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Generic initial ideals and graded Artinian-level algebras not having the Weak-Lefschetz Property[☆]

Jeaman Ahn[a], Yong Su Shin[b,*]

[a] *Korea Institute for Advanced Study, Seoul, 130-722, Republic of Korea*
[b] *Department of Mathematics, Sungshin Women's University, Seoul, 136-742, Republic of Korea*

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Abstract

We find a sufficient condition that **H** is not level based on a reduction number. In particular, we prove that a graded Artinian algebra of codimension 3 with Hilbert function $\mathbf{H} = (h_0, h_1, \dots, h_{d-1} > h_d = h_{d+1})$ cannot be level if $h_d \le 2d + 3$, and that there exists a level O-sequence of codimension 3 of type **H** for $h_d \ge 2d + k$ for $k \ge 4$. Furthermore, we show that **H** is not level if $\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}})$, and also prove that any codimension 3 Artinian graded algebra $A = R/I$ cannot be level if $\beta_{1,d+2}(\text{Gin}(I)) = \beta_{2,d+2}(\text{Gin}(I))$. In this case, the Hilbert function of A does not have to satisfy the condition $h_{d-1} > h_d = h_{d+1}$.

Moreover, we show that every codimension n graded Artinian level algebra having the Weak-Lefschetz Property has a strictly unimodal Hilbert function having a growth condition on $(h_{d-1} - h_d) \le (n-1)(h_d - h_{d+1})$ for every $d > \theta$ where

$$h_0 < h_1 < \cdots < h_\alpha = \cdots = h_\theta > \cdots > h_{s-1} > h_s.$$

In particular, we show that if A is of codimension 3, then $(h_{d-1} - h_d) < 2(h_d - h_{d+1})$ for every $\theta < d < s$ and $h_{s-1} \le 3h_s$, and prove that if A is a codimension 3 Artinian algebra with an h-vector $(1, 3, h_2, \dots, h_s)$ such that

$$h_{d-1} - h_d = 2(h_d - h_{d+1}) > 0 \quad \text{and} \quad \text{soc}(A)_{d-1} = 0$$

for some $r_1(A) < d < s$, then $(I_{\le d+1})$ is $(d+1)$-regular and $\dim_k \text{soc}(A)_d = h_d - h_{d+1}$.

MSC: Primary: 13D40; secondary: 14M10

1. Introduction

Let $R = k[x_1, \dots, x_n]$ be an n-variable polynomial ring over an infinite field with characteristic 0. In this article, we shall study Artinian quotients $A = R/I$ of R where I is a homogeneous ideal of R. These rings are often referred

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* Corresponding author.
E-mail addresses: ajman@kias.re.kr (J. Ahn), ysshin@sungshin.ac.kr (Y.S. Shin).

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to as standard graded algebras. Since $R = \oplus_{i=0}^{\infty} R_i$ (R_i: the vector space of dimension $\binom{i+(n-1)}{n-1}$ generated by all the monomials in R having degree i) and $I = \oplus_{i=0}^{\infty} I_i$, gives

$$A = R/I = \oplus_{i=0}^{\infty}(R_i/I_i) = \oplus_{i=0}^{\infty} A_i$$

as a graded ring. The numerical function

$$\mathbf{H}_A(t) := \dim_k A_t = \dim_k R_t - \dim_k I_t$$

is called the *Hilbert function* of the ring A.

Given an O-sequence $\mathbf{H} = (h_0, h_1, \ldots)$, we define the *first difference* of $\mathbf{H}$ as

$$\Delta\mathbf{H} = (h_0, h_1 - h_0, h_2 - h_1, h_3 - h_2, \ldots).$$

If I is a homogeneous ideal of R of height n, then $A = R/I$ is an *Artinian k-algebra*, and hence $\dim_k A < \infty$. We associate the graded algebra A with a vector of nonnegative integers which is an $(s+1)$-tuple, called the *h-vector* of A and denoted by

$$h(A) = (h_0, h_1, \ldots, h_s),$$

where $h_i = \dim_k A_i$. Thus, we can write $A = k \oplus A_1 \oplus \cdots \oplus A_s$ where $A_s \neq 0$. We call s the *socle degree* of A. The *socle* of A is defined by the annihilator of the maximal homogeneous ideal, namely

$$\operatorname{ann}_A(m) := \{a \in A \mid am = 0\} \quad \text{where } m = \sum_{i=1}^{s} A_i.$$

Moreover, an h-vector $(h_0, h_1, \ldots, h_s)$ is called

unimodal if $h_0 \leq \cdots \leq h_t = \cdots = h_\ell \geq \cdots \geq h_s$,
strictly unimodal if $h_0 < \cdots < h_t = \cdots = h_\ell > \cdots > h_s$.

A graded Artinian k-algebra $A = \bigoplus_{i=0}^{s} A_i$ ($A_s \neq 0$) is said to have the *Weak-Lefschetz Property* (WLP for short) if there is an element $L \in A_1$ such that the linear transformations

$$A_i \stackrel{\times L}{\rightarrow} A_{i+1}, \quad 1 \leq i \leq s-1,$$

which are defined by a multiplication by L, are either injective or surjective. This implies that the linear transformations have maximal ranks for every i. In this case, we call L a *Lefschetz element*.

A monomial ideal I in R is *stable* if the monomial

$$\frac{x_j w}{x_{m(w)}}$$

belongs to I for every monomial $w \in I$ and $j < m(w)$ where

$$m(u) := \max\{j \mid a_j > 0\}$$

for $u = x_1^{a_1} \cdots x_n^{a_n}$. Let S be a subset of all monomials in $R = \bigoplus_{i \geq 0} R_i$ of degree i. We call S a *Boreal fixed set* if

$$u = x_1^{a_1} \cdots x_n^{a_n} \in S, \quad a_j > 0, \quad \text{implies} \quad \frac{x_i u}{x_j} \in S$$

for every $1 \leq i \leq j \leq n$.

A monomial ideal I of R is called a *Borel-fixed ideal or strongly stable ideal* if the set of all monomials in I_i is a Borel set for every i. There are two Borel-fixed monomial ideals canonically attached to a homogeneous ideal I of R: the generic initial ideal $\operatorname{Gin}(I)$ with respect to the reverse lexicographic order and the lex-segment ideal I^{lex}. The ideal I^{lex} is defined as follows. For the vector space I_d of forms of degree d in I, one defines $(I^{\text{lex}})_d$ to be the vector space generated by the largest, in lexicographical order, $\dim_k(I_d)$ monomials of degree d. By construction, I^{lex} is a strongly stable ideal and it only depends on the Hilbert function of I.

In the case of the generic initial ideal, it has been proved by Galligo [13] that they are Borel-fixed in characteristic zero, and then by Bayer and Stillman [2] that they are generalized to every characteristic.

In [1], Ahn and Migliore gave some geometric results using generic initial ideals for the degree reverse lexicographic order, which improved a well-known result of Bigatti, Geramita, and Migliore concerning geometric consequences of maximal growth of the Hilbert function of the Artinian reduction of a set of points in [6]. In [15], Geramita, Harima, Migliore, and Shin gave a homological reinterpretation of a level Artinian algebra and explained the combinatorial notion of Cancellation of Betti numbers of the minimal free resolution of the lex-segment ideal associated to a given homogeneous ideal. We shall explain the new result when we carry out the analogous result using the generic initial ideal instead of the lex-segment ideal. We find some new results on the maximal growth of the difference of Hilbert function in degree d larger than the reduction number $r_1(A)$ if there is no socle element in degree $d-1$ using some recent result given by Ahn and Migliore [1]. As an application, we give the condition if some O-sequences are "either level or non-level sequences of Artinian graded algebras with the WLP.

Let $\mathcal{F}$ be the graded minimal resolution of R/I, i.e.,

$$\mathcal{F}:\ 0 \to \mathcal{F}_n \to \mathcal{F}_{n-1} \to \cdots \to \mathcal{F}_1 \to R \to R/I \to 0.$$

We can write

$$\mathcal{F}_i = \bigoplus_{j=1}^{\gamma_i} R^{\beta_{ij}}(-\alpha_{ij})$$

where $\alpha_{i1} < \alpha_{i2} < \cdots < \alpha_{i\gamma_i}$. The numbers α_{ij} are called the *shifts* associated to R/I, and the numbers β_{ij} are called the *graded Betti numbers* of R/I. For I as above, the *Betti diagram* of R/I is a useful device to encode the graded Betti numbers of R/I (and hence of I). It is constructed as follows:

$$\begin{array}{c} \\ 0\\ 1\\ \vdots\\ t\\ \vdots\\ d-2\\ d-1\\ d\\ \vdots \end{array}
\begin{array}{c} \begin{array}{ccccc} \quad & 0 & 1 & \cdots & n-1 \end{array} \\
\left(\begin{array}{cccccc}
1 & 0 & 0 & \cdots & 0 \\
0 & * & * & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \beta_{0,t+1} & \beta_{1,t+2} & * & \beta_{n-1,t+n} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \beta_{0,d-1} & \beta_{1,d} & * & \beta_{n-1,d-2+n} \\
0 & \beta_{0,d} & \beta_{1,d+1} & * & \beta_{n-1,d-1+n} \\
0 & \beta_{0,d+1} & \beta_{1,d+2} & * & \beta_{n-1,d+n} \\
\vdots & \vdots & \vdots & \vdots &
\end{array}\right) \end{array}$$

When we need to emphasize the ideal I, we shall use $\beta_{i,j}(I)$ for $\beta_{i,j}$.

Recall that if the last free module of the minimal free resolution of a graded ring A with Hilbert function $\mathbf{H}$ is of the form $\mathcal{F}_n = R^{\beta}(-s)$ for some $s > 0$, then the Hilbert function $\mathbf{H}$ and the graded ring A are called *level*. For a special case, if $\beta = 1$, then we call a graded Artinian algebra A *Gorenstein*. In [32], Stanley proved that any graded Artinian Gorenstein algebra of codimension 3 is unimodal. In fact, he proved a stronger result than unimodality using the structure theorem of Buchsbaum and Eisenbud for the Gorenstein algebra of codimension 3 in [8]. Since then, the graded Artinian Gorenstein algebras of codimension 3 have been much studied (see [9,15,16,20,21,27,28,31,33]). In [3], Bernstein and Iarrobino showed how to construct non-unimodal graded Artinian Gorenstein algebras of codimension higher than or equal to 5. Moreover, in [7], Boij and Laksov showed another method on how to construct the same graded Artinian Gorenstein algebras. Unfortunately, it is unknown if there exists a graded non-unimodal Gorenstein algebra of codimension 4. For unimodal Artinian Gorenstein algebras of codimension 4, how to construct some of them using the link-sum method has been shown by Shin in [31]. It has also been shown by Geramita, Harima, and Shin [16] and Harima [20] how to obtain some unimodal Artinian Gorenstein algebras of any codimension n (≥ 3). An SI-sequence is a finite sequence of positive integers which is symmetric, unimodal, and satisfies a certain growth condition. In [28], Migliore and Nagel showed how to construct a reduced, arithmetically Gorenstein configuration G of linear varieties of arbitrary dimension whose Artinian reduction has the given SI-sequence as Hilbert function and has the Weak Lefschetz Property. For graded Artinian-level algebras, it has been recently studied (see [3,5,7,10,15,17,27,33,34]). In [15], they proved the following result. Let

$$\mathbf{H} \quad : \quad h_0 \quad h_1 \quad \cdots \quad h_{d-1} \quad h_d \quad h_d \quad \cdots \tag{1.1}$$

with $h_{d-1} > h_d$. If $h_d \le d+1$ with any codimension h_1, then $\mathbf{H}$ is *not* level.

