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A holographic model for hall viscosity

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ABSTRACT

We have modified the holographic model of Saremi and Son [12] by using a charged black brane, instead of a neutral one, such that when the bulk pseudo scalar (θ) potential is made of θ^2 and θ^4 terms, parity can still be broken spontaneously in the boundary theory. In our model, the $3+1$ dimensional bulk has a pseudo scalar coupled to the gravitational Chern–Simons term in the anti de Sitter charged black brane back ground. Parity could be broken spontaneously in the bulk by the pseudo scalar hairy solution and give rise to non-zero Hall viscosity at the boundary theory.

1. Introduction

In recent years, the AdS/CFT correspondence [1–3] has been applied to study strongly coupled phenomena in condensed matter physics at finite temperature and chemical potential. In particular, inspired by the idea of spontaneous symmetry breaking in the presence of horizon [4,5], holographic superconductors [6,7] and superfluids [8] are two remarkable examples where the gauge/gravity duality plays an important role.

On the other hand, the hydrodynamic limit of AdS/CFT correspondence has also attracted much attention recently. Computations of the ratio of shear viscosity to entropy density for a big class of gauge field theories with gravitational duals yields the same number $1/4\pi$ which is not far away from that observed in the strong interacting quark-gluon plasma created in RHIC [9,10]. Later it has been shown that by using the boundary derivative expansion, one can consistently solve the Einstein equation order by order and compute various hydrodynamics transport coefficients of the boundary fluid [11]. Recently, a holographic model for the parity violating Hall viscosity was proposed. Like the other transport coefficients, Hall viscosity is also found to be uniquely determined by the near horizon data of the bulk black brane [12]. This is yet another example of the membrane paradigm. In the original construction the $(3+1)$ dimensional bulk action has a negative cosmological constant, a real scalar field coupled to the gravitational Chern–Simons term.[1]

While it has been shown that a non-trivial profile of the bulk scalar field is important to obtain a non-vanishing Hall viscosity of the $(2+1)$ dimensional boundary field theory, from the holography point of view it would be interesting to further investigate what role this bulk scalar plays at the boundary. One possible interpretation is to identify the boundary value of this scalar as an order parameter field which condensates at low temperature in the boundary field theory. From the condensed matter point of view the physical realization of this order parameter, which leads a system to the spontaneously parity breaking phase is not clear. But interestingly in terms of physical quantity such as hall viscosity one might get information about how the system breaks parity spontaneously. So, effectively in the hydrodynamic regime, Hall viscosity can play the role of order parameter which is non-zero only below a critical temperature. To be ready for such a boundary theory interpretation, one shall look for a sourceless boundary condition for the hairy scalar if parity is only broken spontaneously.

However, it has been shown that a neutral scalar hair with quadratic and quartic potential that satisfies the usual sourceless boundary condition in a Schwarzschild-AdS black hole spacetime does not satisfy the positive energy theorem [16]. This essentially means that a Schwarzschild-AdS black hole with a sourceless neutral scalar hair is intrinsically unstable.

While it is still possible to find a sourced solution which minimizes the free energy, we will take a different approach to modification of the model by including a gauge field in the bulk. The scalar in the original theory is identified as a pseudo scalar now, so its coupling to the gravitational Chern–Simons term does not

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[1] There exist early studies of Chern–Simons term in the holographic models. To mention a few: the effect of Maxwell Chern–Simons term $\theta^* FF$ was studied in the holographic superconductor [13,14]. The spectrum of quasinormal modes was studied in the dynamic Chern–Simons gravity and correction to some hydrodynamic quantities was discussed [15].

