Viable control for uncertain nonlinear dynamical systems described by differential inclusions

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Abstract

In this paper, we will study the viable control problem for a class of uncertain nonlinear dynamical systems described by a differential inclusion. The goal is to construct a feedback control such that all trajectories of the system are viable in a map. Moreover, for any initial states no viable in the map, under the feedback control, all solutions of the system are steered to the map with an exponential convergence rate and viable in the map after a finite time $T$. In this case, an estimate of the time $T$ of all trajectories attaining the map is given. In the nanomedicine system, an example inspired from cerebral embolism and cerebral thrombosis problems illustrates the use of our main results.

Keywords: Viable control problem; Differential inclusion; Feedback-controlled system; Uncertain dynamical systems; Exponential convergence rate

1. Introduction

Recently, the problem of designing a state feedback control to stabilize a dynamical system with significant uncertainties has been widely studied over the last decade. A common approach is to describe the dynamics of the control system by nonlinear ordinary differential equations or
differential inclusions (see [4–9]). Then Lyapunov techniques are used constructively to design a feedback control such that certain stability performance for the uncertain dynamical system is achieved. In this paper, we will apply the Lyapunov techniques to study the viable control problem for a class of uncertain nonlinear dynamical systems described by a differential inclusion as follows:

\[
\dot{x}(t) \in F(t, x(t), u(t)),
F(t, x(t), u(t)) : = f(t, x(t)) + F_\alpha(t, x(t)) + Q(t, x(t))[u(t) + F_\beta(t, x(t), u(t))],
\]

(1.1)

where \( t \in [0, \infty) \) is the time variable, \( u(t) \in \mathbb{R}^p \) is the control input, and \( x(t) \in \mathbb{R}^n \) denotes the state of the system. The set-valued maps \( F_\alpha(x) \subseteq \mathbb{R}^p \) and \( F_\beta(u) \subseteq \mathbb{R}^p \) model the system uncertainty. The functions \( f: \mathbb{R}^n \to \mathbb{R}^n \) and \( Q: \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^p \) are single-valued continuous functions and have linear growth. With the state feedback \( u(t) = u(t, x(t)) \), the feedback-controlled system (1.1) becomes the closed-loop system described by the differential inclusion as follows:

\[
\dot{x}(t) \in F_c(t, x(t)): = F_\alpha(t, x(t)) + F_\beta(t, x(t), u(t)).
\]

(1.2)

When \( F_\alpha(t, x(t)) = F_\beta(t, x(t), u(t)) = \{0\} \), observe that the original system (1.1) may be regarded as the model of the nominal system (1.3) without uncertainty described as

\[
\dot{x}(t) = f(t, x(t)) + Q(t, x(t))u(t).
\]

(1.3)

This implies that the nominal system (1.3) is a special case of the nonlinear dynamical system (1.1) subject to uncertainty.

Throughout this paper, let \( h: [0, \infty) \to \mathbb{R}^n \) be a single-valued continuous differentiable map, where \( h(\cdot) \) is a Lipschitz map; that is, there exists a constant \( K_h \geq 0 \) such that

\[
\|h(s) - h(t)\| \leq K_h \|s - t\| \quad \text{for all } s, t \in [0, \infty).
\]

In this paper, we will consider the completely viable control problem of an uncertain nonlinear dynamical system described by a differential inclusion. The goal is to find a feedback control \( u(t) = u(t, x(t)) \) such that the closed-loop system (1.2) is completely viable controllable for \( h(\cdot) \).

In this case, for any initial state \( x_0 = h(0) \), all solutions \( x(\cdot) \) of the system (1.2) satisfy \( x(t) = h(t) \) for all \( t \geq 0 \). Furthermore, if \( x_0 \neq h(0) \), namely \( x(\cdot) \) is not viable in the map \( h(\cdot) \) at the initial state, under the feedback control, all solutions of the uncertain nonlinear dynamical system (1.2) are viable for \( h(\cdot) \) after a finite time \( T \), that is, \( x(t) = h(t) \) for all \( t \geq T \). An estimate of the time \( T \) of all trajectories \( x(\cdot) \) attaining the map \( h(\cdot) \) is given. Moreover, all trajectories \( x(\cdot) \) of the system (1.2) are steered to the map \( h(\cdot) \) with an exponential convergence rate.

2. Assumptions and definitions

For convenience, denote \( \|\cdot\| \) as the Euclidean norm or the corresponding induced norm of a matrix. Let

\[
\|F(x)\| : = \sup_{y \in F(x)} \|y\|,
\]

where \( F \) is a set-valued map. For the existence of solutions of differential inclusions (1.2), in general case, \( F_c(\cdot, \cdot) \) needs to satisfy the assumption of upper semicontinuity. More precisely, if \( F_c(\cdot, \cdot) \) is upper semicontinuous with convex and compact values, for any initial state \( x_0 \), then there exist a positive \( T \) and a solution \( x(\cdot) \) defined on \([0, T]\) for the system (1.2) such that either
T = ∞ \quad \text{or} \quad T < ∞ \quad \text{and} \quad \limsup_{t \to T^-} \| x(t) \| = ∞

(cf. [2, p. 98, Theorem 3]; [3, p. 390, Theorem 10.1.3]). Further more adequate information—a priori estimates on the growth of $F_c(\cdot, \cdot)$—allows us to exclude the case when \( \limsup_{t \to T^-} \| x(t) \| = ∞ \). This is the case for instance when \( F_c(\cdot, \cdot) \) is bounded for all \( t \geq 0 \).