In [33], Zanello constructed a non-unimodal level O-sequence of codimension 3 as follows:

$$\mathbf{H} = (h_0, h_1, \ldots, h_d, t, t, t+1, t, t, \ldots, t+1, t, t)$$

where the sequence $t, t, t+1$ can be repeated as many times as we want. Thus there exists a graded Artinian-level algebra of codimension 3 of type in Eq. (1.1) which does not have the WLP.

In Section 2, preliminary results and notations on lex-segment ideals and generic initial ideals are introduced. In Section 3, we show that any codimension n graded Artinian level algebra A having the WLP has the Hilbert function which is strictly unimodal (see Theorem 3.6). In particular, we prove that if A has the Hilbert function such that

$$h_0 < h_1 < \cdots < h_{r_1(A)} = \cdots = h_\theta > \cdots > h_{s-1} > h_s,$$

then $h_{d-1} - h_d \le (n-1)(h_d - h_{d+1})$ for every $\theta < d \le s$ (see Theorem 3.6). Furthermore, we show that if A is of codimension 3, then $h_{d-1} - h_d < 2(h_d - h_{d+1})$ for every $\theta < d < s$ and $h_{s-1} \le 3h_s$ (see Theorem 3.23). We also prove that if A is a codimension 3 Artinian graded algebra with socle degree s and

$$\beta_{1,d+2}(\mathrm{Gin}(I)) = \beta_{2,d+2}(\mathrm{Gin}(I)) > 0$$

for some $d < s$, then A cannot be level (see Theorem 3.14). Moreover, if $A = R/I$ is a codimension 3 Artinian graded algebra with an h-vector $(1, 3, h_2, \ldots, h_s)$ such that $h_{d-1} - h_d = 2(h_d - h_{d+1}) > 0$ for some $r_1(A) < d < s$ and $\mathrm{soc}(A)_{d-1} = 0$, then $(I_{\le d+1})$ is $(d+1)$-regular and $\dim_k \mathrm{soc}(A)_d = h_d - h_{d+1}$ (see Theorem 3.19).

One of the main topics of this paper is to study O-sequences of type in Eq. (1.1) and find an answer to the following question.

Question 1.1. Let $\mathbf{H}$ be as in Eq. (1.1) with $h_1 = 3$. What is the minimum value for h_d when $\mathbf{H}$ is level?

Finally in Section 4, we show that if R/I is a graded Artinian algebra of codimension 3 having Hilbert function $\mathbf{H}$ in Eq. (1.1) and $\beta_{1,d+2}(I^{\mathrm{lex}}) = \beta_{2,d+2}(I^{\mathrm{lex}})$, then R/I is *not* level, i.e., $\mathbf{H}$ cannot be level (see Theorem 4.5). Furthermore, we prove that any O-sequence $\mathbf{H}$ of codimension 3 in Eq. (1.1) cannot be level when $h_d \le 2d+3$ and there exists a level O-sequence of codimension 3 of the type in Eq. (1.1) having $h_d \ge 2d+k$ for every $k \ge 4$ (see Theorem 4.1, Proposition 4.9, and Remark 4.10), which is a complete answer to Question 1.1.

A computer program CoCoA (see [30]) was used for all examples in this article.

2. Some preliminary results

In this section, we introduce some preliminary results and notations on lex-segment ideals and generic initial ideals. We only consider the degree reverse lexicographic order.

Theorem 2.1 ([1,2,19]). *Let L be a general linear form and let $J = (I + (L))/(L)$ be considered as a homogeneous ideal of $S = k[x_1, \ldots, x_{n-1}]$. Then*

$$\mathrm{Gin}(J) = (\mathrm{Gin}(I) + (x_n))\,/(x_n).$$

Let I be a homogeneous ideal of R. For a monomial term ordering τ there exists a flat family of ideals I_t with $I_0 = \mathrm{in}_\tau(I)$ (the initial ideal of I) and I_t canonically isomorphic to I for all $t \ne 0$ (this implies that $\mathrm{in}_\tau(I)$ has the same Hilbert function as that of I). Using this result, gives us the following theorem:

Theorem 2.2 (*The Cancelation Principle, [1,19]*). *For any homogeneous ideal I and any i and d, there is a complex of $k \cong R/m$-modules $V_\bullet^d$ such that*

$$V_i^d \cong \mathrm{Tor}_i^R(\mathrm{in}_\tau(I), k)_d$$
$$H_i(V_\bullet^d) \cong \mathrm{Tor}_i^R(I, k)_d.$$

Remark 2.3. One way to paraphrase this theorem is to say that the minimal free resolution of I is obtained from that of $\mathrm{in}_\tau(I)$, the *initial ideal* of I, by canceling some adjacent terms of the same degree.

Theorem 2.4 (*Eliahou and Kervaire, [11]*). *Let I be a stable monomial ideal of R. Denote by $\mathcal{G}(I)$ the set of minimal (monomial) generators of I and $\mathcal{G}(I)_d$ the elements of $\mathcal{G}(I)$ having degree d. Then*

$$\beta_{q,i}(I) = \sum_{T \in \mathcal{G}(I)_{i-q}} \binom{m(T)-1}{q}.$$

This theorem gives all the graded Betti numbers of the lex-segment ideal and the generic initial ideal just from an intimate knowledge of the generators of that ideal. Since the minimal free resolution of the ideal of a k-configuration in $\mathbb{P}^n$ is extremal [16,18], we may apply this result to those ideals. It is an immediate consequence of the Eliahou–Kervaire theorem that if I is a lex-segment ideal, a generic initial ideal, or the ideal of a k-configuration in $\mathbb{P}^n$ which has *no* generators in degree d, then $\beta_{q,i} = 0$ whenever $i - q = d$.

Remark 2.5. Let I be any homogeneous ideal of $R = k[x_1, \ldots, x_n]$ and $J = \mathrm{Gin}(I)$. Then, by Theorem 2.2, we have

$$\beta_{q,i}(I) \le \beta_{q,i}(J).$$

In particular, if $\beta_{q,i}(J) = 0$, then $\beta_{q,i}(I) = 0$.

Let I be a homogeneous ideal of $R = k[x_1, \ldots, x_n]$ such that $\dim(R/I) = d$. In [23], they defined the *s-reduction number* $r_s(R/I)$ of R/I for $s \ge d$ and have shown the following theorem.

Theorem 2.6 (*[1,23]*). *For a homogeneous ideal I of R,*

$$r_s(R/I) = r_s(R/\mathrm{Gin}(I)).$$

If I is a Borel-fixed monomial ideal of $R = k[x_1, \ldots, x_n]$ with $\dim(R/I) = n - d$, then we know that there are positive numbers $a_1, \ldots, a_d$ such that $x_i^{a_i}$ is a minimal generator of I. In [23], they have also proved that if a monomial ideal I is strongly stable, then

$$r_s(R/I) = \min\{\ell \mid x_{n-s}^{\ell+1} \in I\}.$$

Furthermore, the following useful lemma has been proved in [1].

Lemma 2.7 (*Lemma 2.15, [1]*). *For a homogeneous ideal I of R and for $s \ge \dim(R/I)$, the s-reduction number $r_s(R/I)$ can be given as the following:*

$$\begin{aligned} r_s(R/I) &= \min\{\ell \mid x_{n-s}^{\ell+1} \in \mathrm{Gin}(I)\} \\ &= \min\{\ell \mid \text{ Hilbert function of } R/(I+J) \text{ vanishes in degree } \ell+1\} \end{aligned}$$

where J is generated by s general linear forms of R.

For a homogeneous ideal I of $R = k[x_1, \ldots, x_n]$, we recall that I^{lex} is a lex-segment ideal associated with I. In Section 4, we shall use the following two useful lemmas.

Lemma 2.8. *Let I be a homogeneous ideal of $R = k[x_1, \ldots, x_n]$ and let $\bar{I} = (I_{\le d+1})$ for some $d > 0$. Then,*

(a) $\beta_{i,j}(I) \le \beta_{i,j}(\mathrm{Gin}(I)) \le \beta_{i,j}(I^{\mathrm{lex}})$ *for all i, j.*
(b) $\beta_{0,d+2}(\bar{I}^{\mathrm{lex}}) = \beta_{0,d+2}(I^{\mathrm{lex}}) - \beta_{0,d+2}(I)$,
(c) $\beta_{0,d+2}(\mathrm{Gin}(\bar{I})) = \beta_{0,d+2}(\mathrm{Gin}(I)) - \beta_{0,d+2}(I)$.

Proof. (a) The first inequality can be proved by Theorem 2.2. The second one is directly obtained from the theorem of Bigatti, Hulett, and Pardue [4,24,29].