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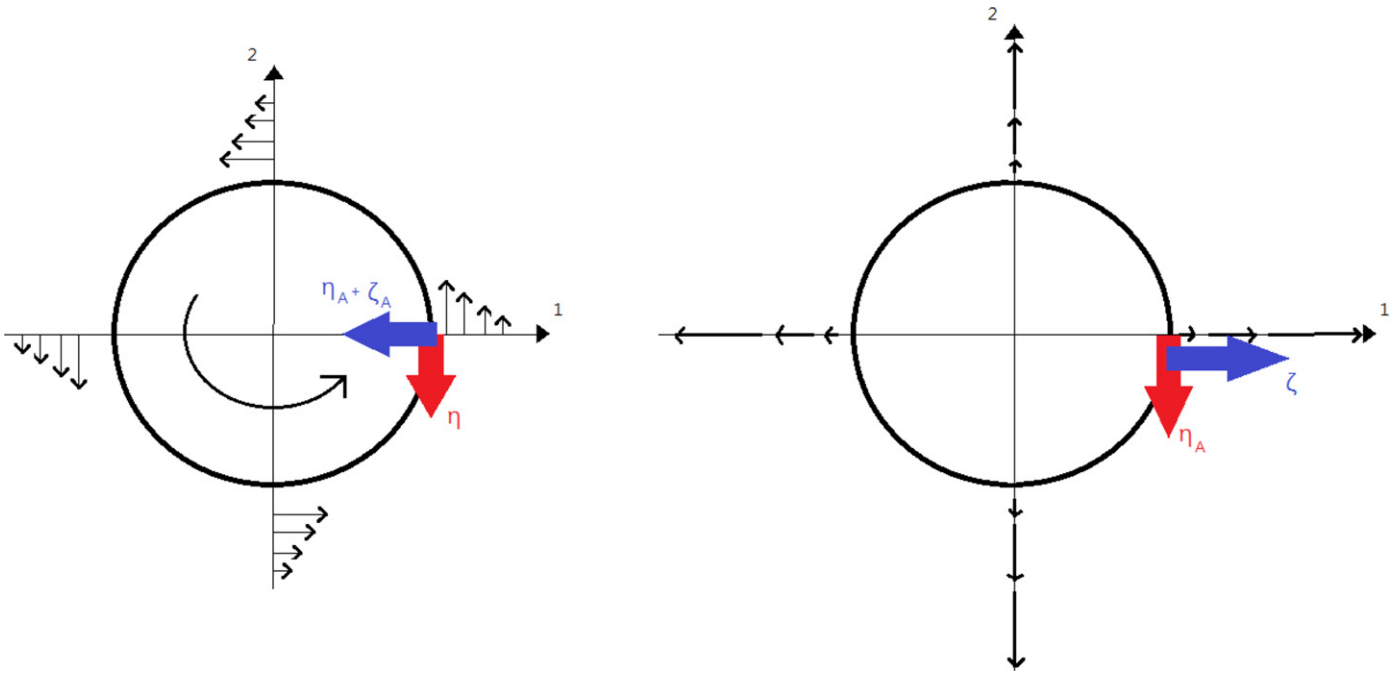

Fig. 1. Cartoon pictures showing the force directions associated with various viscosities against non-uniform fluid velocity in the (x_1, x_2) plane. The usual bulk viscosity (ζ), the sum of Hall viscosity (η_A) and curl viscosity (ζ_A) are generated by non-zero curl of velocity, while the shear viscosity (η) and Hall viscosity (η_A) along are generated by non-zero divergence of velocity. Notice that the force flips its direction when the direction of the velocity field, which is depicted by the arrows outside the spheres, is reversed.

break parity. The pseudo scalar hair, however, breaks parity spontaneously and gives a pseudo scalar condensate in the boundary field theory which, as we will demonstrate in the next section, is important for Hall viscosity. In the probed limit, this pseudo scalar hair solution in the charged black brane background is known to be stable [17].

The Letter is organized as follows: in Section 2, we present a general discussion of the parity violating viscosities and set up the holographic model. We then compute the Hall viscosity and commend on the boundary field theory in Section 3. We then conclude our results in Section 4. A detailed derivation of Hall viscosity together with an analytical approximation are given in Appendices A and B.

2. General properties of viscosities

It is instructive to classify viscosities by considering the general relation between the energy momentum tensor and the spatial derivative of the fluid velocity

$$T_{ij} = \eta_{ijkl}\partial_{(k}V_{l)} + \xi_{ijkl}\partial_{[k}V_{l]}, \tag{1}$$

where i, j, k, l are spatial indices and $\partial_{(k}V_{l)} = (\partial_k V_l + \partial_l V_k)$ and $\partial_{[k}V_{l]} = (\partial_k V_l - \partial_l V_k)$ are just the symmetric and anti-symmetric combinations of the derivatives, respectively. We have $T_{ij} = T_{ji}$. In two spatial dimensional systems, η_{ijkl} and ξ_{ijkl} can be constructed by δ_{ij} and the two dimensional anti-symmetric tensor ϵ_{ij}. Taking $\eta_{ijkl} \propto \delta_{ij}\delta_{kl}$, $\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}$ and $\epsilon_{ik}\epsilon_{jl} + \epsilon_{jk}\epsilon_{il}$ give rise to the usual shear (η) and bulk (ζ) viscosity contributions

$$\delta T_{ij} = -\eta(\partial_i V_j + \partial_j V_i - \text{trace}) + \zeta\delta_{ij}\nabla\cdot\mathbf{V}. \tag{2}$$