In general, we can take \( T = ∞ \) when \( F_c(\cdot, \cdot) \) enjoys linear growth as the following definition.

**Definition 2.1** [1, p. 62]. Let \( F : [0, ∞) \times X \mapsto Y \) be a set-valued map from the domain \( [0, ∞) \times X \), denoted by \( \text{Dom}(F) \), into the codomain \( Y \). We say that \( F \) has linear growth if there exists a positive constant \( c \) such that

\[
\| F(t, x) \| \equiv \sup_{y \in F(t, x)} \| y \| \leq c(\| x \| + 1)
\]

for each \( (t, x) \in \text{Dom}(F) \).

We say that \( F \) is a Marchaud map if it is nontrivial, upper semicontinuous, has compact convex images and linear growth. Clearly, any single-valued Lipschitz map is a Marchaud map.

**Assumption 2.1.** Throughout the paper the following assumptions are made.

(A1) \( F_\alpha(t, x) \) and \( F_\beta(t, x, u) \) are upper semicontinuous with convex and compact values for all \( t \in [0, ∞) \), \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^p \);

(A2) \( \| F_\alpha(t, x) \| \leq k_\alpha(x) \) for all \( t \in [0, ∞) \), \( x \in \mathbb{R}^n \);

(A3) \( \| F_\beta(t, x, u) \| \leq k_\beta(x) + \gamma \| u \| \) for all \( t \in [0, ∞) \), \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^p \);

(A4) \( \| Q(t, x) u \| \leq k_q(x) + r \| u \| \) for all \( t \in [0, ∞) \), \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^p \);

(A5) \( \text{rank}[Q(t, x)] = n \leq p \) for all \( t \in [0, ∞) \), \( x \in \mathbb{R}^n \), and

\[
0 < \| (Q Q^T)^{-1} Q \|_\infty \equiv \sup_{t \in [0, ∞), x \in \mathbb{R}^n} \{ \| [Q(t, x) Q^T(t, x)]^{-1} Q(t, x) \| \} < ∞,
\]

where \( k_\alpha(x) \), \( k_\beta(x) \) and \( k_q(x) \) are nonnegative real-valued function with linear growth, \( 0 < γ < 1 \) and \( r \) are known positive constants.

**Remark 2.1.** Note that the existence of solutions \( x(\cdot) \) defined on \( [0, ∞) \) for the closed-loop system (1.2) satisfying Assumption 2.1 is guaranteed. More precisely, assumptions (A2)–(A4) imply that \( F_c(\cdot, \cdot) \) enjoys linear growth (see Section 3). Assumption 2.1(A5) shows that all solutions \( x(t) \) of the feedback-controlled system (1.2) are steered to the map \( h(t) \) with an exponential convergence rate (see the Section 4).

Now, let \( h : [0, ∞) \to \mathbb{R}^n \) be a single-valued continuous differentiable Lipschitz map. We define an uncertain dynamical system which is completely viable controllable for \( h \) as follows.

**Definition 2.2.** We say that the system (1.2) is viable controllable for \( h \) if for all initial states \( x_0 \in \mathbb{R}^n \), there exist \( T \geq 0 \), a feedback control \( u(t) = u(t, x(t)) \) and a solution \( x(\cdot) \) of the closed-loop differential inclusion (1.2) satisfying \( x(t) = h(t) \) for all \( t \geq T \).

**Definition 2.3.** We say that the system (1.2) is completely viable controllable for \( h \) if for all initial states \( x_0 \in \mathbb{R}^n \), there exist \( T \geq 0 \) and a feedback control \( u(t) = u(t, x(t)) \) such that all solutions \( x(\cdot) \) of the closed-loop differential inclusions (1.2) satisfying \( x(t) = h(t) \) for all \( t \geq T \).
Remark 2.2. Clearly, by Definitions 2.2 and 2.3, we obtain that the completely viable controllable system for $h$ is also viable controllable for $h$.

