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Note

Star-factorization of symmetric complete bipartite multi-digraphs

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Abstract

We show that a necessary and sufficient condition for the existence of an S_k-factorization of the symmetric complete bipartite multi-digraph $\lambda K^*_{m,n}$ is $m = n \equiv 0 \,(\mathrm{mod}\, k(k-1)/d)$, where $d = (\lambda, k-1)$.

Keywords: Star-factorization; Symmetric complete bipartite multi-digraph

1. Introduction

The *symmetric complete bipartite multi-digraph* $\lambda K^*_{m,n}$ is the symmetric complete bipartite digraph $K^*_{m,n}$ in which every arc is taken λ times. Let S_k $(k \geqslant 3)$ denote the *orientation of the star* $K_{1,k-1}$ in which all arcs are directed away from the center-vertex to end-vertices. A spanning subgraph F of $\lambda K^*_{m,n}$ is called an *S_k-factor* if each component of F is isomorphic to S_k. If $\lambda K^*_{m,n}$ is expressed as an arc-disjoint sum of S_k-factors, then this sum is called an *S_k-factorization* of $\lambda K^*_{m,n}$. In this paper, it is shown that a necessary and sufficient condition for the existence of such a factorization is $m = n \equiv 0 \,(\mathrm{mod}\, k(k-1)/d)$, where $d = (\lambda, k-1)$.

Let K_{n_1,n_2}, $K^*_{n_1,n_2}$, K_{n_1,n_2,n_3}, $K^*_{n_1,n_2,n_3}$, and $K^*_{n_1,n_2,\dots,n_m}$ denote the complete bipartite graph, the symmetric complete bipartite digraph, the complete tripartite graph, the symmetric complete tripartite digraph, and the symmetric complete multipartite digraph, respectively. Let $\hat{C}_k$, $\hat{S}_k$, $\hat{P}_k$, and $\hat{K}_{p,q}$ denote the cycle or the directed cycle, the star or the directed star, the path or the directed path, and the complete bipartite graph or the complete bipartite digraph, respectively, on two partite sets V_i and V_j. Let $\bar{S}_k$

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and $\tilde{S}_k$ denote the evenly partite star and semi-evenly partite star, respectively, on three partite sets V_i, V_{j_1}, V_{j_2}. Then the problems of giving the necessary and sufficient conditions of $\hat{C}_k$-factorization of K_{n_1,n_2}, $K^*_{n_1,n_2}$, $K^*_{n_1,n_2,n_3}$, and $K^*_{n_1,n_2,\dots,n_m}$ have been completely solved by Enomoto et al. [3] and Ushio [10,13]. $\hat{S}_k$-factorization of K_{n_1,n_2}, $K^*_{n_1,n_2}$, and $K^*_{n_1,n_2,n_3}$ have been studied by Du [2], Martin [5,6], Ushio and Tsuruno [17], Ushio [12], and Wang [18]. Ushio [9] gives the necessary and sufficient condition of $\hat{S}_k$-factorization of $K^*_{n_1,n_2}$. Ushio [14,15] gives the necessary and sufficient conditions of $\bar{S}_k$-factorization and $\tilde{S}_k$-factorization of $K^*_{n_1,n_2,n_3}$. $\hat{P}_k$-factorization of K_{n_1,n_2} and $K^*_{n_1,n_2}$ have been studied by Ushio and Tsuruno [16], and Ushio [7,8]. $\hat{K}_{p,q}$-factorization of K_{n_1,n_2} has been studied by Martin [5]. Ushio [11] gives the necessary and sufficient condition of $\hat{K}_{p,q}$-factorization of $K^*_{n_1,n_2}$. For graph theoretical terms, see [1,4].

2. S_k-factor of $\lambda K^*_{m,n}$

The following theorem is on the existence of S_k-factors of $\lambda K^*_{m,n}$.

Theorem 1. *$\lambda K^*_{m,n}$ has an S_k-factor if and only if* (i) $m + n \equiv 0 \pmod{k}$, (ii) $(k-1)n - m \equiv 0 \pmod{k(k-2)}$, (iii) $(k-1)m - n \equiv 0 \pmod{k(k-2)}$, (iv) $m \leqslant (k-1)n$ *and* (v) $n \leqslant (k-1)m$.

Proof. Suppose that $\lambda K^*_{m,n}$ has an S_k-factor F. Let t be the number of components of F. Then $t = (m+n)/k$. Hence, Condition (i) is necessary. Among these t components, let t_1 and t_2 be the number of components whose center-vertices are in V_1 and V_2, respectively. Then, since F is a spanning subgraph of $\lambda K^*_{m,n}$, we have $t_1 + (k-1)t_2 = m$ and $(k-1)t_1 + t_2 = n$. Hence $t_1 = ((k-1)n - m)/k(k-2)$ and $t_2 = ((k-1)m - n)/k(k-2)$. From $0 \leqslant t_1 \leqslant m$ and $0 \leqslant t_2 \leqslant n$, we must have $m \leqslant (k-1)n$ and $n \leqslant (k-1)m$. Conditions (ii)–(v) are, therefore, necessary.