Taking $\eta_{ijkl} \propto \delta_{ik}\epsilon_{jl} + \delta_{jk}\epsilon_{il}$ gives rise to the Hall viscosity (η_A) and "curl" viscosity (ζ_A) contributions

$$\delta T^A_{ij} = -\eta_A \begin{pmatrix} (\partial_1 V_2 + \partial_2 V_1) & (-\partial_1 V_1 + \partial_2 V_2) \\ (-\partial_1 V_1 + \partial_2 V_2) & -(\partial_1 V_2 + \partial_2 V_1) \end{pmatrix} + \zeta_A\delta_{ij}(\partial_1 V_2 - \partial_2 V_1). \tag{3}$$

The curl viscosity can also arise from taking $\xi_{ijkl} \propto \delta_{ij}\epsilon_{kl}$. The curl structure naturally reminds us vortices. It is interesting that the bulk and curl viscosities are associated with the divergence and the curl of the velocity. Both of them can only exist in systems without scaling invariance due to its trace like structure in the energy momentum tensor. It is easy to generalize the above discussion to higher dimensions. However, the Hall and curl viscosities can only exist in two dimensions as depicted in Fig. 1.

The Hall and curl viscosities have distinct transformation properties from the shear and bulk viscosities under parity. Under the coordinate reflection $(x_1, x_2) \to (-x_1, x_2)$, $\delta T_{ij} \to (-1)^{i+j}\delta T_{ij}$ while $\delta T^A_{ij} \to (-1)^{i+j+1}\delta T^A_{ij}$. Since δT_{ij} exists in parity conserving systems, δT^A_{ij} only exists in parity violating systems.

In summary, we need to work in $(2+1)$ dimensional parity violating systems to study the Hall and curl viscosities. In the following section we will explicitly construct a holographic model and calculate the Hall viscosity of the boundary fluid.

3. The holographic set up

Following the discussion of the previous section, we will consider a four dimensional bulk action as the holographic dual to a three dimensional boundary theory. It is given by a four dimensional Einstein action with a negative cosmological constant; the matter sector includes an abelian Yang–Mills $F_{\mu\nu}$ and a pseudo scalar field θ:

$$\mathcal{L} = R - \frac{6}{L^2} - \frac{1}{4}F^2 - \frac{1}{2}(\partial\theta)^2 - V(\theta) - \frac{\lambda}{4}\theta^* RR. \tag{4}$$

The bulk action conserves parity, so θ is a pseudo scalar from the last term on the Lagrangian which is important to introduce parity violation to the boundary theory through the θ condensate. The F^2 term is the only difference between our model and Saremi and Son's. In our model a charged black hole solution is allowed. We will recover their result by taking the black hole charge to zero.

In this Letter, we will only focus on the following form of the potential,

$$V(\theta) = \frac{1}{2}m^2\theta^2 + \frac{1}{4}c\theta^4. \tag{5}$$

As discussed in [17], the second term is necessary to have consistent solution at $T = 0$. We will study the probed limit of the scalar field by sending $\theta \to \varepsilon\theta$ and $\lambda \to \varepsilon\lambda$ for small ε. Thus, at leading order in ε, we only need to solve for the equation of motion governed by the upper line of (4). Then the background is exactly a charged black brane in AdS_4 spacetime and the Hall viscosity η_A can be recovered at the $\mathcal{O}(\varepsilon^2)$ order, since $\eta_A \to \varepsilon^2\eta_A$ in this probe limit.

The charged black brane solution is given by the metric:

$$ds^2 = 2\,dv\,dr - r^2 f(r)\,dv^2 + r^2\left(dx^2 + dy^2\right), \tag{6}$$

where

$$f(r) = \frac{1}{L^2} - \frac{M}{r^3} + \frac{Q^2}{r^4} \tag{7}$$

and the abelian gauge field[2]:

$$A = 2\frac{Q}{r_H}\left(1 - \frac{r_H}{r}\right)dv - \frac{Q}{r^3 f}\,dr. \tag{8}$$

Here black brane mass and electric charge are M and Q. The horizon is at $r = r_H$. The metric is asymptotically AdS_4 with curvature radius L. It is convenient to work in the units of $L = 1$ and rescale the horizon to $r_H = 1$[3]

$$f(r) = 1 - \frac{1+3\kappa}{r^3} + \frac{3\kappa}{r^4}, \qquad A = 2\sqrt{3\kappa}\left(1 - \frac{1}{r}\right)dv. \tag{10}$$

The charged black brane in the bulk corresponds to a boundary field theory at finite temperature T and chemical potential μ, that is

$$T = \frac{3}{4\pi}(1-\kappa), \qquad \mu = 2\sqrt{3\kappa}. \tag{11}$$

We remark that $\kappa = 0$ corresponds to a neutral black brane with zero chemical potential and $\kappa = 1$ corresponds to an extremal black brane at zero temperature.