3. Design of feedback control inputs

Consider now the uncertain nonlinear dynamical system (1.1) with the control $u(t)$:

\[ u(t) = u_n(t) + u_c(t), \]

where $u_n(t)$ is given by

\[ u_n(t) = Q^T(t, x(t)) \left[ Q(t, x(t)) Q^T(t, x(t)) \right]^{-1} \left[ A(x(t) - h(t)) - f(x(t)) + \frac{dh(t)}{dt} \right], \]

and

\[ u_c(t) = -k(x(t)) \cdot \Psi \left( Q^T(t, x(t)) M(x(t) - h(t)) \right), \]

where $M$ is the positive definite symmetric $n \times n$ matrix satisfying the following Lyapunov equation:

\[ A^T M + MA = -L, \]

where $L$ is an arbitrary positive definite symmetric $n \times n$ matrix and $A$ is an Hurwitz $n \times n$ matrix; $k(x(t))$ is a positive real-valued continuous function with linear growth satisfying

\[ k(x(t)) \geq k_0(x(t)), \]

\[ k_0(x(t)) := (1 - \gamma)^{-1} \left[ k_\beta(x(t)) + \gamma \| u_n(t) \| + k_\alpha(x(t)) \left\| (Q Q^T)^{-1} Q \right\|_\infty + \delta \right], \]

where $\delta$ is any positive constant, and

\[ \Psi(\xi) := \begin{cases} \xi / \| \xi \| & \text{if } \xi \neq 0, \\ \{ \theta \in \mathbb{R}^p : \| \theta \| \leq 1 \} & \text{if } \xi = 0, \end{cases} \]

is an upper semicontinuous function on $\mathbb{R}^p$.

For the existence of solutions $x(\cdot)$ defined on $[0, \infty)$ for the closed-loop system (1.2), we need to show that $F_c(t, x) := F(t, x, u(t, x))$ is a Marchaud map as follows.

Lemma 3.1. Let the feedback-controlled system (1.2) satisfy assumptions (A1)–(A5), subject to the controller (3.1) with (3.2)–(3.6). Then, in the system (1.2), $F_c(t, x)$ is a Marchaud map.

Proof. By assumption (A1), for all $x \in \mathbb{R}^n$, $F_c(t, x)$ is upper semicontinuous with convex and compact value. From Definition 2.1, we only check that $F_c(t, x)$ is dominated by a linear growth map, which implies $F_c(t, x)$ is a Marchaud map. By (A2)–(A5), we have

\[
\| F_c(t, x) \|
\leq \| A \| (\| x \| + \| h(t) \|) + K_h + k_\alpha(x) + k_\beta(x) + r \| u_c + F_\beta(t, x, u) \|
\leq \| A \| (\| x \| + \| h(t) \|) + K_h + k_\alpha(x) + k_\beta(x) + r \left[ \| u_c \| + k_\beta(x) + \gamma \| u_n + u_c \| \right]
\leq \| A \| (\| x \| + \| h(t) \|) + K_h + k_\alpha(x) + k_\beta(x) + r \left[ (1 + \gamma) \| u_c \| + k_\beta(x) + \gamma \| u_n \| \right]
\leq \| A \| (\| x \| + \| h(t) \|) + K_h + k_\alpha(x) + k_\beta(x) + r \left[ (1 + \gamma) \| u_c \| + k_\beta(x) \right] + r \gamma \| u_n \|
\[ \leq \|A\| (\|x\| + \|h(t)\|) + K_h + k_\alpha(x) + k_q(x) + r \left( (1 + \gamma)(k(x) + 1) + k_\beta(x) \right) \]
\[ + r\gamma \left[ \|QQ^T\|^{-1} \|A\| (\|x\| + \|h(t)\|) + \|f(t, x)\| + \|K_h\| \right]. \]

where \( K_h \geq 0 \) is a Lipschitz constant of \( h \). This shows that \( F_c(t, x) \) is dominated by a linear growth map. This implies that \( F_c(t, x) \) has linear growth. \( \square \)

4. Main results

For convenience, the Euclidean inner product is denoted by \( \langle \cdot, \cdot \rangle \). We also define \( \langle x, S \rangle \) to be the subset \( \{ \langle x, s \rangle \mid s \in S \} \) of \( \mathbb{R} \) and define \( \langle x, S \rangle \leq K \) to mean \( \langle x, s \rangle \leq b \) for all \( s \in S \), where \( b \in \mathbb{R} \). Denote \( \lambda_m(M) \) and \( \lambda_M(M) \) as the minimum and the maximum eigenvalues of the real symmetric matrix \( M \), respectively.

In this paper, to obtain the main theorem, we need the following two lemmas.

**Lemma 4.1.** Let \( x(t) \) be any trajectory of the feedback-controlled system (1.2) satisfying assumptions (A1)–(A5), subject to the controller (3.1) with (3.2)–(3.6). Given any real number \( T_0 \in [0, \infty) \) with \( x(T_0) \neq h(T_0) \), there exists a positive real number \( T_1 \in (T_0, \infty) \) such that all trajectories \( x(t) \) of the feedback-controlled system (1.2) are steered to the map \( h(t) \) with an exponential convergence rate on \([T_0, T_1]\) and \( x(T_1) = h(T_1) \), where \( x(t) \neq h(t) \) for all \( t \in [T_0, T_1) \).