(b) Firstly, note that

$$\begin{aligned}\beta_{0,d+2}(I^{\text{lex}}) &= \dim_k(I^{\text{lex}})_{d+2} - \dim_k(R_1(I^{\text{lex}})_{d+1})\\ &= [\dim_k R_{d+2} - \dim_k(R_1(I^{\text{lex}})_{d+1})] - [\dim_k R_{d+2} - \dim_k(I^{\text{lex}})_{d+2}]\\ &= \mathbf{H}_{R/I^{\text{lex}}}(d+1)^{\langle d+1\rangle} - \mathbf{H}_{R/I^{\text{lex}}}(d+2)\\ &= \mathbf{H}_{R/I}(d+1)^{\langle d+1\rangle} - \mathbf{H}_{R/I}(d+2) \quad (\because \mathbf{H}_{R/I}(t) = \mathbf{H}_{R/I^{\text{lex}}}(t) \text{ for every } t).\end{aligned} \tag{2.1}$$

It follows from Eq. (2.1) that

$$\begin{aligned}\beta_{0,d+2}(I) &= \dim_k(I_{d+2}) - \dim_k(\bar{I}_{d+2})\\ &= [\dim_k R_{d+2} - \dim_k(\bar{I}_{d+2})] - [\dim_k R_{d+2} - \dim_k(I_{d+2})]\\ &= \mathbf{H}_{R/\bar{I}}(d+2) - \mathbf{H}_{R/I}(d+2)\\ &= (\mathbf{H}_{R/I}(d+1)^{\langle d+1\rangle} - \mathbf{H}_{R/I}(d+2)) - (\mathbf{H}_{R/I}(d+1)^{\langle d+1\rangle} - \mathbf{H}_{R/\bar{I}}(d+2))\\ &= (\mathbf{H}_{R/I}(d+1)^{\langle d+1\rangle} - \mathbf{H}_{R/I}(d+2)) - (\mathbf{H}_{R/\bar{I}}(d+1)^{\langle d+1\rangle} - \mathbf{H}_{R/\bar{I}}(d+2))\\ &\quad (\because \mathbf{H}_{R/I}(d+1) = \mathbf{H}_{R/\bar{I}}(d+1))\\ &= \beta_{0,d+2}(I^{\text{lex}}) - \beta_{0,d+2}(\bar{I}^{\text{lex}}) \quad (\because (2.1)).\end{aligned}$$

(c) Note that $\text{Gin}(I)_{d+1} = \text{Gin}(\bar{I})_{d+1}$. Hence we have

$$\begin{aligned}\beta_{0,d+2}(I) &= \dim_k(I_{d+2}) - \dim_k(\bar{I}_{d+2})\\ &= \dim_k(\text{Gin}(I)_{d+2}) - \dim_k(\text{Gin}(\bar{I})_{d+2})\\ &= [\dim_k(\text{Gin}(I)_{d+2}) - \dim_k(R_1\text{Gin}(I)_{d+1})] - [\dim_k(\text{Gin}(\bar{I})_{d+2}) - \dim_k(R_1\text{Gin}(\bar{I})_{d+1})]\\ &\quad (\because \text{Gin}(I)_{d+1} = \text{Gin}(\bar{I})_{d+1})\\ &= \beta_{0,d+2}(\text{Gin}(I)) - \beta_{0,d+2}(\text{Gin}(\bar{I})),\end{aligned}$$

which completes the proof. □

Lemma 2.9. *Let $I \subset R = k[x_1, x_2x_3]$ be a homogeneous ideal and let that $A = R/I$ be a graded Artinian algebra. Then, for every $d > 0$,*

(a) $\beta_{1,d}(I^{\text{lex}}) - \beta_{1,d}(I) = [\beta_{0,d}(I^{\text{lex}}) - \beta_{0,d}(I)] + [\beta_{2,d}(I^{\text{lex}}) - \beta_{2,d}(I)]$.
(b) $\beta_{1,d}(\text{Gin}(I)) - \beta_{1,d}(I) = [\beta_{0,d}(\text{Gin}(I)) - \beta_{0,d}(I)] + [\beta_{2,d}(\text{Gin}(I)) - \beta_{2,d}(I)]$.

Proof. (a) Recall the Betti diagram of R/I^{lex}:

$$\begin{array}{c} \\ 0\\ 1\\ \vdots\\ d-3\\ d-2\\ d-1\\ \vdots \end{array}\begin{array}{cccc} & 0 & 1 & 2\\ \end{array}$$

$$\begin{array}{c|cccc} & & 0 & 1 & 2\\ \hline 0 & 1 & 0 & 0 & 0\\ 1 & 0 & * & * & *\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ d-3 & 0 & \beta_{0,d-2}(I^{\text{lex}}) & \beta_{1,d-1}(I^{\text{lex}}) & \beta_{2,d}(I^{\text{lex}})\\ d-2 & 0 & \beta_{0,d-1}(I^{\text{lex}}) & \beta_{1,d}(I^{\text{lex}}) & \beta_{2,d+1}(I^{\text{lex}})\\ d-1 & 0 & \beta_{0,d}(I^{\text{lex}}) & \beta_{1,d+1}(I^{\text{lex}}) & \beta_{2,d+2}(I^{\text{lex}})\\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

and let $\gamma_{i,d} = \beta_{i,d}(I^{\text{lex}}) - \beta_{i,d}(I)$. Then, by Theorem 2.2, we have that

$$\begin{array}{ccccc} \gamma_{1,d} & = & \gamma_{0,d} & + & \gamma_{2,d}\\ \| & & \| & & \|\\ \beta_{1,d}(I^{\text{lex}}) - \beta_{1,d}(I) & = & [\beta_{0,d}(I^{\text{lex}}) - \beta_{0,d}(I)] & + & [\beta_{2,d}(I^{\text{lex}}) - \beta_{2,d}(I)], \end{array}$$

as we desired.

(b) In the same way as above, (b) holds immediately. □

3. An *h*-vector of a graded Artinian-level algebra having the WLP

In this section, we consider h-vectors of a graded Artinian level algebra with the WLP and we prove that some of graded Artinian O-sequences are not level using generic initial ideals. Moreover, we assume that $R = k[x_1, \ldots, x_n]$ is an n-variable polynomial ring over a field k with characteristic 0.

For positive integers h and i, h can be written uniquely in the form

$$h = h_{(i)} := \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j}$$

where $m_i > m_{i-1} > \cdots > m_j \geq j \geq 1$. This expansion for h is called the *i-binomial expansion* of h. For such h and i, we define

$$(h_{(i)})^- := \binom{m_i - 1}{i} + \binom{m_{i-1} - 1}{i-1} + \cdots + \binom{m_j - 1}{j},$$
$$(h_{(i)})^+_+ := \binom{m_i + 1}{i+1} + \binom{m_{i-1} + 1}{i} + \cdots + \binom{m_j + 1}{j+1}.$$

Let $\mathbf{H} = \{h_i\}_{i \geq 0}$ be the Hilbert function of a graded ring A. For simplicity in the notation we usually rewrite $((h_i)_{(i)})^-$ and $((h_i)_{(i)})^+_+$ as $(h_i)^-$ and $(h_i)^+_+$, respectively. Recall that we sometimes use another simpler notation $h^{\langle i \rangle}$ for $(h_i)^+_+$ and define $0^{\langle i \rangle} = 0$.

A well-known result of Macaulay is the following theorem.

Theorem 3.1 (*Macaulay*). *Let $\mathbf{H} = \{h_i\}_{i \geq 0}$ be a sequence of non-negative integers such that $h_0 = 1$, $h_1 = n$, and $h_i = 0$ for every $i > e$. Then $\mathbf{H}$ is the h-vector of some standard graded Artinian algebra if and only if, for every $1 \leq d \leq e - 1$,*

$$h_{d+1} \leq (h_d)^+_+ = h_d^{\langle d \rangle}.$$

We use a generic initial ideal with respect to the reverse lexicographic order to obtain the results in Section 3. Note that, by Green's hyperplane restriction theorem (see [12,19]), we have

$$\mathbf{H}(R/(J + x_n), d) \leq (\mathbf{H}(R/J, d))^-, \tag{3.1}$$

where J is either a generic initial ideal with respect to the reverse lexicographic order, or a lex-segment ideal. The equality holds when J is a lex-segment ideal of R (see [12]).

The following lemma will be used often in this section.

Lemma 3.2. *Let $A = R/I$ be an Artinian k-algebra and let L be a general linear form.*

(a) *If*

$$\dim_k(0 : L)_d > (n - 1) \dim_k(0 : L)_{d+1}$$

for some $d > 0$, then A has a socle element in degree d.

(b) *Let $h(A) = (h_0, h_1, \ldots, h_s)$ be the h-vector of A. Then, we have*

$$h_d - h_{d+1} \leq \dim_k(0 : L)_d \leq h_d - h_{d+1} + (h_{d+1})^-. \tag{3.2}$$

In particular, $\dim_k(0 : L)_d = h_d - h_{d+1}$ *if and only if* $d \geq r_1(A)$.

Proof. (a) Consider a map $\varphi : (0 : L)_d \to \bigoplus^{n-1}(0 : L)_{d+1}$, defined by $\varphi(F) = (x_1 F, \ldots, x_{n-1} F)$. Since L is a general linear form, we may assume that the kernel of this map is exactly $\operatorname{soc}(A)_d$. Since $\dim_k(0 : L)_d > (n - 1) \dim_k(0 : L)_{d+1}$, the map φ is not injective and we obtain the desired result.

(b) Consider the following exact sequence

$$0 \to (0 : L)_d \to A_d \stackrel{\times L}{\to} A_{d+1} \to (A/LA)_{d+1} \to 0.$$

Then we have

$$\dim_k(0 : L)_d = h_d - h_{d+1} + \dim_k[A/(L)A]_{d+1}, \tag{3.3}$$

and thus $h_d - h_{d+1} \le \dim_k(0 : L)_d$. The right-hand side of the inequality (3.2) follows from Green's hyperplane restriction theorem, i.e., $\dim_k[A/(L)A]_{d+1} \le (h_{d+1})^-$.

Moreover, $\dim_k(0 : L)_d = h_d - h_{d+1}$ if and only if $\dim_k[A/(L)A]_{d+1} = 0$, and it is equivalent to $d \ge r_1(A)$ by the definition of $r_1(A)$. □

Remark 3.3. Let $\mathbf{H} = (h_0, h_1, \ldots, h_s)$ be the h-vector of a graded Artinian-level algebra $A = R/I$ and L is a general linear form of A. In general, it is not easy to find the reduction number $r_1(A)$ based on its h-vector. However, if $h_{d+1} \le d + 1$ then $(h_{d+1})^- = 0$, and thus $\dim_k(0 : L)_d = h_d - h_{d+1}$. Hence $d \ge r_1(A)$ by Lemma 3.2. In other words,

$$r_1(A) \le \min\{k \mid h_{k+1} \le k + 1\}.$$

Proposition 3.4. *Let $R = k[x_1, \ldots, x_n]$ and let $\mathbf{H} = (h_0, h_1, \ldots, h_s)$ be the h-vector of a graded Artinian-level algebra $A = R/I$ with socle degree s. Suppose that $h_{d-1} > h_d$ for some $d \ge r_1(A)$. Then*

(a) $h_{d-1} > h_d > \cdots > h_{s-1} > h_s > 0$, *and*
(b) $h_{t-1} - h_t \le (n - 1)(h_t - h_{t+1})$ *for all* $d \le t \le s$.

Proof. (a) First of all, note that, by Lemma 3.2(b), $h_t - h_{t+1} = \dim_k(0 : L)_t$ for every $t \ge r_1(A)$. Hence we have that

$$h_{d-1} > h_d \ge h_{d+1} \ge \cdots \ge h_s.$$

Now assume that there is $t \ge d$ such that $h_{t-1} > h_t = h_{t+1}$. Since $t \ge r_1(A)$, we know that, by Lemma 3.2(b),

$$\dim_k(0 : L)_{t-1} \ge h_{t-1} - h_t > 0 \quad \text{and} \quad \dim_k(0 : L)_t = 0.$$

Hence there is a socle element of A in degree $t - 1$, which is a contradiction as A is level. This means that $h_t > h_{t+1}$ for every $t \ge d - 1$.