The equation of motion the probed neutral pseudo scalar reads

$$\theta'' + \left(\frac{f'}{f} + \frac{4}{r}\right)\theta' - \frac{V'(\theta)}{r^2 f} = 0. \tag{12}$$

Near the boundary, the asymptotic behavior of pseudo scalar is

$$\theta = \frac{J}{r^{\Delta_-}} + \frac{\mathcal{O}}{r^{\Delta_+}} + \cdots, \tag{13}$$

with

$$\Delta_\pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} + m^2L^2}. \tag{14}$$

[2] The A_r component of gauge potential can be gauged away with no contribution to the equation of motion. However, we keep it here to show that under a proper coordinate transformation, $v = t + h(r)$ where $h'(r) = \frac{1}{r^2 f(r)}$, this black brane solution can be brought to the usual diagonal coordinate given by (t, r, x, y).

[3] The action and equations of motion are invariant under the following scaling:

$$r \to cr, \quad (v, x, y) \to c^{-1}(v, x, y),$$

$$Q \to c^2 Q, \quad M \to c^3 M,$$

$$\theta \to \theta, \quad A \to A, \quad f \to f. \tag{9}$$

Here we adopt the convention of [17] where $Q \to Qr_H^2$, $r \to r_H r$ and $(v, x, y) \to r_H^{-1}(v, x, y)$.

We remark that in our construction, the mode J can be consistently turned off and $\mathcal{O}$ is identified as the condensate in the boundary.[4] However, this was not possible in the original construction with neutral black brane [12] where J can be turned off only if $c < -\frac{3}{4}$ [16] which violates the positive energy theorem and hence it is not a stable solution. In our model, the θ^4 term is required to make the θ solution regular at the horizon [17].

4. The Hall viscosity

The detail derivation of the viscosities is presented in Appendix A. We have first included the back reaction of θ as was done in [12], then take the probe limit ($\theta \to \varepsilon\theta$ and $\lambda \to \varepsilon\lambda$) to the final result of the viscosity expression. The expression for η_A, which appears at $\mathcal{O}(\varepsilon^2)$, is identical to that obtained in [12].

From the derivation in Appendix A, we obtain the shear viscosity of the universal value as expected:

$$\frac{\eta}{s} = \frac{1}{4\pi}, \tag{15}$$

where s is the entropy density. Theses combinations are dimensionless and are invariant under the scaling of Eq. (9).

The Hall viscosity in our charged black brane background takes the same form as the case of the neutral black brane background [12], that is

$$\eta_A = -\frac{1}{8\pi G_N}\frac{\lambda}{4}\frac{r^4 f'(r)\theta'(r)}{H(r)^2}\bigg|_{r=r_H}. \tag{16}$$

The dimensionless and scale invariant combination yields

$$\frac{\eta_A}{s} = -\frac{\lambda}{8\pi}\frac{r^4 f'(r)\theta'(r)}{H(r)^2}\bigg|_{r=r_H}. \tag{17}$$

In Eq. (17), η_A/s vanishes when the solution of θ is trivial ($\theta(r) = 0$), which happened in the symmetric phase, or when θ is a constant field. In the former case, parity is not broken in the bulk. Then by the correspondence, it will not be broken at the boundary either. Likewise, in the latter case, when θ is a constant, the ${}^*RR$ term is just a surface term in the action which has no effect to the bulk equations of motion. Hence it does not contribute to η_A either. Therefore, it should not be a surprise that the phase diagram for η_A/s is very similar to that with the neutral scalar hair of Ref. [17] with just one difference—η_A/s vanishes when $T = 0$. This comes from the factor $f'(r_H) \propto T$. One peculiar feature of this model is that the entropy of the charged black hole does not vanish at zero temperature. Perhaps in models with zero entropy at zero temperature, η_A/s stays finite at zero temperature. We show $\eta_A/s\lambda L$ as a function of T/μ and m^2L^2 in Fig. 2. These three quantities are all scale invariant and dimensionless.

In Fig. 3, the dependence of the scale invariant, dimensionless quantities $\eta_A/s\lambda L$ and T/μ is shown for $m^2L^2 = -2$. $\eta_A/s \propto T$ as $T \to 0$ due to $f'(r_H) \propto T$ in Eq. (16). When $T \to T_c$, η_A/s vanishes. The analytic approximation performed in Appendix B suggests that critical exponent is of mean field value: $\eta_A/s \propto (1 - T/T_c)^{1/2}$ as $T \to T_c$. One can also see that $\eta_A \to 0$ as we take $\mu \to 0$ and the black hole becomes charge neutral.

In Fig. 4, $\eta_A/s\lambda L$ vs. m^2L^2 is plotted for $T/\mu = 7.55 \times 10^{-6}$. The critical m^2L^2 is smaller than the critical value $m^2L^2 = -1.5$ at zero T because it is harder to form the condensate at higher T.