**Proof.** First, let \( x(t) \) be any trajectory of the system (1.2). We claim that there exists a positive real number \( T_1 \in (T_0, \infty) \) such that \( x(T_1) = h(T_1) \), where \( T_1 \in (T_0, \infty) \) is the smallest positive real number such that \( x(T_1) = h(T_1) \). Suppose that \( T_1 \) does not exist, that is, \( T_1 \) is infinite. This implies that \( x(t) \neq h(t) \) for all \( t \geq T_0 \). Let \( e(t) = x(t) - h(t) \) be the deviation of the state \( x(t) \) from the map \( h(t) \). In terms of state \( x \) and error \( e \), the closed-loop system (1.2) becomes

\[
\dot{x}(t) = f(t, x(t)) + F_\alpha(t, x(t)) + Q(t, x(t)) [u(t) + F_\beta(t, x(t), u(t))],
\]
\[
\dot{e}(t) = \dot{x}(t) - \frac{dh(t)}{dt} = Ae(t) + F_\alpha(t, x(t)) + Q(t, x(t))u_c(t) + Q(t, x(t))F_\beta(t, x(t), u(t)).
\]

Let \( V(e) = (1/2)e^T Me \) for all \( e \in \mathbb{R}^n \). Then for all \( x(t) \neq h(t), t \geq T_0 \), we have

\[
\frac{dV(e(t))}{dt} = \frac{1}{2} \langle \dot{e}(t), Me(t) + e(t)M\dot{e}(t) \rangle = e^T(t)M\dot{e}(t)
\]
\[
\leq \langle Me(t), Ae(t) \rangle + \|Me(t), Q(t, x(t))u_c(t)\| + \|Me(t), F_\alpha(t, x(t))\|
\]
\[
+ \|Me(t), Q(t, x(t))F_\beta(t, x(t), u(t))\|
\]
\[
= \langle Me(t), Ae(t) \rangle + \|Q^T(t, x(t))Me(t), u_c(t)\|
\]
\[
+ \|Q^T(t, x(t))Me(t), Q^T(t, x(t))[Q(t, x(t))Q^T(t, x(t))]^{-1}F_\alpha(t, x(t))\|
\]
\[
+ \|Q^T(t, x(t))Me(t), F_\beta(t, x(t), u(t))\|
\]
\[
\leq -\frac{1}{2} e^T(t)Le(t) - k(x(t))\|Q^T(t, x(t))Me(t)\|
\]
\[
+ k_\alpha(x(t))\|Q^T(t, x(t))Me(t)\|\|Q^T(t, x(t))[Q(t, x(t))Q^T(t, x(t))]^{-1}\|
\]
\[
+ [k_\beta(x(t)) + \gamma]\|u_n(t) + u_c(t)\|\|Q^T(t, x(t))Me(t)\|
\]
\[
\begin{align*}
    &\leq -\frac{1}{2}e^T(t)Le(t) - k(x(t))\|Q^T(t, x(t))Me(t)\| \\
    &\quad + k_\alpha(x(t))\|Q^T(t, x(t))Me(t)\|\|Q^T(t, x(t))[Q(t, x(t))Q^T(t, x(t))]^{-1}\| \\
    &\quad + [k_\beta(x(t)) + \gamma\|u_n(t)\| + \gamma k(x(t))]\|Q^T(t, x(t))Me(t)\| \\
    &\leq -\frac{1}{2}e^T(t)Le(t) - (1 - \gamma)k(x(t))\|Q^T(t, x(t))Me(t)\| \\
    &\quad + k_\alpha(x(t))\|Q^T(t, x(t))Me(t)\|\|Q^TQQ^T)^{-1}\|_\infty \\
    &\quad + [k_\beta(x(t)) + \gamma\|u_n(t)\| + k_\alpha(x(t))((QQ^T)^{-1}Q^\infty)]\|Q^T(t, x(t))Me(t)\| \\
    &= -\frac{1}{2}e^T(t)Le(t) - \delta\|Q^T(t, x(t))Me(t)\|. \quad (4.1)
\end{align*}
\]

Since \((1/2)\lambda_m(M) \leq (1/2)e^T(t)Me(t) = V(e(t))\), we have

\[
\begin{align*}
    V(e(t)) &= \frac{1}{2}\{e(t), [Q(t, x(t))Q^T(t, x(t))]^{-1}Q(t, x(t))Q^T(t, x(t))Me(t)\} \\
    &\leq \frac{1}{2}\|[Q(t, x(t))Q^T(t, x(t))]^{-1}Q(t, x(t))\|e(t)\|Q^T(t, x(t))Me(t)\| \\
    &\leq \frac{1}{2}\|(QQ^T)^{-1}Q\|_\infty \left(\frac{2V(e(t))}{\lambda_m(M)}\right)^{1/2}\|Q^T(t, x(t))Me(t)\|.
\end{align*}
\]

Combining the above result and (4.1), for all \(x(t) \neq h(t), t \geq T_0\), we obtain

\[
\begin{align*}
    \frac{dV(e(t))}{dt} &\leq -\frac{1}{2}e^T(t)Le(t) - \delta\frac{\sqrt{2\lambda_m(M)}}{\|(QQ^T)^{-1}Q\|_\infty}(V(e(t)))^{1/2}. \quad (4.2)
\end{align*}
\]

