (i_1, j_1), \ldots, (i_m, j_m)$, such that no edge repeats. Since $\bar{T}\{i, j\}$ is even for $0 < i < r$, $0 < j < s$, we have $(i_m, j_m) \in \{0, r\} \times \{0, 1, \ldots, s\} \cup \{0, 1, \ldots, r\} \times \{0, s\}$. Since $\bar{T}\{0, 0\}$ is one element, $(i_m, j_m) \neq (0, 0)$. Evidently, the length of q coincides with the lengths of its projections $0, i_1, i_2, \ldots, i_m$ and $0, j_1, \ldots, j_m$. Thus, if $i_m = 0$, this common length is 0 and hence also the length of $0, 1, 2, \ldots, j_m - 1, j_m$ is 0, so that, finally, the length of $j_m, j_m + 1, \ldots, s - 1, s$ is n in contradiction with the assumption on quasipaths. Thus, $i_m = r$ and hence the length of q is n.

5.9. Proof of Theorem 5.5. We have $A \in (\mathscr{C} \not\to L_n)$ iff there is a quasipath q of length $n+1$ and a homomorphism $\varphi: q \to A$. (Indeed, A contains either a quasipath of length $n+1$ or a cycle of nontrivial length; in the latter case we obtain the required homomorphism by "winding up" along the cycle.) Now, if $A_i \in (\mathscr{C} \not\to L_n)$, $i = 1, 2$, consider homomorphisms $\varphi_i : q_i \to A_i$ with length of q_i equal to $n+1$. By Lemma 5.8 we have an embedding $\psi : q \to q_1 \times q_2$ of an equally long quasipath. Thus, due to $(\varphi_1 \times \varphi_2) \circ \psi$, $A_1 \times A_2$ is in $\mathscr{C} \not\to L_n$.

5.10. Lemma. *Let q_1, q_2 be quasicycles with lengths n_1, n_2 respectively. Then the product $q_1 \times q_2$ can be decomposed into a disjoint system of quasicycles with the sum of lengths equal to n_1, n_2.*

Proof. By 5.7(1), $q_1 \times q_2$ obviously decomposes into a disjoint system of quasicycles. We have to prove that we can choose and orient them in such a way that the sum of their lengths will be as required. This follows from 5.7(2): If in cases (d) and (f), we observe the rule of proceeding from each arrow to a neighboring one (rather than to the opposite one), the position of the arrows is as indicated in Table 1. Thus, if q_i consists of n_i^+ direct and n_i^- reversed arrows, we have $n_i = n_i^+ - n_i^-$, and the sum of the lengths of the quasicycles in our decomposition is $(n_1^+ n_2^+ + n_1^- n_2^-) - (n_1^+ n_2^- + n_1^- n_2^+) = n_1 n_2$.

5.11. Proof of Theorem 5.6. One sees easily that $A \in (\mathscr{C} \not\to C_n)$ iff there is a quasicycle in A the length of which is not divisible by n. Consider such a quasicycle q_i in A_i, with length n_i. Let every quasicycle in $q_1 \times q_2$ have length divisible by n. By 5.10, $n_1 n_2$ is divisible by n, in contradiction with n being prime and dividing neither n_1 nor n_2.

5.12. Remark. On the other hand, one sees easily that no $\mathscr{C} \not\to C_n$ with $n = p_1^{\alpha_1} p_2^{\alpha_2} x$ where p_1 and p_2 are distinct primes, is productive.

The remaining case of $n =$ power of a prime is more complicated. We have a good indication that here again a productive class is obtained, but do not see any proof as simple as those above. We suspect, however, there may be one.

5.13. Little is known about productive classes $\mathscr{C} \not\to A$, where $\mathscr{C}$ is the class of (undirected) graphs. In [1] it was shown that productivity occurs with $A = K_1$, K_2 and K_3. The following problem is due to L. Lovász:

Problem 3. Is every class $\mathscr{C} \not\to K_n$ productive? (Equivalently, does it hold that $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$?)

So far, even the possibility that there are only finitely many A with productive $\mathscr{C} \not\to A$ is not excluded. Also, it would be interesting to find a productive $\mathscr{C} \not\to A$ with a non-complete A.

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