[4] For $-9/4 < m^2L^2 < -5/4$, J and $\mathcal{O}$ are both renormalizable and one is free to choose either one as source and the other as condensate [18].

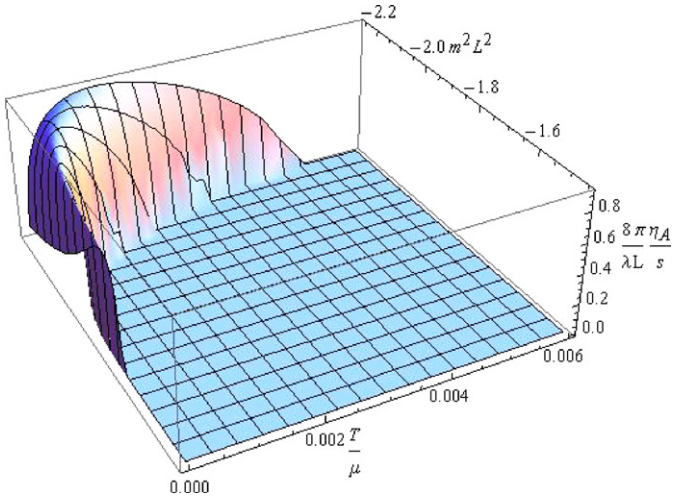

Fig. 2. $\eta_A/s\lambda L$ as a function of T/μ and m^2L^2.

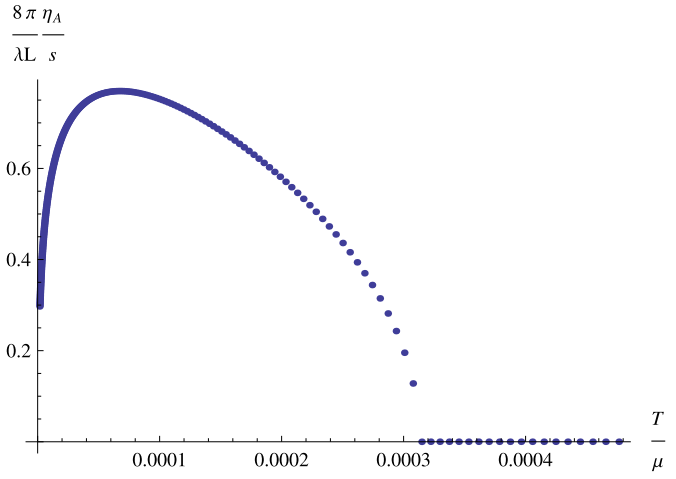

Fig. 3. $\eta_A/s\lambda L$ vs. T/μ for $m^2L^2 = -2$.

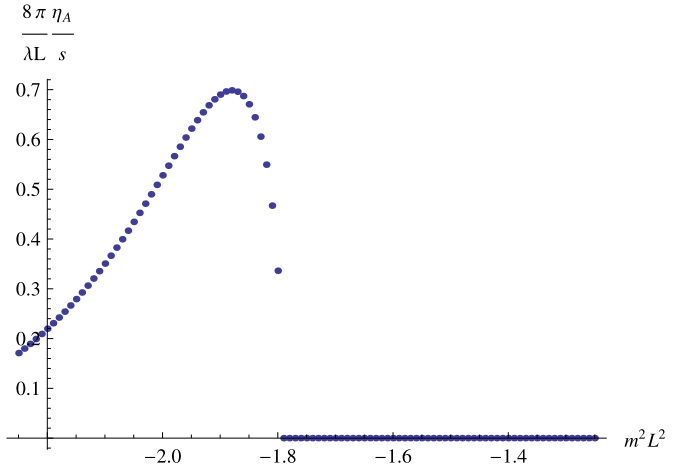

Fig. 4. $\eta_A/s\lambda L$ vs. m^2L^2 for $T/\mu = 7.55\times10^{-6}$.

In our model, the non-zero Hall viscosity arises because parity is broken spontaneously. The non-zero classical solution (or equivalently, vacuum expectation value) of θ yields a pseudo scalar condensate at the boundary which is a necessary condition to have non-zero Hall viscosity.

5. Conclusion

We have modified the holographic model of Saremi and Son [12] by using a charged black brane, instead of a neutral one, such that when the bulk pseudo scalar (θ) potential is made of θ^2 and θ^4 terms, parity can still be broken spontaneously in the boundary theory. In our model, the $3+1$ dimensional bulk has a pseudo scalar coupled to the gravitational Chern–Simons term in the anti de Sitter charged black brane back ground. Parity could be broken spontaneously in the bulk by the pseudo scalar hairy solution and give rise to non-zero Hall viscosity at the boundary theory.

This study does not exclude a non-vanishing Hall viscosity in Saremi and Son's model be found with a more general potential. It is interesting to investigate the Hall viscosity in other parity-broken holographic condensed matter systems, such as the D-wave superconductors [19,20]. We will report it in a future project.

