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The Characteristic Maxpolynomial of a Matrix

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An analog of the characteristic polynomial is defined for a matrix over the algebraic structure ($\mathbb{R}$, max, +) and its properties are discussed.

1. INTRODUCTION

The algebraic structure $\mathscr{E} = (\mathscr{E}; \oplus, \otimes)$, where $\mathscr{E}$ is the real-number set and

$$x \oplus y = \max(x, y), \qquad x \otimes y = x + y \qquad (x, y \in \mathscr{E}), \tag{1.1}$$

can mimic a wide range of the algebraic properties of the usual real-number system $\mathbb{R} = (\mathscr{E}, +, \times)$ and also provides a formulation language for some important classes of applied mathematical problems.

For instance, in [1] we give an extensive account of the *matrix algebra* constructible over $\mathscr{E}$, together with its application to certain classes of scheduling, path finding, and Boolean problems; and in [2] *polynomial algebra* over $\mathscr{E}$ was considered, with application to minimax problems and combinatorial optimisation.

In classical algebra, the link between matrices and polynomials comes through the *characteristic polynomial* of a matrix, which has the eigenvalues of the matrix as its roots. We shall show that a closely analogous result holds in algebra over $\mathscr{E}$. The discussion will then be extended from polynomials to infinite series. For the sake of brevity, we shall draw extensively on the results in [1, 2].

2. BACKGROUND RESULTS

Let the notations $\sum_\oplus$ and $\prod_\otimes$, respectively, stand for iterated use of the associative "addition" $\oplus$ and the associative "multiplication" $\otimes$, by analogy with the usual notations $\sum$ and $\prod$. Let $\mathscr{M}_{mn}$ denote the set of $(m \times n)$ arrays

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over $\mathscr{E}$, and let $[a_{ij}] = \mathbf{A} \in \mathscr{M}_{nn}$; $[u_j] = \mathbf{u} \in \mathscr{M}_{n1}$; $\lambda \in \mathscr{E}$. Then $\mathbf{u}$, λ are, respectively, an *eigenvector* and *eigenvalue* of $\mathbf{A}$ if

$$\mathbf{A} \otimes \mathbf{u} = \lambda \otimes \mathbf{u}, \tag{2.1}$$

where for $[b_{ij}] \in \mathscr{M}_{mn}$ and $[c_{jk}] \in \mathscr{M}_{np}$ the matrix product $[d_{ik}] = [b_{ij}] \otimes [c_{jk}]$ is defined by analogy with traditional linear algebra by

$$d_{ik} = \sum_{j=1}^{n}{}_{\oplus} (b_{ij} \otimes c_{jk}) \qquad (i = 1,..., m; k = 1,..., p). \tag{2.2}$$

Thus (2.2) and (1.1) imply that (2.1) is equivalent to

$$\lambda + u_i = \max_{j=1,...,n} (a_{ij} + u_j) \qquad (i = 1,..., n). \tag{2.3}$$

LEMMA 1. *For any* $\mathbf{A} \in \mathscr{M}_{nn}$, *there exist* $\mathbf{u}$ *and* λ *satisfying* (2.1). *Also,* λ *is unique and equals* $\lambda(\mathbf{A})$, *the greatest of the "circuit averages"*:

$$a_{ii}; \quad \frac{a_{ij} + a_{ji}}{2}; \quad \frac{a_{ij} + a_{jk} + a_{ki}}{3}; ... \qquad (i, j, k,... \in \{1,..., n\}). \tag{2.4}$$

Proof [1, Lemma 23-2 and Theorem 24-9].

Now let x be a variable ranging over $\mathscr{E}$ and for positive integers p define $x^{(p)} = x \otimes \cdots \otimes x$ (p-fold) with $x^{(0)} = 0$. Let $\alpha_0,..., \alpha_N \in \mathscr{E}$ $(N \geqslant 1)$ be given. Then the function $Q(x)$, where

$$Q(x) = \max_{r=0,...,N} (\alpha_r + rx) \tag{2.5}$$

$$= \sum_{r=0}^{N}{}_{\oplus} \alpha_r \otimes x^{(r)} \tag{2.6}$$

is called a *maxpolynomial.* The following analog of the fundamental theorem of algebra holds:

LEMMA 2. *Any maxpolynomial* (2.6) *may be written as a "product of N linear factors"*

$$Q(x) = \alpha_N \otimes \theta_1(x) \otimes \cdots \otimes \theta_N(x), \tag{2.7}$$

where each $\theta_k(x)$ *is either of the form* x, *or of the form* $(x \oplus \beta_s)$ *for some* $\beta_s \in \mathscr{E}$. *Apart from order, factorisation* (2.7) *is unique.*

Proof [2, Theorem 11].

The constants β_s in Lemma 2 are called the *corners* of $Q(x)$.