(b) Since A is a level algebra and $\dim_k(0 : L)_t = h_{t-1} - h_t$, the result follows directly from Lemma 3.2(a). □

Remark 3.5. Let I be a homogeneous ideal of $R = k[x_1, \ldots, x_n]$ such that R/I has the WLP with a Lefschetz element L and let $\mathbf{H}(R/I, d - 1) > \mathbf{H}(R/I, d)$ for some d. Now we consider the following exact sequence

$$(R/I)_{d-1} \stackrel{\times L}{\to} (R/I)_d \to (R/(I + (L)))_d \to 0. \tag{3.4}$$

Since R/I has the WLP and $\mathbf{H}(R/I, d - 1) > \mathbf{H}(R/I, d)$, the above multiplication map cannot be injective, but surjective. In other words, $(R/(I + (L)))_d = 0$. This implies that $d > r_1(R/I)$ by Lemma 2.7.

The following theorem shows a useful condition to be a level O-sequence with the WLP.

Theorem 3.6. *Let $R = k[x_1, \ldots, x_n]$, $n \ge 3$ and let $\mathbf{H} = (h_0, h_1, \ldots, h_s)$ be the Hilbert function of a graded Artinian-level algebra $A = R/I$ having the WLP. Then,*

(a) *the Hilbert function* $\mathbf{H}$ *is a strictly unimodal O-sequence*

$$h_0 < h_1 < \cdots < h_{r_1(A)} = \cdots = h_\theta > \cdots > h_{s-1} > h_s$$

such that the positive part of the first difference $\Delta\mathbf{H}$ *is an O-sequence, and*

(b) $h_{d-1} - h_d \le (n - 1)(h_d - h_{d+1})$ *for* $s \ge d > \theta$.

Proof. (a) First, note that, by Proposition 3.5 in [22], $\mathbf{H}$ is a unimodal O-sequence such that the positive part of the first difference is an O-sequence. Hence it suffices to show that $\mathbf{H}$ is strictly unimodal.

If $d \le r_1(A)$, then $\mathbf{H}_{R/(I+L)}(d) \neq 0$ by the definition of $r_1(A)$, and so the multiplication map $\times L$ is not surjective in Eq. (3.4). In other words, the multiplication map $\times L$ is injective since A has the WLP. Thus, we have a short exact sequence as follows

$$0 \to (R/I)_{d-1} \xrightarrow{\times L} (R/I)_d \to (R/(I+(L)))_d \to 0.$$

Hence we obtain that

$$\begin{aligned}\mathbf{H}_A(d) &= \mathbf{H}_A(d-1) + \mathbf{H}_{R/(I+L)}(d)\\ &> \mathbf{H}_A(d-1) \quad (\because \mathbf{H}_{R/(I+L)}(d) \neq 0),\end{aligned}$$

and so the Hilbert function of A is strictly increasing up to $r_1(A)$.

Moreover, by Proposition 3.4(a), $\mathbf{H}$ is strictly decreasing in degrees $d \ge \theta$, where

$$\theta := \min\{t \mid h_t > h_{t+1}\}.$$

(b) The result follows directly from Proposition 3.4(b). □

Remark 3.7. Theorem 3.6 gives us a necessary condition when a numerical sequence becomes a level O-sequence with the WLP. In general, this condition is not sufficient. One can find many non-level sequences satisfying the inequality of Theorem 3.6 in [15].

In [15], they gave some 'non-level sequences' using the homological method, which is the combinatorial notion of the cancellation of shifts in the minimal free resolutions of the lex-segment ideals associated with the given homogeneous ideals.

In this section, we use generic initial ideals, instead of the lex-segment ideals. Firstly, note that, by the Bigatti-Hulett-Pardue theorem, the worst minimal free resolution of a homogeneous ideal I depends on only the Hilbert function of I. Unfortunately, we cannot apply their theorem to obtain the minimal free resolutions of the generic initial ideals. However, we can find Betti numbers $\beta_{i,d+i}(\mathrm{Gin}(I))$ for $d > r_1(A)$ and $i \ge 0$, which depend on only the given Hilbert function (see Corollary 3.10).

For the remainder of this section, we need the following useful results.

Lemma 3.8. *Let J be a stable ideal of R and let $T_1, \ldots, T_r$ be the monomials which form a k-basis for $((J : x_n)/J)_{d-1}$, then*

$$\{x_n T_1, \ldots, x_n T_r\} = \{T \in \mathcal{G}(J)_d \mid x_n \text{ divides } T\}.$$

In particular,

$$\dim_k ((J : x_n)/J)_{d-1} = |\{T \in \mathcal{G}(J)_d \mid x_n \text{ divides } T\}|.$$

Proof. For every $T = x_n T' \in \mathcal{G}(J)_d$, we have that $x_n T' \in J_d \subset J$, i.e., $T' \in (J : x_n)_{d-1}$, and thus $\overline{T}' \in ((J : x_n)/J)_{d-1} = \langle \overline{T}_1, \ldots, \overline{T}_r \rangle$. However, since T' and T_i are all monomials of $(J : x_n)_{d-1}$ in degree $d-1$, we have that $T' = T_i$ for some i, and hence $T = x_n T' \in \{x_n T_1, \ldots, x_n T_r\}$.

Conversely, note that $T_i \notin J_{d-1}$ and $x_n T_i \in J_d$ for every $i = 1, \ldots, r$. If $x_n T_i \notin \mathcal{G}(J)_d$ for some $i = 1, \ldots, r$, then $x_n T_i \in R_1 J_{d-1}$. Since $T_i \notin J_{d-1}$, we see that

$$x_n T_i = x_j U$$

for some monomial $U \in J_{d-1}$ and $j < n$. Hence, we have that

$$x_n \mid U.$$

Moreover, since J is a stable monomial ideal, for every $\ell < n$,

$$\frac{x_\ell}{x_n} U \in J_{d-1}.$$

In particular, we have

$$T_i = \frac{x_j}{x_n} U \in J_{d-1},$$

which is a contradiction. Therefore, $x_n T_i \in \mathcal{G}(J)_d$, for every $i = 1, \ldots, r$, as we desired. □

Using the previous lemma, we obtain the following proposition, where we know the difference between h_d and h_{d+1} when $d > r_1(A)$.

Proposition 3.9. *Let $A = R/I$ be a graded Artinian algebra with Hilbert function $\mathbf{H} = (h_0, h_1, \ldots, h_s)$ and let $J = \mathrm{Gin}(I)$. If $d \geq r_1(A)$ then,*

$$|\{T \in \mathcal{G}(J)_{d+1} \mid x_n \textit{ divides } T\}| = h_d - h_{d+1}.$$

Moreover, if $d > r_1(A)$,

$$|\mathcal{G}(J)_{d+1}| = |\{T \in \mathcal{G}(J)_{d+1} \mid x_n \textit{ divides } T\}| = h_d - h_{d+1}.$$

Proof. Consider the following exact sequence:

$$0 \to ((J : x_n)/J)_d \to (R/J)_d \xrightarrow{\times x_n} (R/J)_{d+1} \to (R/J + (x_n))_{d+1} \to 0.$$

Note that $\mathbf{H}(R/I, t) = \mathbf{H}(R/J, t)$ for every $t \geq 0$. Therefore,

$$\begin{aligned} &\dim_k ((J : x_n)/J)_d + \dim_k (R/J)_{d+1} = \dim_k (R/J)_d + \dim_k (R/J + (x_n))_{d+1}, \\ &\Leftrightarrow \dim_k ((J : x_n)/J)_d + h_{d+1} = h_d + \dim_k (R/J + (x_n))_{d+1}. \end{aligned} \tag{3.5}$$

Moreover, by Theorems 2.1, 2.6, and Lemma 2.7, we have

$$\begin{aligned} r_1(R/I) &= r_1(R/J) \\ &= \min\{\ell \mid \mathbf{H}(R/J + (x_n), \ell + 1) = 0\}, \end{aligned}$$

which means $\mathbf{H}(R/J + (x_n), d + 1) = 0$ for every $d \geq r_1(R/I)$. Hence, from Eq. (3.5), we obtain

$$\dim_k ((J : x_n)/J)_d = |\{T \in \mathcal{G}(J)_{d+1} \mid x_n \text{ divides } T\}| = h_d - h_{d+1}. \tag{3.6}$$

Now suppose that $d > r_1(A)$. Then it is obvious that

$$\{T \in \mathcal{G}(J)_{d+1} \mid x_n \text{ divides } T\} \subseteq \mathcal{G}(J)_{d+1}. \tag{3.7}$$

Conversely, note that $x_{n-1}^d \in J$ from the first equality of Lemma 2.7. Since J is a strongly stable ideal, J_d has to contain all monomials U of degree d such that

$$\mathrm{supp}(U) := \{i \mid x_i \text{ divides } U\} \subseteq \{1, \ldots, n-1\}.$$

This implies $\overline{\mathbf{m}}_d \subseteq J_d$ where $\overline{\mathbf{m}} = (x_1, \ldots, x_{n-1})^d$. Thus we have

$$R_1 \overline{\mathbf{m}}_d \subseteq J_{d+1}.$$

Therefore, for every $T \in \mathcal{G}(J)_{d+1}$, we have $x_n \mid T$, and so

$$\mathcal{G}(J)_{d+1} \subseteq \{T \in \mathcal{G}(J)_{d+1} \mid x_n \text{ divides } T\}. \tag{3.8}$$

It follows from Eqs. (3.7) and (3.8) that

$$\mathcal{G}(J)_{d+1} = \{T \in \mathcal{G}(J)_{d+1} \mid x_n \text{ divides } T\}, \tag{3.9}$$

and hence

$$|\mathcal{G}(J)_{d+1}| = \dim_k ((J : x_n)/J)_d = h_d - h_{d+1},$$

as we hoped. □

Corollary 3.10. *Let $A = R/I$ be a graded Artinian algebra with Hilbert function $\mathbf{H} = (h_0, h_1, \ldots, h_s)$. If $d > r_1(A)$ then, for all $i \geq 0$,*

$$\beta_{i,\, i+(d+1)}(\mathrm{Gin}(I)) = (h_d - h_{d+1}) \binom{n-1}{i}.$$

Proof. By Proposition 3.9,

$$|\mathcal{G}(\mathrm{Gin}(I))_{d+1}| = |\{T \in \mathcal{G}(\mathrm{Gin}(I))_{d+1} \mid x_n \text{ divides } T\}| = h_d - h_{d+1}$$

for every $d > r_1(A)$, and thus the result follows from Theorem 2.4. □

Recall that a homogeneous ideal I is *m-regular* if, in the minimal free resolution of I, for all $p \geq 0$, every pth syzygy has degree $\leq m + p$. The regularity of I, $\mathrm{reg}(I)$, is the smallest such m.