Applying \(V(e(t)) \leq (1/2)\lambda_M(M)\|e(t)\|^2\) and \((1/2)\lambda_m(L)\|e(t)\|^2 \leq (1/2)e^T(t)Le(t)\) to (4.2) yields

\[
\begin{align*}
    \frac{dV(e(t))}{dt} &\leq -\frac{1}{2}e^T(t)Le(t) \leq -\frac{\lambda_m(L)}{\lambda_M(M)}V(e(t)) \quad \text{for all } t \geq T_0, \\
    V(e(t)) &\leq V(e(0))e^{-\frac{\lambda_m(L)}{\lambda_M(M)}t} \to 0 \quad \text{as } t \to \infty. \quad (4.3)
\end{align*}
\]

From (4.2), we also have

\[
\begin{align*}
    \frac{dV(e(t))}{dt} &\leq -\delta\frac{\sqrt{2\lambda_m(M)}}{\|(QQ^T)^{-1}Q\|_\infty}(V(e(t)))^{1/2} \quad \text{for all } t \geq T_0.
\end{align*}
\]

To solve the above inequality, it is easy to get

\[
\begin{align*}
    \int_{V(e(T_0))}^{V(e(t))} (V)^{-1/2} dV &\leq \int_{T_0}^{t} \delta\left(\frac{2\lambda_m(M)}{\|(QQ^T)^{-1}Q\|_\infty^2}\right)^{1/2} dt, \\
    2\left[(V(e(t)))^{1/2} - (V(e(T_0)))^{1/2}\right] &\leq -\delta\left(\frac{2\lambda_m(M)}{\|(QQ^T)^{-1}Q\|_\infty^2}\right)^{1/2}(t - T_0),
\end{align*}
\]
\begin{equation}
2\left( [V(e(T_0))]^{1/2} - (V(e(t)))^{1/2} \right) \geq \delta \left( \frac{2\lambda_m(M)}{\|QQ^T\|^{-1}Q\|_\infty^2} \right)^{1/2} (t - T_0). \tag{4.4}
\end{equation}

Taking the limit \( t \to \infty \) on the two sides of (4.4), by (4.3), we obtain
\begin{equation}
2(V(e(T_0)))^{1/2} = \lim_{t \to \infty} 2\left( [V(e(T_0))]^{1/2} - (V(e(t)))^{1/2} \right) \geq \lim_{t \to \infty} \left( \delta \left( \frac{2\lambda_m(M)}{\|QQ^T\|^{-1}Q\|_\infty^2} \right)^{1/2} (t - T_0) \right) = \infty.
\end{equation}

This contradicts the fact that \( 2(V(e(T_0)))^{1/2} < \infty \). Thus, we conclude that \( T_1 \) is finite.

Finally, we claim that the trajectory \( x(\cdot) \) of the system (1.2) is steered to the map \( h(\cdot) \) with an exponential convergence rate on \( [T_0, T_1] \). Since \( x(t) \neq h(t) \) for all \( t \in [T_0, T_1] \), (4.2) induces that
\begin{equation}
\dot{V}(e) \leq -\frac{1}{2} e^T L e - \delta \frac{\sqrt{2\lambda_m(M)}}{\|QQ^T\|^{-1}Q\|_\infty} (V(e))^{1/2} < 0.
\end{equation}

This shows that \( V(e(\cdot)) \) is decreasing on \( [T_0, T_1] \). Applying \( V(e) \leq (1/2)\lambda_M(M)\|e\|^2 \) and \((1/2)\lambda_m(L)\|e\|^2 \leq (1/2)e^T L e\), we obtain
\begin{equation}
\frac{dV(e(t))}{dt} \leq -\frac{1}{2} e^T Le(t) \leq -\frac{\lambda_m(L)}{\lambda_M(M)} V(e(t)),
\end{equation}
\begin{equation}
V(e(t)) \leq V(e(T_0)) e^{-\frac{\lambda_m(L)}{\lambda_M(M)} (t - T_0)} \quad \text{for all } t \in [T_0, T_1].
\end{equation}

Since \((1/2)\lambda_m(M)\|e\|^2 \leq V(e) \leq (1/2)\lambda_M(M)\|e\|^2 \), from the above inequality, we get
\begin{equation}
\|x(t) - h(t)\| \leq \sqrt{\frac{\lambda_M(M)}{\lambda_m(M)}} \|x(T_0) - h(T_0)\| e^{-\frac{\lambda_m(L)}{2\lambda_M(M)} (t - T_0)}. \tag{4.5}
\end{equation}

This shows that the trajectory \( x(\cdot) \) of the feedback-controlled system (1.2) is steered to the map \( h(t) \) with an exponential convergence rate on \( [T_0, T_1] \).

**Lemma 4.2.** Let \( x(t) \) be any trajectory of the feedback-controlled system (1.2) satisfying assumptions (A1)–(A5) subject to the controller (3.1) with (3.2)–(3.6). If \( x(T) = h(T) \), then \( x(t) = h(t) \) for all \( t > T \).