Acknowledgements

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Appendix A. Derivation of Hall viscosity

Here we detail the Hall viscosity derivation with the charged black brane solution. The hydrodynamics of charged fluid has been extensively studied in the holographic set up [21]. The general procedure to calculate the holographic hydrodynamic transport coefficients has been given in [11]. We largely follow the procedures adopted in [11,12] with the neutral black brane solution. The equations of motion by varying the action (4) with respect to the metric, the scalar and the gauge field are as

$$\begin{aligned}
&R_{MN} - \frac{1}{2}g_{MN}R + \Lambda g_{MN} - \lambda C_{MN} = T_{MN}(\theta) + T_{MN}(A),\\
&\nabla^2\theta = \frac{dV}{d\theta} + \frac{\lambda}{4}{}^*RR,\\
&\nabla_M F^{MN} = 0,
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
T_{MN}(\theta) &= \frac{1}{2}\partial_M\theta\partial_N\theta - \frac{1}{4}g_{MN}(\partial\theta)^2 - \frac{1}{2}g_{MN}V(\theta),\\
T_{MN}(A) &= \frac{1}{2}F_M{}^A F_{NA} - \frac{1}{8}g_{MN}F_{AB}F^{AB},
\end{aligned}$$

and C_{MN} is called Cotton tensor coming from the gravitational Chern–Simons term

$$\begin{aligned}
C^{MN} = \frac{1}{2}\big[&\partial_A\theta\big(\epsilon^{AMBC}\nabla_B R_C^N + \epsilon^{ANBC}\nabla_B R_C^M\big)\\
&+ \nabla_A\partial_B\theta\big({}^*R^{AMBN} + {}^*R^{ANBM}\big)\big],
\end{aligned}$$

where ϵ^{AMBC} is the usual four dimensional Levi-Civita tensor.

An ansatz satisfying the equations of motion is

$$\begin{aligned}
ds^2 &= -2H(r,b,q)u_\mu\,dx^\mu\,dr - r^2 f(r,b,q)u_\mu u_\nu\,dx^\mu\,dx^\nu\\
&\quad + r^2 P_{\mu\nu}\,dx^\mu\,dx^\nu,\\
\theta &= \theta(r,b,q),\\
A &= A(r,b,q)u_\mu\,dx^\mu.
\end{aligned} \tag{19}$$

This ansatz describes a boosted black brane solution alone the boundary coordinates. Then following the standard procedure of

the fluid/gravity correspondence, we perturb the system away from equilibrium by promoting the velocity u^μ, mass b and charge q to vary slowly with respect to the boundary coordinates. In the co-moving frame where the fluid two-velocity is zero at the origin of the boundary coordinates ($x^\mu = 0$), we Taylor expand quantities near the origin to the first derivative order:

$$
\begin{aligned}
&u^\mu = \big(1, x^\mu \partial_\mu \beta^i\big),\\
&b = b_0 + x^\mu \partial_\mu b,\\
&q = q_0 + x^\mu \partial_\mu q,\\
&f(r,b,q) = f(r) + \frac{\partial f}{\partial b} x^\mu \partial_\mu b + \frac{\partial f}{\partial q} x^\mu \partial_\mu q = f(r) + \delta f,\\
&H(r,b,q) = H(r) + \frac{\partial H}{\partial b} x^\mu \partial_\mu b + \frac{\partial H}{\partial q} x^\mu \partial_\mu q = H(r) + \delta H,\\
&A(r,b,q) = A(r) + \frac{\partial A}{\partial b} x^\mu \partial_\mu b + \frac{\partial A}{\partial q} x^\mu \partial_\mu q = A(r) + \delta A,\\
&\theta(r,b,q) = \theta(r) + \frac{\partial \theta}{\partial b} x^\mu \partial_\mu b + \frac{\partial \theta}{\partial q} x^\mu \partial_\mu q = \theta(r) + \delta\theta. \qquad (20)
\end{aligned}
$$

Substitute these into the ansatz, we get

$$
\begin{aligned}
ds^2 &= 2H(r)\,dv\,dr - r^2 f(r)\,dv^2 + r^2\,dx^i\,dx^i\\
&\quad + \epsilon\big[2\delta H\,dv\,dr - r^2\delta f\,dv^2 - 2H(r)x^\mu\partial_\mu\beta^i\,dx^i\,dr\\
&\quad - 2r^2\big(1 - f(r)\big)x^\mu\partial_\mu\beta^i\,dx^i\,dv\big], \qquad (21)\\
\theta &= \theta(r) + \epsilon\delta\theta,\\
A &= -A(r)\,dv + \epsilon\big(-\delta A\,dv + A(r)x^\mu\partial_\mu\beta^i\,dx^i\big), \qquad (22)
\end{aligned}
$$

where we have added the parameter ϵ to keep track of how many derivatives on the boundary coordinates each term has.