For those parameters m and n satisfying (i)–(v), let $t_1 = ((k-1)n - m)/k(k-2)$ and $t_2 = ((k-1)m - n)/k(k-2)$. Then t_1 and t_2 are integers such that $0 \leqslant t_1 \leqslant m$ and $0 \leqslant t_2 \leqslant n$. Hence, $t_1 + (k-1)t_2 = m$ and $(k-1)t_1 + t_2 = n$. Using t_1 vertices in V_1 and $(k-1)t_1$ vertices in V_2, consider t_1 S_k's whose end-vertices are in V_2. Using the remaining $(k-1)t_2$ vertices in V_1 and the remaining t_2 vertices in V_2, consider t_2 S_k's whose end-vertices are in V_1. Then these $t_1 + t_2$ S_k's are arc-disjoint and they form an S_k-factor of $\lambda K^*_{m,n}$. □

Corollary 2. *$\lambda K^*_{n,n}$ has an S_k-factor if and only if $n \equiv 0 \pmod{k}$.*

3. S_k-factorization of $\lambda K^*_{m,n}$

We use the following notation.

Notation. Given an S_k-factorization of $\lambda K^*_{m,n}$, let
r be the number of factors
t be the number of components of each factor
b be the total number of components.

Among t components of each factor, let t_1 and t_2 be the numbers of components whose center-vertices are in V_1 and V_2, respectively.

Among r components having vertex x in V_i, let r_{ij} be the numbers of components whose center-vertices are in V_j.

We give the following necessary condition for the existence of an S_k-factorization of $\lambda K^*_{m,n}$.

Theorem 3. *Let $d = (\lambda, k-1)$. If $\lambda K^*_{m,n}$ has an S_k-factorization, then $m = n \equiv 0 \pmod{k(k-1)/d}$.*

Proof. Suppose that $\lambda K^*_{m,n}$ has an S_k-factorization. Then $b = 2\lambda mn/(k-1)$, $t = (m+n)/k$, $r = b/t = 2\lambda kmn/(k-1)(m+n)$, $t_1 = ((k-1)n-m)/k(k-2)$, $t_2 = ((k-1)m-n)/k(k-2)$, $m \leqslant (k-1)n$, and $n \leqslant (k-1)m$. Moreover, $(k-1)r_{11} = \lambda n$, $r_{12} = \lambda n$, $(k-1)r_{22} = \lambda m$, and $r_{21} = \lambda m$. Thus we have $r = r_{11} + r_{12} = \lambda kn/(k-1)$ and $r = r_{21} + r_{22} = \lambda km/(k-1)$. Therefore, $m = n$ holds. Moreover, when $m = n$, we have $b = 2\lambda n^2/(k-1)$, $t = 2n/k$, $r = \lambda kn/(k-1)$, $t_1 = t_2 = n/k$, $r_{11} = r_{22} = \lambda n/(k-1)$, and $r_{12} = r_{21} = \lambda n$. Therefore, $n \equiv 0 \pmod{k(k-1)/d}$ holds, too. □

We prove the following extension theorems, which we use later in this paper.

Theorem 4. *If $\lambda K^*_{n,n}$ has an S_k-factorization, then $s\lambda K^*_{n,n}$ has an S_k-factorization for every positive integer s.*

Proof. Obvious. Construct an S_k-factorization of $\lambda K^*_{n,n}$ repeatly s times. Then we have an S_k-factorization of $s\lambda K^*_{n,n}$. □

Theorem 5. *If $\lambda K^*_{n,n}$ has an S_k-factorization, then $\lambda K^*_{sn,sn}$ has an S_k-factorization for every positive integer s.*

Proof. S_k can be denoted as $K_{1,k-1}$. When $\lambda K^*_{n,n}$ has an S_k-factorization, $\lambda K^*_{n,n}$ has a $K_{1,k-1}$-factorization. Therefore, $\lambda K^*_{sn,sn}$ has a $K_{s,(k-1)s}$-factorization. Obviously, $K_{s,(k-1)s}$ has an S_k-factorization. Therefore, $\lambda K^*_{sn,sn}$ has an S_k-factorization. □

We use the following notation for an S_k.

Notation. For an S_k whose center-vertex is u and end-vertices are $v_1, v_2, \ldots, v_{k-1}$, we denote $(u; v_1, v_2, \ldots, v_{k-1})$.

We give the following sufficient conditions for the existence of an S_k-factorization of $\lambda K^*_{n,n}$.