**Proof.** Suppose that Lemma 4.2 is not true, that is, there exists a positive real number \( T_0 \in (T, \infty) \) such that \( x(T_0) \neq h(T_0) \). By Lemma 4.1, there exists a positive real number \( T_1 \in (T_0, \infty) \) such that \( x(T_1) = h(T_1) \) and \( x(t) \neq h(t) \) for all \( t \in [T_0, T_1] \). Without loss of generality, we can assume that there exists an interval \([t_1, t_2] \subset [T, T_1]\) such that \( x(t_1) = h(t_1), x(t_2) = h(t_2) \) and \( x(t) \neq h(t) \) for all \( t \in (t_1, t_2) \), where \( t_1 < t_2 \). By the same argument of Lemma 4.1, (4.2) induces the following result:
\begin{equation}
\dot{V}(e) \leq -\frac{1}{2} e^T L e - \delta \frac{\sqrt{2\lambda_m(M)}}{\|QQ^T\|^{-1}Q\|_\infty} (V(e))^{1/2} < 0 \quad \text{for all } t \in (t_1, t_2).
\end{equation}

Obviously, the above result implies that
\begin{equation}
\dot{V}(e(t)) \leq -\delta \left( \frac{2\lambda_m(M)}{\|QQ^T\|^{-1}Q\|_\infty^2} \right)^{1/2} (V(e(t)))^{1/2} \quad \text{holds for all } t \in (t_1, t_2).
\end{equation}

To solve the above inequality, it is easy to get
\[
\int_{V(e(t_1))}^{V(e(t_2))} (V)^{-1/2} dV \leq -\int_{t_1}^{t_2} \delta \left( \frac{2\lambda_m(M)}{\|(QQ^T)^{-1}Q\|_\infty^2} \right)^{1/2} dt,
\]
\[
2 \left[ (V(e(t_2)))^{1/2} - (V(e(t_1)))^{1/2} \right] \leq -\delta \left( \frac{2\lambda_m(M)}{\|(QQ^T)^{-1}Q\|_\infty^2} \right)^{1/2} (t_2 - t_1),
\]
\[
0 = 2 \left[ (V(0))^{1/2} - (V(0))^{1/2} \right] \leq -\delta \left( \frac{2\lambda_m(M)}{\|(QQ^T)^{-1}Q\|_\infty^2} \right)^{1/2} (t_2 - t_1).
\]

This shows that \( t_2 = t_1 \). This contradicts the fact that \( t_2 > t_1 \). Thus Lemma 4.2 is true.

**Remark 4.1.** Let \( x(t) \) be any trajectory of the feedback-controlled system (1.2) satisfying assumptions (A1)–(A5), subject to the controller (3.1) with (3.2)–(3.6). By Lemma 4.2, if any initial state \( x(0) = h(0) \) then \( x(t) = h(t) \) for all \( t \geq 0 \).

Combining Lemmas 4.1 and 4.2, we obtain the following theorem.

**Theorem 4.1.** Let \( x(t) \) be any trajectory of the feedback-controlled system (1.2) satisfying assumptions (A1)–(A5), subject to the controller (3.1) with (3.2)–(3.6). If any initial state \( x(0) \neq h(0) \), then there exists a positive real number \( T > 0 \) such that all trajectories \( x(t) \) of the system (1.2) are steered to the map \( h(\cdot) \) with an exponential convergence rate, that is,
\[
\|x(t) - h(t)\| \leq \sqrt{\frac{\lambda_M(M)}{\lambda_m(M)}} \cdot \|x(0) - h(0)\| \cdot e^{-\frac{\lambda_m(L)}{2\lambda_M(M)} t} \quad \text{for all } t \in [0, T)
\]
and \( x(t) = h(t) \) for all \( t \geq T \).

**Proof.** In Lemma 4.1, taking \( T_0 = 0 \), then there exists a positive real number \( T > 0 \) such that the trajectory \( x(t) \) of the system (1.2) is steered to the map \( h(\cdot) \) with an exponential convergence rate on \([0, T]\) and \( x(T) = h(T) \). Moreover, applying Lemma 4.2 and (4.5), we conclude that
\[
\|x(t) - h(t)\| \leq \sqrt{\frac{\lambda_M(M)}{\lambda_m(M)}} \cdot \|x(0) - h(0)\| \cdot e^{-\frac{\lambda_m(L)}{2\lambda_M(M)} t} \quad \text{for all } t \in [0, T)
\]
and \( x(t) = h(t) \) for all \( t \geq T \). \( \square \)