Note that after we promote the parameter to be dependent on the boundary coordinates, the ansatz no longer satisfies the equations of motion. Hence we add corrections order by order to the metric, scalar and gauge fields such that, order by order, the whole metric, scalar and gauge fields still satisfy the equations of motion. To calculate the Hall viscosity, it suffices to consider the symmetric traceless part of the correction to the metric:

$$
\begin{aligned}
ds^2 &= \epsilon\bigg(\frac{k(r)}{r^2}\,dv^2 + 2h(r)\,dv\,dr - r^2 h(r)\,dx^i\,dx^i + \frac{2}{r}a^i(r)\,dv\,dx^i\\
&\quad + r^2\alpha_{ij}(r)\,dx^i\,dx^j\bigg),\\
\theta &= \epsilon\theta_{cor},\\
A &= \epsilon\big(A^v_{cor}(r)\,dv + A^x_{cor}(r)\,dx + A^y_{cor}(r)dy\big). \qquad (23)
\end{aligned}
$$

In this case, the trace-reversed form of the Einstein equations is more convenient, which is given by

$$
E_{MN} = R_{MN} - \Lambda g_{MN} - \lambda C_{MN} - d_{MN} = 0, \qquad (24)
$$

where

$$
\begin{aligned}
&d_{MN} = d_{MN}(\theta) + d_{MN}(A) = T_{MN} - \frac{1}{2}g_{MN}T,\\
&d_{MN}(\theta) = \frac{1}{2}\partial_M\theta\partial_N\theta + \frac{1}{2}g_{MN}V(\theta),\\
&d_{MN}(A) = -\frac{1}{2}F_M{}^A F_{AN} - \frac{1}{8}g_{MN}F_{AB}F^{AB}. \qquad (25)
\end{aligned}
$$

Substitute all into the Einstein equations and collect the first order term from the xy-component of the trace-reversed Einstein equation, we get

$$
\begin{aligned}
&\frac{1}{H}\frac{d}{dr}\bigg[-\frac{1}{2}\frac{r^4 f}{H}\frac{d}{dr}\alpha_{xy}\bigg]\\
&\quad + \bigg[\frac{r^3 H' f}{H^3} - \frac{r^3 f'}{H^2} - \frac{3r^2 f}{H^2} + 3r^2 - \frac{r^2}{2}V(\theta) - \frac{r^2 A'^2}{4H^2}\bigg]\alpha_{xy}\\
&= \frac{r}{H}(\partial_x\beta_y + \partial_y\beta_x) + \frac{\lambda}{4H}\frac{d}{dr}\bigg(\frac{r^4 f'\theta'}{H^2}\bigg)(\partial_x\beta_x - \partial_y\beta_y). \qquad (26)
\end{aligned}
$$

However, the zeroth order of the xx-component of the trace-reversed Einstein equation yields

$$
\frac{r^3 H' f}{H^3} - \frac{r^3 f'}{H^2} - \frac{3r^2 f}{H^2} + 3r^2 - \frac{r^2}{2}V(\theta) - \frac{r^2 A'^2}{4H^2} = 0. \qquad (27)
$$

Therefore we obtain a differential equation for α_{xy},

$$
\begin{aligned}
\frac{d}{dr}\bigg[-\frac{1}{2}\frac{r^4 f}{H}\frac{d}{dr}\alpha_{xy}\bigg] &= r(\partial_x\beta_y + \partial_y\beta_x)\\
&\quad + \frac{\lambda}{4}\frac{d}{dr}\bigg(\frac{r^4 f'\theta'}{H^2}\bigg)(\partial_x\beta_x - \partial_y\beta_y). \qquad (28)
\end{aligned}
$$

And hence

$$
\begin{aligned}
\alpha_{xy}(r) &= \int_r^\infty \frac{2H(t)\,dt}{t^4 f(t)}\int_{r_H}^t dz\bigg[z(\partial_x\beta_y + \partial_y\beta_x)\\
&\quad + \frac{\lambda}{4}\frac{d}{dz}\bigg(\frac{z^4 f'\theta'}{H^2}\bigg)(\partial_x\beta_x - \partial_y\beta_y)\bigg]. \qquad (29)
\end{aligned}
$$

As in [12], we use the following formula to compute the asymptotic form,

$$
r^n\alpha_{xy}(r) \to -\frac{r^{n+1}}{n}\frac{d}{dr}\alpha_{xy}(r) \quad \text{as } r\to\infty. \qquad (30)
$$

