THE ENUMERATION OF HAMILTONIAN POLYGONS IN ROOTED PLANAR TRIANGULATIONS

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1. Introduction

A finite nonseparable topological graph in $E^2$ is said to be a triangular map if all its finite faces are triangular. If no two edges of the graph have the same pair of ends, the graph and its embedding is said to be a triangulation. The maps considered here are rooted by distinguishing an external vertex and an incident external edge. A polygon in such a graph $G$ of such a map is Hamiltonian if it includes all the vertices of $G$.

In [1], the average number of Hamiltonian polygons in a member of the class of nonisomorphic rooted triangular maps with $n$ internal and $k$ external edges is treated. An analogous result for triangulations is dealt with here. For a more complete treatment of the definitions involved in the introduction, the reader is referred to [1, Section 1].

2. The enumeration of Hamiltonian rooted maps

A triangulation or triangular map is Hamiltonian rooted if one of its Hamiltonian polygons is distinguished as root polygon. Let $f_{n,k}$ denote the number of Hamiltonian rooted triangulations with $k$ external and $n$ internal vertices. In future, we refer to maps with $n$ internal and $k$ external vertices as maps of type $(n, k)$.

We define the generating function $F(x, y)$ by the formal power series,

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\[ R(X) = \sum_{n=1}^{\infty} r_n x^n, \]

where \( r_n \) is the number of Hamiltonian triangular maps of type \((n, 2)\) in which the root polygon contains the root edge. This generating function will be useful in defining \( F(x, y) \) in an implicit form. Using the notation and results of \([1]\), it is evident that

\[ r_n = \frac{1}{2} \left( p_{n,2} - q_{n,2} \right) = \left( \frac{2^n}{n} \right) \left( n + 1 \right)^2. \]

Let us refer to any proper maximal submap of type \((n, 2)\) of a triangular map as a 'link map. A link map of a Hamiltonian map must contain a Hamiltonian arc connecting its two external vertices. Looking at such a map in its own right, we may root it by distinguishing one of its external vertices and edges in accordance with some convention, then convert it to a Hamiltonian rooted map of type \((n, 2)\) whose root polygon passes through its root edge lay adjoining the root edge to the Hamiltonian arc previously mentioned. If we replace all such maps in a Hamiltonian rooted triangular map of type \((n, k)\), \(k \geq 3\), by single edges, we obtain a Hamiltonian rooted triangulation. Conversely, every Hamiltonian rooted map may be obtained in a unique fashion by reversing the procedure. Thus we obtain the functional equation

\[ \sum_{n=0}^{\infty} \sum_{k=3}^{\infty} f_{n,k} x^n y^k \left( 1 + R(x) \right)^{n+k} = \sum_{n=0}^{\infty} \sum_{k=3}^{\infty} p_{n,k} x^n y^k, \]

where \( p_{n,k} \) is the number of Hamiltonian rooted triangular maps of type \((n, k)\). It is shown in \([1]\) that

\[ p_{n,k} = \frac{k \left( 2n + 2k - 4 \right)! \left( 2n + k - 1 \right)!}{(n + k - 1)! (n + k - 2)! n! (n + k)!}. \]

If we define \( P(x, y) \) by the formal series
\[ P(x, y) = \sum_{n=0}^{\infty} \sum_{k=3}^{\infty} p_{n,k} x^n y^k, \]

we have \( F(x, y) \) defined implicitly by the relation

\[ F(x (1 + R(x)), y (1 + R(x))) = P(x, y). \]

3. Determination of \( f_{n,k} \)

In this section, we develop a method to determine \( f_{n,k} \). Some explicit expressions for \( f_{0,k}, f_{1,k}, f_{2,k}, f_{3,k} \) are given in closed form. A table for \( f_{n,k} \), where \( n = 0, 1, \ldots, 10 \) and \( k = 3, 4, \ldots, 10 \) is given at the end of this section.