In [2,19], it was proved that the regularity of $\mathrm{Gin}(I)$ is the largest degree of a generator of $\mathrm{Gin}(I)$. Moreover, Bayer and Stillman [2] showed the regularity of I to be equal to the regularity of $\mathrm{Gin}(I)$.

Theorem 3.11 (*[2,19]*). *For any homogeneous ideal I, using the reverse lexicographic order,*

$$\mathrm{reg}(I) = \mathrm{reg}(\mathrm{Gin}(I)).$$

Theorem 3.12 (*Crystallization Principle, [1,19]*). *Let I be a homogeneous ideal generated in degrees $\leq d$. Assume that there is a monomial order τ such that $\mathrm{Gin}_\tau(I)$ has no generator in degree $d + 1$. Then $\mathrm{Gin}_\tau(I)$ is generated in degrees $\leq d$ and I is d-regular.*

Lemma 3.13. *Let $R = k[x_1, x_2, x_3]$ and let $A = R/I$ be an Artinian algebra and let $\mathbf{H} = (h_0, h_1, \ldots, h_s)$ be the Hilbert function of $A = R/I$. Suppose that, for $t > 0$,*

(a) $\mathrm{soc}(A)_{t-2} = 0$,
(b) $\beta_{1,\, t+1}(\mathrm{Gin}(I)) = \beta_{2,\, t+1}(\mathrm{Gin}(I))$.

Then $(I_{\leq t})$ is t-regular and

$$h_{t-1} - h_t \leq \dim_k \mathrm{soc}(A)_{t-1} \leq h_{t-1} - h_t + (h_t)^-. \tag{3.10}$$

In particular, if $t > r_1(A)$ then

$$\dim_k(\mathrm{soc}(A)_{t-1}) = h_{t-1} - h_t.$$

Proof. Let $\bar{I} = (I_{\leq t})$. Note that $\beta_{i,t+1}(\mathrm{Gin}(I)) = \beta_{i,t+1}(\mathrm{Gin}(\bar{I}))$ for $i = 1, 2$ and $\beta_{0,t+1}(\bar{I}) = 0$. Furthermore, since I and $\bar{I}$ agree in degree $\leq t$ and $\mathrm{soc}(A)_{t-2} = 0$, we see that $\beta_{2,t+1}(I) = \beta_{2,t+1}(\bar{I}) = 0$.

Applying Lemma 2.9(b) the ideal $\bar{I}$, we have that

$$\begin{aligned}
&\beta_{1,t+1}(\mathrm{Gin}(\bar{I})) - \beta_{1,t+1}(\bar{I}) = (\beta_{0,t+1}(\mathrm{Gin}(\bar{I})) - \beta_{0,t+1}(\bar{I})) + (\beta_{2,t+1}(\mathrm{Gin}(\bar{I})) - \beta_{2,t+1}(\bar{I}))\\
&\Rightarrow -\beta_{1,t+1}(\bar{I}) = (\beta_{0,t+1}(\mathrm{Gin}(\bar{I})) - \beta_{0,t+1}(\bar{I})) - \beta_{2,t+1}(\bar{I}) \quad (\because \beta_{1,\,t+1}(\mathrm{Gin}(\bar{I})) = \beta_{2,\,t+1}(\mathrm{Gin}(\bar{I})))\\
&\Rightarrow -\beta_{1,t+1}(\bar{I}) = \beta_{0,t+1}(\mathrm{Gin}(\bar{I})) \quad (\because \beta_{0,t+1}(\bar{I}) = \beta_{2,t+1}(\bar{I}) = 0)\\
&\Rightarrow \beta_{0,\,t+1}(\mathrm{Gin}(\bar{I})) = 0.
\end{aligned}$$

Thus, by Theorem 3.12, the ideal $\bar{I} = (I_{\leq t})$ is t-regular.

Let $\bar{A} = R/\bar{I}$. For a general linear form L, consider the following exact sequence

$$0 \to \left(0 :_{\bar{A}} L\right)_{t-1} \to (R/\bar{I})_{t-1} \xrightarrow{\times L} (R/\bar{I})_t \to (R/\bar{I} + (L))_t \to 0. \tag{3.11}$$

After we replace $\bar{I}$ and $\bar{A}$ by $\mathrm{Gin}(\bar{I})$ and $\tilde{A} = R/\mathrm{Gin}(\bar{I})$, respectively, we can rewrite Eq. (3.11) as

$$0 \to \left(0 :_{\tilde{A}} x_3\right)_{t-1} \to (R/\mathrm{Gin}(\bar{I}))_{t-1} \xrightarrow{\times x_3} (R/\mathrm{Gin}(\bar{I}))_t \to (R/\mathrm{Gin}(\bar{I}) + (x_3))_t \to 0. \tag{3.12}$$

Then, by Theorem 2.1, we know that

$$\begin{aligned}\dim_k \left(0:_{\bar{A}} x_3\right)_{t-1} &= \dim_k((\mathrm{Gin}(\bar{I}) : x_3)/\mathrm{Gin}(\bar{I}))_{t-1} \\ &= h_{t-1} - h_t + \dim_k(R/\mathrm{Gin}(\bar{I}) + (x_3))_t \\ &= h_{t-1} - h_t + \dim_k(R/\bar{I} + (L))_t \\ &= \dim_k(0:_{\bar{A}} L)_{t-1}.\end{aligned}$$

On the other hand, by Lemma 3.8,

$$\begin{aligned}\dim_k((\mathrm{Gin}(\bar{I}) : x_3)/\mathrm{Gin}(\bar{I}))_{t-1} &= \left|\{T \in \mathcal{G}(\mathrm{Gin}(\bar{I}))_t \mid x_3 \text{ divides } T\}\right| \\ &= \beta_{2,\,t+2}(\mathrm{Gin}(\bar{I})),\end{aligned}$$

and by Lemma 3.2(b)

$$h_{t-1} - h_t \le \dim_k((0:_{\bar{A}} L)_{t-1}) \le h_{t-1} - h_t + (h_t)^-. \tag{3.13}$$

Note that, by Theorem 3.12, $\beta_{1,t+2}(\mathrm{Gin}(\bar{I})) = 0$ since $\bar{I} = (I_{\le t})$ is t-regular. Moreover, since I and $\bar{I}$ agree in degree $\le t$, we have that $\beta_{2,t+2}(I) = \beta_{2,t+2}(\bar{I})$. Hence, by Theorem 2.2,

$$\begin{aligned}\dim_k \mathrm{soc}(A)_{t-1} &= \beta_{2,\,t+2}(I) \\ &= \beta_{2,\,t+2}(\bar{I}) \\ &= \beta_{2,\,t+2}(\mathrm{Gin}(\bar{I})) \quad (\because \beta_{1,t+2}(\mathrm{Gin}(\bar{I})) = 0) \\ &= \dim_k(0:_{\bar{A}} L)_{t-1}.\end{aligned} \tag{3.14}$$

Hence it follows from Eqs. (3.13) and (3.14), that we obtain the inequality (3.10). Moreover, by Lemma 3.2(b), we have

$$\dim_k(\mathrm{soc}(A)_{t-1}) = h_{t-1} - h_t \quad \text{for } t > r_1(A),$$

as we anticipated. □

Theorem 3.14. *Let $A = R/I$ be an Artinian algebra of codimension 3 with socle degree s. If*

$$\beta_{1,\,d+2}(\mathrm{Gin}(I)) = \beta_{2,\,d+2}(\mathrm{Gin}(I)) > 0 \tag{3.15}$$

for some $d < s$, then A is not level.

Proof. Assume A is level. Then $\beta_{2,d+2}(I) = \mathrm{soc}(A)_{d-1} = 0$, and hence, by Lemma 3.13, $\bar{I} = (I_{\le d+1})$ is $(d+1)$-regular.

Let $\bar{A} = R/\bar{I}$. Note that $\mathrm{soc}(A)_d = \mathrm{soc}(\bar{A})_d$ since A and $\bar{A}$ agree in degree $\le d+1$, i.e.

$$\dim_k \mathrm{soc}(A)_d = \beta_{2,d+3}(I) = \beta_{2,d+3}(\bar{I}) = \dim_k \mathrm{soc}(\bar{A})_d.$$

For a general linear form L, by Lemmas 3.2(a) and 3.8, we have that

$$\begin{aligned}0 &< \beta_{2,d+2}(\mathrm{Gin}(I)) \quad (\because \text{ by assumption}) \\ &= \sum_{T \in \mathcal{G}(\mathrm{Gin}(I))_d} \binom{m(T)-1}{2} \\ &= \dim_k\,[(\mathrm{Gin}(I) : x_3)/\mathrm{Gin}(I)]_{d-1} \quad (\because \text{by Lemma 3.8}) \\ &= \dim_k\,[(I : L)/I]_{d-1} \\ &\le 2\dim_k\,[(I : L)/I]_d \quad (\because \text{by Lemma 3.2(a) and } \mathrm{soc}(A)_{d-1} = 0).\end{aligned}$$

Note that, in the similar way, we have $\beta_{2,\,d+3}(\mathrm{Gin}(I)) = \dim_k\,[(I : L)/I]_d$. Hence

$$\beta_{2,\,d+3}(\mathrm{Gin}(I)) > 0.$$

Since $\bar{I} = (I_{\le d+1})$ is $(d+1)$-regular and $\operatorname{reg}(\bar{I}) = \operatorname{reg}(\operatorname{Gin}(\bar{I}))$ by Theorem 3.11, we have that

$$\beta_{0,d+3}(\operatorname{Gin}(\bar{I})) = \beta_{1,d+3}(\operatorname{Gin}(\bar{I})) = 0,$$
$$\beta_{0,d+3}(\bar{I}) = \beta_{1,d+3}(\bar{I}) = 0.$$

Thus, by Lemma 2.9(b),

$$\beta_{2,d+3}(\bar{I}) = \beta_{2,d+3}(\operatorname{Gin}(\bar{I})) > 0,$$

whereby it follows that as $R/\bar{I}$ has a socle element in degree d, so does R/I. This is a contradiction, and thus we complete the proof. □

Remark 3.15. Now we shall show that there is a level O-sequence satisfying Theorem 3.6(a) and (b), but it cannot be the Hilbert function of an Artinian algebra with the WLP.

Consider an h-vector $\mathbf{H} = (1, 3, 6, 10, 8, 7)$, which was given in [15]. Furthermore, it has been shown that there is a level algebra of codimension 3 with Hilbert function $\mathbf{H}$ in [15]. They also raised a question if there exists a codimension 3 graded level algebra having the WLP with Hilbert function $\mathbf{H}$. Note that this is a codimension 3 level O-sequence which satisfies the condition in Theorem 3.6.

Now suppose that there is an Artinian-level algebra $A = R/I$ having the WLP with Hilbert function $\mathbf{H}$. In [15], they gave several results about level or non-level sequences of graded Artinian algebras. One of the tools they used was the fact that Betti numbers of a homogeneous ideal I can be obtained by cancellation of the Betti numbers of I^{lex}. However, in this case, it is not available if $\mathbf{H}$ can be the Hilbert function of an Artinian-level algebra having the WLP based on the Betti numbers of I^{lex}.