Theorem 6. *When* $k-1=p\lambda$ *and* $n=p(p\lambda+1)$, $\lambda K^*_{n,n}$ *has an* S_k*-factorization.*

Proof. Let $V_1=\{1,2,\ldots,n\}$, $V_2=\{1',2',\ldots,n'\}$. For $i=1,2,\ldots,p\lambda+1$ and $j=1,2,\ldots,p\lambda+1$, construct $(p\lambda+1)^2$ S_k-factors F_{ij} as follows:

$$
\begin{aligned}
F_{ij}=\{&((A+1);(B+p+1,B+p+2,\ldots,B+p+p\lambda)')\\
&((A+2);(B+p+p\lambda+1,B+p+p\lambda+2,\ldots,B+p+2p\lambda)')\\
&\cdots\\
&((A+p);(B+p+(p-1)p\lambda+1,B+p+(p-1)p\lambda\\
&\quad+2,\ldots,B+p+p^2\lambda)')\\
&((B+1)';(A+p+1,A+p+2,\ldots,A+p+p\lambda))\\
&((B+2)';(A+p+p\lambda+1,A+p+p\lambda+2,\ldots,A+p+2p\lambda))\\
&\cdots\\
&((B+p)';(A+p+(p-1)p\lambda+1,A+p+(p-1)p\lambda\\
&\quad+2,\ldots,A+p+p^2\lambda))\},
\end{aligned}
$$

where $A=(i-1)p$, $B=(j-1)p$, and the additions are taken modulo n with residues $1,2,\ldots,n$. Then they comprise an S_k-factorization of $\lambda K^*_{n,n}$. $\square$

Theorem 7. *Let* $d=(\lambda,k-1)$. *When* $n\equiv 0\,(\mathrm{mod}\,k(k-1)/d)$, $\lambda K^*_{n,n}$ *has an* S_k*-factorization.*

Proof. Put $\lambda=\alpha d$, $k-1=pd$, $(\alpha,p)=1$, $n=sk(k-1)/d$, and $N=k(k-1)/d$. Then we have $n=p(pd+1)s$ and $N=p(pd+1)$. By Theorem 6, $dK^*_{N,N}$ has an S_k-factorization. Applying Theorems 4 and 5, $\lambda K^*_{n,n}$ has an S_k-factorization. $\square$

We have the following main theorem.

Main Theorem. *Let* $d=(\lambda,k-1)$. $\lambda K^*_{m,n}$ *has an* S_k*-factorization if and only if* $m=n\equiv 0\,(\mathrm{mod}\,k(k-1)/d)$.

References

[1] G. Chartrand, L. Lesniak, Graphs & Digraphs, 2nd Edition, Wadsworth, Belmont, CA, 1986.

[2] B. Du, K_{1,p^2}-factorisation of complete bipartite graphs, Discrete Math. 187 (1998) 273–279.

[3] H. Enomoto, T. Miyamoto, K. Ushio, C_k-factorization of complete bipartite graphs, Graphs Combin. 4 (1988) 111–113.

[4] F. Harary, Graph Theory, Addison-Wesley, Reading, MA, 1972.

[5] N. Martin, Complete bipartite factorisations by complete bipartite graphs, Discrete Math. 167/168 (1997) 461–480.

[6] N. Martin, Balanced bipartite graphs may be completely star-factored, J. Combin. Des. 5 (1997) 407–415.

[7] K. Ushio, P_3-factorization of complete bipartite graphs, Discrete Math. 72 (1988) 361–366.

[8] K. Ushio, G-designs and related designs, Discrete Math. 116 (1993) 299–311.
[9] K. Ushio, Star-factorization of symmetric complete bipartite digraphs, Discrete Math. 167/168 (1997) 593–596.
[10] K. Ushio, $\hat{C}_k$-factorization of symmetric complete bipartite and tripartite digraphs, J. Fac. Sci. Technol. Kinki Univ. 33 (1997) 221–222.
[11] K. Ushio, $\hat{K}_{p,q}$-factorization of symmetric complete bipartite digraphs, in: Y. Alavi, D.R. Lick, A. Schwenk (Eds.), Graph Theory, Combinatorics, Algorithms and Applications, New Issues Press, Kalamazoo, MI, 1999, pp. 823–826.
[12] K. Ushio, $\hat{S}_k$-factorization of symmetric complete tripartite digraphs, Discrete Math. 197/198 (1999) 791–797.
[13] K. Ushio, Cycle-factorization of symmetric complete multipartite digraphs, Discrete Math. 199 (1999) 273–278.
[14] K. Ushio, $\bar{S}_k$-factorization of symmetric complete tripartite digraphs, Discrete Math. 211 (2000) 281–286.
[15] K. Ushio, Semi-evenly partite star-factorization of symmetric complete tripartite digraphs, Australas. J. Combin., 1999, to appear.
[16] K. Ushio, R. Tsuruno, P_3-factorization of complete multipartite graphs, Graphs Combin. 5 (1989) 385–387.
[17] K. Ushio, R. Tsuruno, Cyclic S_k-factorization of complete bipartite graphs, in: Y. Alavi, F.R.K. Chung, R.L. Graham, U.F. Hsu (Eds.), Graph Theory, Combinatorics, Algorithms and Applications SIAM, Philadelphia, PA, 1991, pp. 557–563.
[18] H. Wang, On $K_{1,k}$-factorizations of a complete bipartite graph, Discrete Math. 126 (1994) 359–364.