3. Characteristic Maxpolynomial

Although we cannot define a determinant for $[b_{ij}] \in \mathscr{M}_{nn}$, we can define the *permanent* of $[b_{ij}]$ by

$$\operatorname{perm}[b_{ij}] = \sum_{\sigma} {}_{\oplus} \left(\prod_{t=1}^{n} {}_{\otimes} b_{t\sigma(t)} \right), \tag{3.1}$$

where the "summation" is over all permutations σ in the symmetric group of order n! Then we may define the *characteristic maxpolynomial* $\pi_{\mathbf{A}}(x)$ of a given square matrix $\mathbf{A} \in \mathscr{M}_{nn}$ by

$$\pi_{\mathbf{A}}(x) = \operatorname{perm}(\mathbf{A}, x), \tag{3.2}$$

where $(\mathbf{A}, x)$ is a matrix derived from $\mathbf{A}$ by replacing its diagonal elements a_{ii} by $a_{ii} \oplus x$ $(i = 1,..., n)$.

Thus, if

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}, \tag{3.3}$$

then

$$\pi_{\mathbf{A}}(x) = \operatorname{perm} \begin{bmatrix} 2 \oplus x & 1 & 4 \\ 1 & 0 \oplus x & 1 \\ 2 & 2 & 1 \oplus x \end{bmatrix} \tag{3.4}$$

$$\begin{aligned} &= (2 \oplus x) \otimes (0 \oplus x) \otimes (1 \oplus x) \oplus (2 \oplus x) \otimes 1 \otimes 2 \\ &\quad \oplus 1 \otimes 1 \otimes (1 \oplus x) \oplus 1 \otimes 1 \otimes 2 \\ &\quad \oplus 4 \otimes 1 \otimes 2 \oplus 4 \otimes (0 \oplus x) \otimes 2 \\ &= x^{(3)} \oplus 2 \otimes x^{(2)} \oplus 6 \otimes x \oplus 7, \end{aligned} \tag{3.5}$$

by use of the usual associative, distributive, and commutative laws [2]. Lemma 2 gives for maxpolynomial (3.5)

$$\pi_{\mathbf{A}}(x) = (x \oplus 1) \otimes (x \oplus 3)^{(2)}. \tag{3.6}$$

Lemma 1 implies for $\mathbf{A}$ in (3.3)

$$\lambda(\mathbf{A}) = \max\left(2, 0, 1, \frac{1+1}{2}, \frac{4+2}{2}, \frac{1+2}{2}, \frac{1+1+2}{3}, \frac{4+1+2}{3}\right) = 3. \tag{3.7}$$

Then (3.6) and (3.7) illustrate the following new result, which we now prove.

THEOREM 3. *For any* $\mathbf{A} \in \mathscr{M}_{nn}$: *the greatest corner of* $\pi_{\mathbf{A}}(x)$ *is* $\lambda(\mathbf{A})$.

4. PROOF OF THEOREM 3

For a *circuit* C, e.g., $a_{12}, a_{23}, \ldots, a_{r-1r}, a_{r1}$, let $|C|$ denote the *circuit sum*, e.g.,

$$|C| = a_{12} + \cdots + a_{r1} = a_{12} \otimes \cdots \otimes a_{r1} \tag{4.1}$$

and let $\|C\|$ denote the *circuit length* e.g.,

$$\|C\| = r. \tag{4.2}$$

Thus by (2.4),

$$\lambda(\mathbf{A}) = \max_{\text{circuits } C \text{ in } \mathbf{A}} \left(\frac{|C|}{\|C\|} \right). \tag{4.3}$$

Without loss of generality suppose that, after renumbering if necessary, for suitable R

$$\lambda(\mathbf{A}) = \frac{a_{12} + a_{23} + \cdots + a_{R1}}{R}. \tag{4.4}$$

Then

$$\operatorname{perm} \begin{bmatrix} a_{11}, \ldots, a_{1R} \\ \cdots \\ a_{R1}, \ldots, a_{RR} \end{bmatrix} = (a_{12} \otimes \cdots \otimes a_{R1}) \oplus \cdots \geqslant R\lambda(\mathbf{A}). \tag{4.5}$$

But if $\mathbf{P}$ is $\mathbf{A}$ or any principal minor of $\mathbf{A}$, we have with a self-evident notation

$$\operatorname{perm} \mathbf{P} = \sum\nolimits_{\oplus} (|\text{circuit}_1| \otimes \cdots \otimes |\text{circuit}_p|), \qquad \text{where}$$

$$\|\text{circuit}_1\| + \cdots + \|\text{circuit}_p\| = \text{order of } \mathbf{P}. \tag{4.6}$$

But by (4.3), for any circuit in $\mathbf{A}$

$$|\text{circuit}| \leqslant \lambda(\mathbf{A})\, \|\text{circuit}\|. \tag{4.7}$$

Also (4.6) and (4.7) give

$$\operatorname{perm} \mathbf{P} \leqslant \lambda(\mathbf{A}) \,(\text{order of } \mathbf{P}). \tag{4.8}$$

Then (4.5) and (4.8) give as **P** varies over **A** and its principal minors

$$\lambda(\mathbf{A}) = \max_{\mathbf{P}} \left(\frac{\text{perm } \mathbf{P}}{\text{order of } \mathbf{P}} \right). \tag{4.9}$$