And from [22], the boundary energy momentum tensor for odd boundary dimension is given by

$$
\langle T_{ij}\rangle = \frac{d}{16\pi G_N}g_{(d)ij}, \qquad (31)
$$

where

$$
g\big(x^\mu, r\big) = g_{(0)} + \frac{1}{r^2}g_{(2)} + \cdots + \frac{1}{r^d}g_{(d)} + \cdots. \qquad (32)
$$

Therefore, all we have to do is to find the constant part of $r^3\alpha_{xy}$. Hence

$$
\begin{aligned}
r^3\alpha_{xy}(r) &\to -\frac{r^4}{3}\frac{d}{dr}\alpha_{xy}(r)\\
&= -\frac{r^4}{3}\frac{d}{dr}\int_r^\infty \frac{2H(t)\,dt}{t^4 f(t)}\int_{r_H}^t dz\bigg[z(\partial_x\beta_y + \partial_y\beta_x)\\
&\quad + \frac{\lambda}{4}\frac{d}{dz}\bigg(\frac{r^4 f'\theta'}{H^2}\bigg)(\partial_x\beta_x - \partial_y\beta_y)\bigg]\\
&= \frac{r^4}{3}\frac{2H(r)}{r^4 f(r)}\int_{r_H}^r dz\bigg[z(\partial_x\beta_y + \partial_y\beta_x)\\
&\quad + \frac{\lambda}{4}\frac{d}{dz}\bigg(\frac{r^4 f'\theta'}{H^2}\bigg)(\partial_x\beta_x - \partial_y\beta_y)\bigg]\\
&= \frac{2H}{3f}\bigg[\frac{z^2}{2}(\partial_x\beta_y + \partial_y\beta_x)\\
&\quad + \frac{\lambda}{4}\bigg(\frac{r^4 f'\theta'}{H^2}\bigg)(\partial_x\beta_x - \partial_y\beta_y)\bigg]_{rH}^r.
\end{aligned}
$$

Since f and H asymptotically approach 1 and we shift the horizon to $r_H = 1$, we get the xy-component of the boundary energy momentum tensor as

$$\begin{aligned}\langle T_{xy}\rangle &= \frac{3}{16\pi G_N} g_{(3)xy} \\ &= -\frac{1}{16\pi G_N}(\partial_x\beta_y + \partial_y\beta_x) \\ &\quad - \frac{1}{8\pi G_N}\left[\frac{\lambda}{4}\left(\frac{r^4 f'\theta'}{H^2}\right)(\partial_x\beta_x - \partial_y\beta_y)\right]_{r=r_H}.\end{aligned}$$

The first term is the usual shear mode with

$$\eta = \frac{1}{16\pi G_N} \tag{33}$$

which recovers

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{34}$$

The second term is proportional to the Hall viscosity which yields

$$\eta_A = -\frac{1}{8\pi G_N}\frac{\lambda}{4}\frac{r^4 f'(r)\theta'(r)}{H(r)^2}\bigg|_{r=r_H}. \tag{35}$$

The dimensionless combination

$$\frac{\eta_A}{s} = -\frac{\lambda}{8\pi}\frac{r^4 f'(r)\theta'(r)}{H(r)^2}\bigg|_{r=r_H} \tag{36}$$

is independent of the scaling.

Appendix B. Analytic approximation

Here we will apply an approximation to obtain an analytic expression for Hall viscosity in terms of the condensate in the boundary theory. We use the new coordinate $z = 1/r$ for convenience. Near the horizon $z = 1$, we can expand $\theta(z)$ as

$$\theta(z) \simeq \theta(1) - \frac{m^2}{3(1-\kappa)}\theta(1)(1-z). \tag{37}$$

On the other hand, near the boundary $z = 0$, one has

$$\theta(z) \simeq \mathcal{O} z^{\Delta_+}. \tag{38}$$

By matching above expressions in the middle $z = 1/2$, one can identify approximately

$$\theta(1) \simeq \frac{\mathcal{O}}{1 - \frac{m^2}{6(1-\kappa)}} 2^{-\Delta_+}. \tag{39}$$

Therefore one can express the Hall viscosity as a function of temperature and the condensate $\mathcal{O}$:

$$\eta_A = -\frac{\lambda}{32\pi G_4}\frac{m^2\mathcal{O}}{1 - \frac{m^2}{8\pi T}} 2^{-\Delta_+}. \tag{40}$$

Near the critical T_c, the condensate has the mean field T dependence for a second order phase transition [23]:

$$\mathcal{O} \propto T_c^{\Delta_+}\left(1 - \frac{T}{T_c}\right)^{1/2}\theta(T_c - T), \tag{41}$$

hence near T_c one has

$$\eta_A \propto \left(1 - \frac{T}{T_c}\right)^{1/2}\theta(T_c - T). \tag{42}$$

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