**Theorem 4.2.** Let \( x(t) \) be any trajectory of the feedback-controlled system (1.2) satisfying assumptions (A1)–(A5), subject to the controller (3.1) with (3.2)–(3.6). If any initial state \( x(0) \neq h(0) \), then an estimate of the time \( T \) of all trajectories \( x(\cdot) \) attaining \( h(\cdot) \) is bounded by
\[
\| (QQ^T)^{-1}Q \|_\infty \sqrt{\frac{\lambda_M(M)}{\lambda_m(M)}} \|e(0)\|,
\]
where \( \|e(0)\| = \|x(0) - h(0)\| \) denotes the distance from the initial state \( x(0) \) to \( h(0) \).
Proof. Let \( x(t) \) be any trajectory of the system (1.2). Since \( x(0) \neq h(0) \), by Theorem 4.1, there exists a positive real number \( T > 0 \) such that \( x(t) \neq h(t) \) for all \( t \in [0, T) \) and \( x(t) = h(t) \) for all \( t \geq T \). Now, estimate the time \( T \) as follows. Taking \( T_0 = 0 \) in Lemma 4.1, (4.2) induces that
\[
\dot{V}(e(t)) \leq -\frac{1}{2} e^T(t) L e(t) - \frac{\sqrt{2\lambda_m(M)}}{||Q Q^T||_1 Q||_\infty} (V(e(t)))^{1/2} < 0 \quad \text{for all } t \in [0, T). \quad (4.6)
\]
Obviously, by (4.6), we obtain
\[
\dot{V}(e(t)) \leq -\delta \left( \frac{2\lambda_m(M)}{||Q Q^T||_1 Q||_\infty^2} \right)^{1/2} (V(e(t)))^{1/2} \quad \text{for all } t \in [0, T).
\]
To solve the above inequality, it is easy to get
\[
\int_{V(e(0))}^{V(e(T))} (V)^{-1/2} \, dV \leq -\int_0^T \delta \left( \frac{2\lambda_m(M)}{||Q Q^T||_1 Q||_\infty^2} \right)^{1/2} \, dt.
\]
This implies that
\[
-2\sqrt{V(e(0))} = 2 \left[ (V(e(T)))^{1/2} - (V(e(0)))^{1/2} \right] \leq -\delta \left( \frac{2\lambda_m(M)}{||Q Q^T||_1 Q||_\infty^2} \right)^{1/2} T.
\]
From \((1/2)\lambda_m(M) ||e||^2 \leq V(e) \leq (1/2)\lambda_M(M) ||e||^2\), the above inequality implies that
\[
\delta \left( \frac{2\lambda_m(M)}{||Q Q^T||_1 Q||_\infty^2} \right)^{1/2} T \leq 2 \left( \sqrt{\frac{1}{2}} \lambda_M(M) ||e(0)|| \right).
\]
Hence
\[
T \leq \frac{||Q Q^T||_1 Q||_\infty}{\delta \lambda_m(M)} \sqrt{\frac{\lambda_M(M)}{\lambda_m(M)}} ||e(0)||. \quad \square
\]
Combining Theorems 4.1 and 4.2, we obtain the main theorem as follows.

**Theorem 4.3.** Let \( x(t) \) be any trajectory of the feedback-controlled system (1.2) satisfying assumptions (A1)–(A5). If for any initial state \( x(0) \neq h(0) \), namely \( x(\cdot) \) is not viable in the map \( h(\cdot) \) at the initial state, then the controller (3.1) with (3.2)–(3.6) such that the system (1.2) is completely viable controllable for \( h \). Moreover, all trajectories \( x(t) \) of the system (1.2) are steered to the map \( h(t) \) with an exponential convergence rate, i.e.,
\[
\|x(t) - h(t)\| \leq \sqrt{\frac{\lambda_M(M)}{\lambda_m(M)}} \|x(0) - h(0)\| e^{-\frac{\lambda_m(L)}{2\lambda_M(M)}} t \quad \text{for all } t \in [0, T),
\]
and \( x(t) = h(t) \) for all \( t \geq T \). An estimation of the time \( T \) of all trajectories \( x(\cdot) \) attaining \( h(\cdot) \) is bounded by
\[
\frac{||Q Q^T||_1 Q||_\infty}{\delta \lambda_m(M)} \sqrt{\frac{\lambda_M(M)}{\lambda_m(M)}} ||e(0)||.
\]
Corollary 4.1. Let \( x(t) \) be any trajectory of the feedback-controlled system (1.3) satisfying assumptions (A1)–(A5). If for any initial state \( x(0) \neq h(0) \), then the controller (3.1) with (3.2)–(3.6) such that the system (1.3) is completely viable controllable for \( h \), where \( k_\alpha(x) = 0 \) and \( k(x) = \delta \) for all \( x \in \mathbb{R}^n \). Moreover, all trajectories \( x(t) \) of the system (1.3) are steered to the map \( h(t) \) with an exponential convergence rate. An estimation of the time \( T \) of all trajectories \( x(\cdot) \) attaining \( h(\cdot) \) is also bounded by
\[
\frac{\| (Q Q^T)^{-1} Q \|_\infty}{\delta} \sqrt{\lambda_M(M) \lambda_m(M)} \| e(0) \|.
\]

Proof. Note that the system (1.3) is a special case of the uncertain dynamical system (1.2) subject to uncertainty. Taking \( F_\alpha(t,x(t),u(t)) = F_\beta(t,x(t),u(t)) = \{0\} \) in the system (1.2), Assumption 2.1 is also satisfied for \( k_\alpha(x) = 0 \) and \( k(x) = \delta \). Thus Theorem 4.3 implies that the Corollary 4.1 holds.