In fact, the Betti diagram of R/I^{lex} is

Total:	1	–	–	–
0:	1	–	–	–
1:	0	0	0	0
2:	0	0	0	0
3:	0	7	9	3
4:	0	2	4	2
			...	

and thus we cannot decide if there is a socle element of R/I in degree 3.

Note that, by Theorem 3.6, $r_1(A) = 3$ since A has the WLP. Hence, by Corollary 3.10,

$$\beta_{2,6}(\operatorname{Gin}(I)) = (h_4 - h_5)\binom{2}{2} = 2 \cdot 1 = 2, \quad \text{and}$$
$$\beta_{1,6}(\operatorname{Gin}(I)) = (h_5 - h_6)\binom{2}{1} = 1 \cdot 2 = 2.$$

Therefore, by Theorem 3.14, there is a socle element in A in degree 3, which is a contradiction. In other words, any Artinian-level algebra A with Hilbert function $\mathbf{H}$ does not have the WLP.

Remark 3.16. In general, Theorem 3.14 is not true if Eq. (3.15) holds in the socle degree. For example, we consider a Gorenstein sequence

d	0	1	2	3	4
h_d	1	3	6	3	1

By Remark 3.3, $r_1(A) \le 2$. Hence

$$\beta_{1,6}(\operatorname{Gin}(I)) = (h_4 - h_5)\binom{2}{1} = 1 \cdot 2 = 2, \quad \text{and} \quad \beta_{2,6}(\operatorname{Gin}(I)) = (h_3 - h_4)\binom{2}{2} = 2 \cdot 1 = 2.$$

Note that this satisfies the condition of Theorem 3.14 in the socle degree, but it is a level sequence.

Remark 3.17. Let $A = R/I$ be an Artinian algebra and let $\mathbf{H} = (h_0, h_1, \ldots, h_s)$ be the Hilbert function of $A = R/I$. Then an ideal $(I_{\le d+1})$ is $(d+1)$-regular, if the Hilbert function $\mathbf{H}$ of A has the maximal growth in degree $d > 0$, i.e. $h_{d+1} = h_d^{\langle d \rangle}$. In particular, if $h_d = h_{d+1} = \ell \le d$, then we know that $(I_{\le d+1})$ is $(d+1)$-regular. Recently, this result was improved in [1], that is, $(I_{\le d+1})$ is $(d+1)$-regular if $h_d = h_{d+1}$ and $r_1(A) < d$.

Note that, by Lemma 3.2, the k-vector space dimension of $(0 : L)_d$ in degree $d \ge r_1(A)$ is $h_d - h_{d+1}$. By Proposition 3.4, we have a bound for the growth of the Hilbert function of $(0 : L)$ in degree $d \ge r_1(A)$ if an Artinian algebra A has no socle elements in degree d. Theorem 3.19 shows that a similar result still holds on the maximal growth of the Hilbert function of $(0 : L)$ in codimension three case.

Lemma 3.18. *Let $R = k[x_1, \ldots, x_n]$ and let $A = R/I$ be an Artinian algebra with an h-vector $\mathbf{H} = (1, 3, h_2, \ldots, h_s)$. If $h_{d-1} - h_d = (n-1)(h_d - h_{d+1})$ for $r_1(A) < d < s$, then*

$$\beta_{(n-1),(n-1)+d}(\mathrm{Gin}(I)) = \beta_{(n-2),(n-1)+d}(\mathrm{Gin}(I)).$$

Proof. Let $J = \mathrm{Gin}(I)$. By Proposition 3.9, we have that

$$\begin{aligned}\beta_{(n-1),(n-1)+d}(J) &= \sum_{T \in \mathcal{G}(J)_d} \binom{m(T)-1}{n-1} \\ &= h_{d-1} - h_d.\end{aligned}$$

Moreover, by Corollary 3.10,

$$\begin{aligned}\beta_{(n-2),(n-1)+d}(J) &= \beta_{(n-2),(n-2)+(d+1)}(J) \\ &= (h_d - h_{d+1})\binom{n-1}{n-2} \\ &= (n-1)(h_d - h_{d+1}) \\ &= h_{d-1} - h_d \quad (\because \text{ by given condition}) \\ &= \beta_{(n-1),(n-1)+d}(J),\end{aligned}$$

as we desired. □

Theorem 3.19. *Let $R = k[x_1, x_2, x_3]$ and let $A = R/I$ be an Artinian algebra with an h-vector $\mathbf{H} = (1, 3, h_2, \ldots, h_s)$. If $\mathrm{soc}(A)_{d-1} = 0$ and the Hilbert function of $(0 : L)$ has a maximal growth in degree d for $r_1(A) < d < s$, i.e., $h_{d-1} - h_d = 2(h_d - h_{d+1})$, for a general linear form L, then*

(a) $(I_{\le d+1})$ *is* $(d+1)$*-regular, and*
(b) $\dim_k \mathrm{soc}(A)_d = h_d - h_{d+1}$.

Proof. By Lemma 3.18, we have

$$\beta_{1,d+2}(\mathrm{Gin}(I)) = \beta_{2,d+2}(\mathrm{Gin}(I)), \tag{3.16}$$

for $r_1(A) < d < s$, and the result immediately follows from Lemma 3.13. □

Corollary 3.20. *Let $R = k[x_1, x_2, x_3]$ and let $A = R/I$ be an Artinian algebra with an h-vector $\mathbf{H} = (1, 3, h_2, \ldots, h_s)$. If $h_{d-1} - h_d = 2(h_d - h_{d+1}) > 0$ for $r_1(A) < d < s$, then A is not level.*

Proof. By Lemma 3.18, we have

$$\beta_{2,d+2}(\mathrm{Gin}(I)) = \beta_{1,d+2}(\mathrm{Gin}(I)) > 0,$$

and hence, by Theorem 3.14, A cannot be level, as we wanted. □

Remark 3.21. Remark 3.16 shows Corollary 3.20 is not true if $d = s$. However, we know $h_{s-1} \le 3h_s$ by Theorem 3.6.

Example 3.22. Let $A = R/I$ be a codimension 3 Artinian algebra and let $r_1(A) < d < s$. If A has the Hilbert function

d	$\cdots$	$d-1$	d	$d+1$	$\cdots$
h_d	$\cdots$	$a+3k$	$a+k$	a	$\cdots$

such that $a > 0$ and $k > 0$, then by Corollary 3.20 A cannot be level since

$$h_{d-1} - h_d = 2k = 2(h_d - h_{d+1}) \Leftrightarrow \beta_{2,d+2}(\text{Gin}(I)) = \beta_{1,d+2}(\text{Gin}(I)) > 0.$$

For the codimension 3 case, we have the following theorem, which follows from Theorems 3.6 and 3.19 and Corollary 3.20, and so we shall omit the proof here.

Theorem 3.23. *Let $A = R/I$ be a graded Artinian-level algebra of codimension 3 with the WLP and let $\mathbf{H} = (h_0, h_1, \ldots, h_s)$ be the Hilbert function of A. Then,*

(a) *the Hilbert function $\mathbf{H}$ is a strictly unimodal O-sequence*

$$h_0 < h_1 < \cdots < h_{r_1(A)} = \cdots = h_\theta > \cdots > h_{s-1} > h_s$$

such that the positive part of the first difference $\Delta\mathbf{H}$ is an O-sequence, and

(b) $h_{d-1} - h_d < 2(h_d - h_{d+1})$ *for* $s > d > \theta$.

(c) $h_{s-1} \le 3h_s$.

One may ask if the converse of Theorem 3.23 holds. Before the end of this section, we give the following Question.

Question 3.24. Suppose that $\mathbf{H} = (1, 3, h_2, \ldots, h_s)$ is the h-vector of a *level* algebra $A = R/I$ where $R = k[x_1, x_2, x_3]$. Is there a level algebra A with the WLP such that $\mathbf{H}$ is the Hilbert function of A if $\mathbf{H} = (1, 3, h_2, \ldots, h_s)$ satisfies the conditions (a), (b), and (c) in Theorem 3.23?

4. The lex-segment ideals and graded non-level artinian algebras

In this section, we shall find an answer to Question 1.1.

Theorem 4.1. *Let $R = k[x_1, x_2, x_3]$ and let $\mathbf{H} = (h_0, h_1, \ldots, h_s)$ be the h-vector of a graded Artinian algebra $A = R/I$ with socle degree s. If*

$$h_{d-1} > h_d \quad \text{and} \quad h_d = h_{d+1} \le 2d + 3,$$

then $\mathbf{H}$ *is* not *level.*

Before we prove this theorem, we consider the following lemmas and theorems.

Lemma 4.2. *Let J be a lex-segment ideal in $R = k[x_1, x_2, x_3]$ such that*

$$\mathbf{H}(R/J, i) = h_i$$

for every $i \ge 0$. Then

$$\dim_k ((J : x_3)/J)_i = h_i - h_{i+1} + (h_{i+1})^- \tag{4.1}$$

for such an i.

Proof. First of all, we consider the following exact sequence:

$$0 \to ((J : x_3)/J)_i \to (R/J)_i \xrightarrow{\times x_3} (R/J)_{i+1} \to R/(J + (x_3))_{i+1} \to 0. \tag{4.2}$$

Using Eq. (3.1) and the exact sequence (4.2), we see that

$$\dim_k ((J : x_3)/J)_i = h_i - h_{i+1} + (h_{i+1})^- \tag{4.3}$$

for every $i \ge 0$ as we desired. $\square$

Since the following lemma is obtained easily from the property of the lex-segment ideal, we shall omit the proof here.