Comparing the coefficients of \( y^k \) on both sides of (4), we have

\[ \sum_{n=0}^{\infty} f_{n,k} [1 - R(x)]^{n+k} x^n = \sum_{n=0}^{\infty} p_{n,k} x^n \quad \text{for } k = 3, 4, \ldots. \]

Now expanding \([1 + R(x)]^{n+k}\) with the help of [2, p. 189], we get

\[ [1 + R(x)]^{n+k} = \sum_{s=0}^{\infty} A_{n+k,s}(r_1, r_2, \ldots, r_s) x^s, \]

with

\[ A_{n+k,s}(r_1, r_2, \ldots, r_s) = \sum_{m=1}^{s} \binom{n+k}{m} B_{s,m}(r_1, r_2, \ldots, r_s), \]

\[ B_{s,m}(r_1, r_2, \ldots, r_s) = \sum_{\pi(s)} \frac{m!}{m_1! \ldots m_s!} r_1^{m_1} \ldots r_s^{m_s}, \]

where \( \pi(s) \) denotes a partition of \( s \) such that

\[ s = m_1 + 2m_2 + \ldots + s m_s, \]

\[ m = m_1 + m_2 + \ldots + m_s. \]

In particular, (10), (11) and (3) give
From (8) and (9), we have

\[ \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} a_{s,t} a_{u,v} x^{s+t} = \sum_{n=0}^{\infty} \sum_{k+s}^{\infty} x^{n+s} = \sum_{n=0}^{\infty} p_{n,k} x^{n} , \]

which, on using the result

\[ \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} a_{s,t} x^{s+t} = \sum_{\mu=0}^{\infty} \sum_{v=0}^{\infty} a_{\mu,v} x^{\mu} x^{v} \]

give

\[ \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} f_{s,k} A_{k+s,n-s} x^{n} = \sum_{n=0}^{\infty} p_{n,k} x^{n} . \]

Now comparing the coefficients of \( x^{n} \) on both sides of (19), we get

\[ \sum_{s=0}^{n} f_{s,k} A_{k+s,n-s} = p_{n,k} \quad \text{for} \quad n = 0, 1, 2, ..., \quad k = 3, 4, 5, ... . \]

Thus (20) and (14) give the recurrence relation

\[ f_{n,k} = p_{n,k} - \sum_{s=0}^{n-1} f_{s,k} A_{k+s,n-s} \quad \text{for} \quad n = 0, 1, ..., \quad k = 3, 4, ... , \]

from which \( f_{n,k} \) can be computed. In particular, for \( k \geq 3 \), we have

\[ f_{0,k} = p_{0,k} , \]
\[ f_{1,k} = 3(2k - 4)! \left[ (k - 1)! (k - 3)! \right]^{-1} , \]
\[ f_{2,k} = (2k - 5)! k (9k^3 + 8k^2 - 73k + 48) \left[ (k - 3)! (k + 1)! \right]^{-1} , \]
\[ f_{3,k} = \frac{(2k - 5)! k(9k^5 + 78k^4 + 35k^3 - 514k^2 + 184k + 64)}{(k - 3)! (k + 2)!} . \]

Hence
3. Determination of $f_{n, k}$

Table 1

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\[
F(x, y) = y^3 [1 + 3x + 18x^2 + \ldots] + y^4 [2 + 12x + 52x^2 + \ldots] + y^5 [5 + 45x + 420x^2 + \ldots] + \ldots
\]

From (22)–(25) we conjecture that

\[
F_s(k) = [(k - 3)! (k + s - 1)! [(2k - 5)!]^{-1}]_{s, k}
\]

is a polynomial of degree $2s$. It is not difficult to prove that $F_s(k)$ is a polynomial of degree less than or equal to $2s$.

The coefficients $f_{n, k}$ as calculated from the results of this paper are listed in Table 1 for $0 \leq n \leq 10, 3 \leq k \leq 10$. 
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References