Now consider a "monic" maxpolynomial of the form

$$x^{(n)} \oplus \delta_1 \otimes x^{(n-1)} \oplus \cdots \oplus \delta_{n-1} \otimes x \oplus \delta_n. \tag{4.10}$$

Referring to the algorithm in [2, Sect. 9], we see that the greatest corner of the maxpolynomial (4.10) is

$$\max_{r=1,\ldots,n} \left(\frac{\delta_r}{r} \right). \tag{4.11}$$

But evidently, as (3.5) illustrates, $\pi_{\mathbf{A}}(x)$ is a maxpolynomial of the form (4.10) in which for $r = 1,\ldots, n$

$$\begin{aligned} \delta_r &= \sum\nolimits_{\oplus} (\text{all } (r \times r) \text{ principal minor permanents } \mathbf{P} \text{ in } \mathbf{A}) \\ &= \max(\text{perm } \mathbf{P} \mid \text{order of } \mathbf{P} = r). \end{aligned} \tag{4.12}$$

The result now follows from (4.9), (4.11), and (4.12).

5. Power Series Convergence

In [3], we defined (scalar) *power series*

$$\sum_{r=0}^{\infty}{}_{\oplus} b_r \otimes x^{(r)} \tag{5.1}$$

for a given sequence $\{b_r\}$ of coefficients. Defining

$$\rho = \underline{\lim}_{r \to \infty} \left(-\frac{b_r}{r} \right) \tag{5.2}$$

we proved

Lemma 2. *Series (5.1) converges if $x < \rho$ and diverges if $x > \rho$.*

Thus ρ plays a role analogous to that of a radius of convergence.

Furthermore, introducing the *matrix power series* for a given square matrix $\mathbf{A}$

$$\sum_{r=0}^{\infty}{}_{\oplus} b_r \otimes \mathbf{A}^{(r)}, \tag{5.3}$$

where $\mathbf{A}^{(r)} = \mathbf{A} \otimes \cdots \otimes \mathbf{A}$ (r-fold), we proved

LEMMA 5. *Series* (5.3) *converges if* $\lambda(\mathbf{A}) < \rho$ *and diverges if* $\lambda(\mathbf{A}) > \rho$.

Let us now define the sequence $\{\phi_n\}$ of maxpolynomials by

$$\phi_n(x) = \sum_{r=0}^{n}{}_{\oplus} b_r \otimes x^{(r)} \qquad (n = 1, 2, \ldots). \tag{5.4}$$

Let β_n be the greatest corner of ϕ_n and define

$$\beta = \varliminf_{n \to \infty} \beta_n. \tag{5.5}$$

We shall prove the following new result:

THEOREM 6. *With the foregoing notation:* $\beta = \rho$.

Proof. From [2, Sect. 9], the greatest corner of ϕ_n is given by

$$\beta_n = \max_{j<n} \left(\frac{b_j - b_n}{n - j}\right) \qquad (n = 1, 2, \ldots). \tag{5.6}$$

Taking $j = 0$ in (5.6)

$$\beta_n \geqslant \frac{b_0}{n} - \frac{b_n}{n} \qquad (n = 1, 2, \ldots),$$

whence from (5.2) and (5.5) $\beta \geqslant \rho$. On the other hand, from (5.5) and (5.6)

if $\varepsilon > 0$, $\exists N$ such that

$$\max_{j<n} \left(\frac{b_j - b_n}{n - j}\right) > \beta - \frac{1}{2}\varepsilon \qquad \text{for} \quad n \geqslant N,$$

i.e., if $\varepsilon > 0$, $\exists N$ such that

if $n \geqslant N$, $\exists n' < n$ such that

$$\frac{b_{n'} - b_n}{n - n'} > \beta - \frac{1}{2}\varepsilon, \qquad \text{i.e.,} \qquad b_{n'} - b_n > (n - n')\left(\beta - \frac{1}{2}\varepsilon\right). \tag{5.7}$$

And if n' is not less than N, we may apply (5.7) again to get

$$\exists n'' < n' \qquad \text{such that} \qquad b_{n''} - b_{n'} > (n' - n'')(\beta - \tfrac{1}{2}\varepsilon). \tag{5.7$'$}$$

Since $n > n' > n'' > \cdots$, we may develop (5.7) and (5.7′) and so on until we find $n''' = m$ (say) $< N$. Then adding (5.7), (5.7′)... we get

if $\varepsilon > 0$, $\exists N$ such that: if $n \geqslant N$, $\exists m < N$ such that $(b_m - b_n) > (n - m)(\beta - \frac{1}{2}\varepsilon)$,

i.e.,

$$\begin{aligned} -\frac{b_n}{n} &> \left(1 - \frac{m}{n}\right)\left(\beta - \frac{1}{2}\varepsilon\right) - \frac{b_m}{n} \\ &> (\beta - \varepsilon) \qquad \text{as} \quad n \uparrow \infty. \end{aligned} \tag{5.8}$$

Clearly (5.2), (5.5), and (5.8) imply $\rho \geqslant \beta$ and the result follows.

REFERENCES

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