Lemma 4.3. *Let I be the lex-segment ideal in $R = k[x_1, x_2, x_3]$ with Hilbert function $\mathbf{H} = (h_0, h_1, \dots, h_s)$ where $h_d = d + i$ and $1 \le i \le \frac{d^2+d}{2}$. Then the last monomial of I_d is*

$$\begin{aligned}
&x_1x_2^{i-1}x_3^{d-i}, \quad \textit{for } 1 \le i \le d,\\
&x_1^2x_2^{i-(d+1)}x_3^{(2d-1)-i}, \quad \textit{for } d+1 \le i \le 2d-1,\\
&\vdots\\
&x_1^{d-1}x_2^{i-\frac{d^2+d-4}{2}}x_3^{\frac{d^2+d-2}{2}-i}, \quad \textit{for } \frac{d^2+d-4}{2} \le i \le \frac{d^2+d-2}{2},\\
&x_1^d, \quad \textit{for } i = \frac{d^2+d}{2}.
\end{aligned}$$

Theorem 4.4. *Let $R = k[x_1, x_2, x_3]$ and let $\mathbf{H} = (h_0, h_1, \dots, h_s)$ be the h-vector of an Artinian algebra with socle degree s and*

$$h_d = h_{d+1} = d + i, \qquad h_{d-1} > h_d, \quad \text{and} \quad j := h_{d-1} - h_d$$

for $i = 1, 2, \dots, \frac{d^2+d}{2}$. Then,

$$\beta_{1,d+2} = \begin{cases} 2k-1, & \textit{for } (k-1)d - \dfrac{k(k-3)}{2} \le i \le (k-1)d - \dfrac{k(k-3)}{2} + (k-1),\\ 2k, & \textit{for } (k-1)d - \dfrac{k(k-3)}{2} + k \le i \le kd - \dfrac{(k-1)k}{2}. \end{cases}$$

$$\beta_{2,d+2} = j + \ell, \quad \textit{for } (\ell-1)d - \frac{(\ell-2)(\ell-1)}{2} < i \le \ell d - \frac{(\ell-1)\ell}{2}.$$

Proof. Since $h_d = d + i$, the monomials not in I_d are the last $d + i$ monomials of R_d. By Lemma 4.3, the last monomial of $R_1 I_d$ is

$$\begin{aligned}
&x_1x_2^{i-1}x_3^{d-i+1}, \quad \text{for } i = 1, \dots, d,\\
&x_1^2x_2^{i-(d+1)}x_3^{2d-i}, \quad \text{for } i = d+1, \dots, 2d-1,\\
&\vdots\\
&x_1^{d-1}x_2^{i-\frac{d^2+d-4}{2}}x_3^{\frac{d^2+d}{2}-i}, \quad \text{for } i = \frac{d^2+d-4}{2}, \frac{d^2+d-2}{2},\\
&x_1^dx_3, \quad \text{for } i = \frac{d^2+d}{2}.
\end{aligned}$$

In what follows, the first monomial of $I_{d+1} - R_1 I_d$ is

$$\begin{aligned}
&x_2^{d+1}, \quad \text{for } i = 1,\\
&x_1x_2^{i-2}x_3^{(d+2)-i}, \quad \text{for } i = 2, \dots, d,\\
&\vdots\\
&x_1^{d-1}x_2x_3, \quad \text{for } i = \frac{d^2+d-2}{2},\\
&x_1^{d-1}x_2^2, \quad \text{for } i = \frac{d^2+d}{2}.
\end{aligned} \tag{4.4}$$

Note that

$$(d+i)^{\langle d\rangle} = (d+i)+k, \quad \text{for } i=(k-1)d-\frac{k(k-3)}{2},\ldots,kd-\frac{k(k-1)}{2}, \text{ and } k=1,\ldots,d. \tag{4.5}$$

We now calculate the Betti number

$$\beta_{1,d+2} = \sum_{T\in\mathcal{G}(I)_{d+1}} \binom{m(T)-1}{1}.$$

Based on Eq. (4.4), we shall find this Betti number of each two cases for i as follows.

Case 1.1. $i=(k-1)d-\frac{k(k-3)}{2}$ and $k=1,2,\ldots,d$.

By Eq. (4.5), I_{d+1} has k-generators, which are

$$x_1^{k-1}x_2^{(d+2)-k}, x_1^{k-1}x_2^{(d+1)-k}x_3,\ldots,x_1^{k-1}x_2^{(d+3)-2k}x_3^{k-1}.$$

By the similar argument, I_{d+1} has k-generators including the element $x_1^{k-1}x_2^{(d+2)-k}$ for $i=(k-1)d-\frac{k(k-3)}{2}+1,\ldots,(k-1)d-\frac{k(k-3)}{2}+(k-1)$. Hence we have that

$$\beta_{1,d+2} = \sum_{T\in\mathcal{G}(I)_{d+1}} \binom{m(T)-1}{1} = 2\times(k-1)+1 = 2k-1.$$

Case 1.2. $i=(k-1)d-\frac{k(k-3)}{2}+k=(k-1)d-\frac{k(k-5)}{2},\ldots,kd-\frac{k(k-1)}{2}$ and $k=1,2,\ldots,d$.

By Eq. (4.5), I_{d+1} has k-generators, which are

$$x_1^k x_2^{i-\left((k-1)d-\frac{k^2-3k-2}{2}\right)} x_3^{kd-\frac{k^2-k-4}{2}-i},\ldots,x_1^k x_2^{i-\left((k-1)d-\frac{k(k-5)}{2}\right)} x_3^{\left(kd-\frac{k(k-3)}{2}+1\right)-i}.$$

Hence we have that

$$\beta_{1,d+2} = \sum_{T\in\mathcal{G}(I)_{d+1}} \binom{m(T)-1}{1} = 2\times k = 2k.$$

Now we move on to the Betti number:

$$\beta_{2,d+2} = \sum_{T\in\mathcal{G}(I)_{d}} \binom{m(T)-1}{2}.$$

Recall $h_d=d+i$ and $j:=h_{d-1}-h_d$. The computation of the Betti number of this case is much more complicated, and thus we shall find the Betti number of each four cases based on i and j.

Case 2.1. $(\ell-1)d-\frac{(\ell-2)(\ell-1)}{2} < i < \ell d-\frac{(\ell-1)\ell}{2}$ and $\ell=1,2,\ldots,d$.

The last monomial of I_d for this case is

$$x_1^\ell x_2^{i-(\ell-1)d+\frac{\ell(\ell-3)}{2}} x_3^{\ell d-\frac{(\ell-1)\ell}{2}-i}.$$

Case 2.1.1. $(k-1)d-\frac{(k-1)k}{2} < i+j < kd-\frac{k(k+1)}{2}$ and $k=\ell,\ell+1,\ldots,d$.

Since the first monomial of $I_d-R_1I_{d-1}$ is

$$x_1^k x_2^{(i+j)-\left((k-1)d-\frac{(k-2)(k+1)}{2}\right)} x_3^{\left(kd-\frac{(k-1)(k+2)}{2}\right)-(i+j)},$$

we have $(j+k)$-generators in I_d as follows:

$$\begin{array}{l}
x_1^k x_2^{(i+j)-\left((k-1)d-\frac{(k-2)(k+1)}{2}\right)} x_3^{\left(kd-\frac{(k-1)(k+2)}{2}\right)-(i+j)},\ldots,x_1^k x_3^{d-k},\\
x_1^{(k-1)}x_2^{d-(k-1)}, x_1^{(k-1)}x_2^{(d-1)-(k-1)}x_3,\ldots,x_1^{(k-1)}x_3^{d-(k-1)},\\
\vdots\\
x_1^{\ell+1}x_2^{(d-1)-\ell}, x_1^{\ell+1}x_2^{(d-2)-\ell}x_3,\ldots,x_1^{\ell+1}x_3^{(d-1)-\ell}
\end{array}$$

$$x_1^\ell x_2^{d-\ell}, \ldots, x_1^\ell x_2^{i-(\ell-1)d+\frac{\ell(\ell-3)}{2}} x_3^{\ell d-\frac{(\ell-1)\ell}{2}-i}$$

and thus

$$\beta_{2,d+2} = \sum_{T\in\mathcal{G}(I)_d} \binom{m(T)-1}{2} = j+\ell.$$

Case 2.1.2. $i+j=(k-1)d-\frac{(k-1)k}{2}$ and $k=\ell+1,\ldots,d$.
The first monomial of $I_d - R_1 I_{d-1}$ is

$$x_1^{k-1}x_2^{d-(k-1)},$$

and hence we have $(j+k)$-generators in I_d as follows:

$$\begin{aligned}
&x_1^{k-1}x_2^{d-(k-1)}, x_1^{k-1}x_2^{(d-1)-(k-1)}x_3, \ldots, x_1^{k-1}x_3^{d-(k-1)},\\
&\vdots\\
&x_1^{\ell+1}x_2^{(d-1)-\ell}, x_1^{\ell+1}x_2^{(d-2)-\ell}x_3, \ldots, x_1^{\ell+1}x_3^{(d-1)-\ell}\\
&x_1^\ell x_2^{d-\ell}, \ldots, x_1^\ell x_2^{i-(\ell-1)d+\frac{\ell(\ell-3)}{2}} x_3^{\ell d-\frac{(\ell-1)\ell}{2}-i}
\end{aligned}$$

and thus

$$\beta_{2,d+2} = \sum_{T\in\mathcal{G}(I)_d} \binom{m(T)-1}{2} = j+\ell.$$

Case 2.2. $i=\ell d-\frac{(\ell-1)\ell}{2}$ and $\ell=1,2,\ldots,d$.
The last monomial of I_d is

$$x_1^\ell x_2^{d-\ell}.$$

Case 2.2.1. $(k-1)d-\frac{(k-1)k}{2} < i+j < kd-\frac{k(k+1)}{2}$ and $k=\ell+1,\ldots,d$.
Since the first monomial of $I_d - R_1 I_{d-1}$ is

$$x_1^k x_2^{(i+j)-\left((k-1)d-\frac{(k-2)(k+1)}{2}\right)} x_3^{\left(kd-\frac{(k-1)(k+2)}{2}\right)-(i+j)},$$

we have $(j+k)$-generators in I_d as follows:

$$\begin{aligned}
&x_1^k x_2^{(i+j)-\left((k-1)d-\frac{(k-2)(k+1)}{2}\right)} x_3^{\left(kd-\frac{(k-1)(k+2)}{2}\right)-(i+j)}, \ldots, x_1^k x_3^{d-k},\\
&x_1^{(k-1)}x_2^{d-(k-1)}, x_1^{(k-1)}x_2^{(d-1)-(k-1)}x_3, \ldots, x_1^{(k-1)}x_3^{d-(k-1)},\\
&\vdots\\
&x_1^{\ell+1}x_2^{(d-1)-\ell}, x_1^{\ell+1}x_2^{(d-2)-\ell}x_3, \ldots, x_1^{\ell+1}x_3^{(d-1)-\ell}\\
&x_1^\ell x_2^{d-\ell},
\end{aligned}$$

and thus

$$\beta_{2,d+2} = \sum_{T\in\mathcal{G}(I)_d} \binom{m(T)-1}{2} = j+\ell.$$

Case 2.2.2. $i+j=(k-1)d-\frac{(k-1)k}{2}$ and $k=\ell+1,\ldots,d$.
The first monomial of $I_d - R_1 I_{d-1}$ is

$$x_1^{(k-1)}x_2^{d-(k-1)},$$

and hence we have $(j+k)$-generators in I_d as follows:

$$x_1^{(k-1)}x_2^{d-(k-1)}, x_1^{(k-1)}x_2^{(d-1)-(k-1)}x_3, \ldots, x_1^{(k-1)}x_3^{d-(k-1)},$$

$$\vdots$$
$$x_1^{\ell+1}x_2^{(d-1)-\ell}, x_1^{\ell+1}x_2^{(d-2)-\ell}x_3, \ldots, x_1^{\ell+1}x_3^{(d-1)-\ell}$$
$$x_1^{\ell}x_2^{d-\ell},$$

and thus

$$\beta_{2,d+2} = \sum_{T \in \mathcal{G}(I)_d} \binom{m(T)-1}{2} = j + \ell,$$

as we desired. □

Theorem 4.5. *Let* **H** *be as in Eq.* (1.1) *and* $A = R/I$ *be an algebra with Hilbert function* **H** *such that* $\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}})$ *for some* $d < s$. *Then* A *is* not *level.*

Proof. Let L be a general linear form of A. By Lemma 3.2(b), note that if $d \geq r_1(A)$, then

$$\dim_k(0 : L)_{d-1} \geq h_{d-1} - h_d > 0 \quad \text{and} \quad \dim_k(0 : L)_d = h_d - h_{d+1} = 0,$$

and thus, by Lemma 3.2(a), R/I is not level. Hence we assume that $d < r_1(A)$ and A is a graded-level algebra having Hilbert function **H**. Let $\bar{I} = (I_{\leq d+1})$.