5. An illustrative example

An example has been provided to illustrate the use of our main result about the viable control problem for cerebral embolism and cerebral thrombosis diseases as follows.

According to the American Heart Association, a brain attack occurs when a blood vessel bringing oxygen and nutrients to the brain bursts or is clogged by a blood clot or other particle. There are four main types of stroke: cerebral thrombosis—caused by atherosclerotic thrombosis; cerebral embolism—caused by movable emboli; cerebral hemorrhages—caused by hypertensive intracerebral hemorrhages and subarachnoid hemorrhages—caused by aneurysm rupture. The most common types of brain attacks are caused by blood clots that plug an atherosclerotic artery. Cerebral thrombosis and cerebral embolism account for about 70–80 percents of all strokes. Moreover, cerebral thrombosis occurs when a blood clot forms and blocks blood flow in an artery leading to part of the brain. Blood clots usually form in arteries damaged by atherosclerosis.

In Example 5.1, the trajectory of the medicinal carriers satisfying the uncertain dynamical system described by differential inclusions is guided to the mapping \( h(\cdot) \) in the nanomedicine system. Here, let \( x \) be the state of the medicinal carriers system and let \( h(t) = \sin 10t \) be the state of blood clots or other particles in arteries (see Fig. 5.1). In the nanomedicine system, the goal is to find a feedback control \( u(t) = u(t,x(t)) \) such that the closed-loop system is completely viable controllable for \( h(\cdot) \). This implies that the medicinal carriers \( x(\cdot) \) can be controlled to treat this cerebral embolism and cerebral thrombosis diseases \( h(\cdot) \) in arteries. Furthermore, if \( x_0 \neq h(0) \), under the feedback control, the medicinal carriers \( x \) treat the brain attack \( h(\cdot) \) after a finite time \( T \), that is, \( x(t) = h(t) \) for all \( t \geq T \). An estimate of the treatment time \( T \) of all trajectories \( x(\cdot) \) attaining the map \( h(\cdot) \) is given.

Example 5.1. Consider the viable control problem for the following uncertain nonlinear dynamical system by a differential inclusion:

\[
\dot{x}(t) \in F(t,x(t),u(t)),
\]

\[
F(t,x(t),u(t)) \equiv f(t,x(t)) + F_\alpha(t,x(t)) + Q(t,x(t))\left[u(t) + F_\beta(t,x(t),u(t))\right], \tag{5.1}
\]

where
Fig. 5.1. An atherosclerotic plaque, prior to, and viable control treatment after.

\[ f(t, x(t)) = 10 + 2x(t) + \cos 5x(t) + |x(t)| \sin(2x(t)), \]

\[ Q(t, x(t)) = 1 + (\cos x(t))^2, \]

\[ F_\alpha(t, x(t)) = \{ a(1 + 2x(t) + x(t) \cos(3x(t))) + b \text{sign}(x(t)) \mid a, b \in [-1, 1] \}, \]

\[ \text{sign}(x(t)) = \begin{cases} -1, & x(t) < 0, \\ [-1, 1], & x(t) = 0, \\ 1, & x(t) > 0, \end{cases} \]

\[ F_\beta(t, x(t), u(t)) = \{ |x(t)| + \cos(x(t)) + cu(t) \sin(u(t)) \mid c \in [-0.5, 0.5] \}. \]

From assumptions (A2)–(A5), we have

\[ k_\alpha(x(t)) = 2 + 3|x(t)|, \quad k_\beta(x(t)) = |x(t)| + 1, \]

\[ k_q(x(t)) = 0, \quad \gamma = 0.5 \text{ and } r = 2. \]

For example, take \( a = 1, b = 1 \) and \( c = 0.5 \) in (5.1). Some typical phase trajectories of the uncontrolled system are depicted in Fig. 5.2.

If we choose \( A = -1 \) and \( L = 2 \), by (3.4), then we have \( M = 1 \). Furthermore, let \( h(t) = \sin 10t \) and \( \delta = 0.5 \) in (3.5), then we can calculate the explicit form of the controller \( u(t) \) given by (3.1) with (3.2)–(3.6). They are shown as follows:

\[ u(t) = u_n(t) + u_c(t), \]

where

\[ u_n(t) = -\frac{x(t) - \sin 10t}{1 + (\cos x(t))^2} \cdot \frac{10 + 2x(t) + \cos(5x(t)) + |x(t)| \sin(2x(t))}{1 + (\cos x(t))^2} \]

\[ + \frac{10 \cos 10t}{1 + (\cos x(t))^2}, \]

\[ u_c(t) = -k(x(t))\Psi(\xi), \]

\[ k(x(t)) = 2\left[ 3 + 4|x(t)| + 0.5 |u_n(t)| + 0.5 \right], \]

\[ \xi = (x(t) - \sin 10t)[1 + (\cos x(t))^2]. \]
By Theorem 4.3, all trajectories $x(t)$ of the feedback-controlled system reach the mapping $h(t)$ at a finite time and remain on $h(t)$ thereafter. Some typical phase trajectories of the feedback-controlled system are depicted in Fig. 5.3.
References