Claim. $\beta_{1,d+3}(\text{Gin}(\bar{I})) = 0$ *and* $\beta_{2,d+3}(\text{Gin}(\bar{I})) > 0$.

Proof of Claim. First we shall show that $\beta_{1,d+3}(\text{Gin}(\bar{I})) = 0$. By assumption,

$$\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}}),$$

and, by Lemma 2.9(a), we have that

$$\begin{aligned} &\beta_{1,d+2}(I^{\text{lex}}) - \beta_{1,d+2}(I) = [\beta_{0,d+2}(I^{\text{lex}}) - \beta_{0,d+2}(I)] + [\beta_{2,d+2}(I^{\text{lex}}) - \beta_{2,d+2}(I)] \\ &\Rightarrow -\beta_{1,d+2}(I) = [\beta_{0,d+2}(I^{\text{lex}}) - \beta_{0,d+2}(I)] - \beta_{2,d+2}(I). \end{aligned} \tag{4.6}$$

Moreover, since $A = R/I$ is level, we know that $\beta_{2,d+2}(I) = 0$, and hence rewrite Eq. (4.6) as

$$0 \leq [\beta_{0,d+2}(I^{\text{lex}}) - \beta_{0,d+2}(I)] = -\beta_{1,d+2}(I) \leq 0,$$

which follows from Lemma 2.8(b) that

$$\beta_{0,d+2}(I^{\text{lex}}) - \beta_{0,d+2}(I) = \beta_{0,d+2}(\bar{I}^{\text{lex}}) = 0.$$

Also, by Lemma 2.8(a), we have

$$\beta_{0,d+2}(\text{Gin}(\bar{I})) \leq \beta_{0,d+2}(\bar{I}^{\text{lex}}) = 0, \quad \text{i.e.,} \quad \beta_{0,d+2}(\text{Gin}(\bar{I})) = 0.$$

Since $\text{Gin}(\bar{I})$ is a Borel-fixed monomial ideal, by Theorem 2.4,

$$\beta_{1,d+3}(\text{Gin}(\bar{I})) = 0.$$

Now we shall prove that $\beta_{2,d+3}(\text{Gin}(\bar{I})) > 0$. Let $J = \text{Gin}(\bar{I})$. Consider the following exact sequence

$$0 \to ((J : x_3)/J)_d \to (R/J)_d \xrightarrow{\times x_3} (R/J)_{d+1} \to (R/J + (x_3))_{d+1} \to 0.$$

Since $d < r_1(A)$, we know that

$$\begin{aligned} \dim_k((J : x_3)/J)_d &= h_d - h_{d+1} + \dim_k((R/J + (x_3))_{d+1}) \\ &= \dim_k((R/J + (x_3))_{d+1}) \quad (\because h_d = h_{d+1}) \\ &\neq 0. \end{aligned}$$

By Lemma 3.8,

$$\mathcal{G}(J)_{d+1} = \mathcal{G}(\text{Gin}(\bar{I}))_{d+1} \neq \varnothing,$$

Table 1
Betti diagram of R/I^{lex}

Total:	1	–	–	–
0:	1	–	–	–
1:	–	–	–	–
			…	
$d-1$:	–	*	*	3
d:	–	*	4	*
$d+1$:	–	*	*	*
			…	

and so there is a monomial $T \in \mathcal{G}(\text{Gin}(\bar{I}))_{d+1}$ such that $x_3 \mid T$. In other words,

$$\beta_{2,d+3}(\text{Gin}(\bar{I})) > 0,$$

as we desired.

By the above claim and a cancellation principle, $R/\bar{I}$ has a socle element in degree d, and thus R/I has such a socle element in degree d since R/I and $R/\bar{I}$ agree in degrees $\leq d+1$, and hence A cannot be level, as we desired. □

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let **H** and j be as in Theorem 4.4 and let $h_d = d+i$ for $-(d-1) \leq i \leq d+3$.

By the proposition in [15], this theorem holds for $-(d-1) \leq i \leq 1$. It suffices, therefore, to prove this theorem for $2 \leq i \leq d+3$. By Theorem 4.4, we have

$$\begin{aligned}
\beta_{1,d+2}(I^{\text{lex}}) &= \begin{cases} 2, & \text{for } i = 2, \ldots, d, \\ 3, & \text{for } i = d+1, d+2, \quad \text{and} \\ 4, & \text{for } i = d+3, \end{cases} \\
\beta_{2,d+2}(I^{\text{lex}}) &= \begin{cases} j+1, & \text{for } i = 2, \ldots, d, \\ j+2, & \text{for } i = d+1, d+2, d+3. \end{cases}
\end{aligned} \tag{4.7}$$

Note that if either $j \geq 3$ and $2 \leq i \leq d+3$ or $j = 2$ and $2 \leq i \leq d+2$, then **H** is not level since $\beta_{2,d+2}(I^{\text{lex}}) > \beta_{1,d+2}(I^{\text{lex}})$.

Now suppose either $j = 1$ and $2 \leq i \leq d+2$ or $j = 2$ and $i = d+3$. By Eq. (4.7), we have

$$\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}}) = \begin{cases} 2, & \text{for } j = 1 \text{ and } i = 2, \ldots, d, \\ 3, & \text{for } j = 1 \text{ and } i = d+1, d+2, \\ 4, & \text{for } j = 2 \text{ and } i = d+3. \end{cases}$$

Thus, by Theorem 4.5, **H** cannot be level.

It is enough, therefore, to show the case $j = 1$ and $i = d+3$. Assume there exists a level algebra R/I with Hilbert function **H**. Applying Eq. (4.7) again, we have

$$\beta_{1,d+2}(I^{\text{lex}}) = \beta_{2,d+2}(I^{\text{lex}}) + 1 = 4. \tag{4.8}$$

Note that $h_{d-1} = 2d+4$ and $h_d = h_{d+1} = 2d+3$ in this case. By Eq. (4.8), the Betti diagram of R/I^{lex} is given in Table 1.

Moreover, by Lemmas 3.8 and 4.2,

$$\begin{aligned}
\dim_k((I^{\text{lex}} : x_3)/I^{\text{lex}})_d &= |\{T \in \mathcal{G}(I^{\text{lex}})_{d+1} | x_3 | T\}| \\
&= h_d - h_{d+1} + (h_{d+1})^- \\
&= (h_{d+1})^- \\
&= \left(\binom{d+2}{d+1} + \binom{d+1}{d}\right)^- \\
&= 2.
\end{aligned} \tag{4.9}$$

Table 2
Betti diagram of R/I^{lex}

Total:	1	–	–	–
0:	1	–	–	–
1:	–	–	–	–
			…	
$d-1$:	–	*	*	3
d:	–	2	4	2
$d+1$:	–	*	*	*
			…	

Table 3
Betti diagram of R/J

Total:	1	–	–	–
0:	1	–	–	–
1:	–	–	–	–
			…	
$d-1$:	–	*	*	3
d:	–	2	4	2
$d+1$:	–	a	b	*
			…	

Hence, using Eq. (4.9), we can rewrite Table 1 as Table 2.

Let $J := (I_{\leq d+1})^{\text{lex}}$. Note I^{lex} and J agree in degree $\leq d+1$. Hence we can write the Betti diagram of R/J (Table 3).

Since R/I is level and $(I_{\leq d+1})$ has no generators in degree $d+2$, we have

$$\beta_{0,d+2}(I_{\leq d+1}) = \beta_{2,d+2}(I_{\leq d+1}) = 0.$$

By Lemma 2.9(a),

$$\begin{aligned} a &= \beta_{0,d+2}(J) \\ &= \beta_{1,d+2}(J) - \beta_{1,d+2}(I_{\leq d+1}) - \beta_{2,d+2}(J) \\ &\leq \beta_{1,d+2}(J) - \beta_{2,d+2}(J) \\ &= 1. \end{aligned} \tag{4.10}$$

Hence, we have $a = 0$ or 1.

Case 1. Let $a = 0$. Then, by Theorem 2.4, we have $b = 0$. Moreover, by Lemma 2.9(a) again,

$$\begin{aligned} \beta_{2,d+3}(J) - \beta_{2,d+3}((I_{\leq d+1})) &\leq \beta_{1,d+3}(J) - \beta_{1,d+3}((I_{\leq d+1})) \\ &\leq \beta_{1,d+3}(J) \\ &= b \\ &= 0, \end{aligned} \tag{4.11}$$

and hence,

$$\beta_{2,d+3}(J) = \beta_{2,d+3}((I_{\leq d+1})) = 2.$$

This means that $R/(I_{\leq d+1})$ has two-dimensional socle elements in degree d, as does R/I, which is a contradiction.

Case 2. Let $a = 1$, then J has one generator in degree $d+2$. By Lemmas 3.8 and 4.2,

$$\begin{aligned} \dim_k((J : x_3)/J)_{d+1} &= |\{T \in \mathcal{G}(J)_{d+2}\, |x_3|\, T\}| \\ &= h_{d+1} - h_{d+2} + (h_{d+2})^{-} \end{aligned} \tag{4.12}$$

where $h_{d+2} = \mathbf{H}(R/J, d+2) = h_{d+1}^{\langle d+1\rangle} - 1 = (2d+3)^{\langle d+1\rangle} - 1 = 2d+4$. Hence, we obtain $(h_{d+2})^{-} = (2d+4)^{-} = 1$, and by Eq. (4.12)

$$\dim_k((J : x_3)/J)_{d+1} = 0.$$

Applying Theorem 2.4 again, we find

$$b = \beta_{1,d+3}(J) = \sum_{T \in \mathcal{G}(J)_{d+2}} \binom{m(T)-1}{1} = 1$$

since $x_1^{d+2} \notin \mathcal{G}(J)_{d+2}$. Thus R/J has at least one socle element in degree d, and so does $R/(I_{\leq d+1})$. Since R/I and $R/(I_{\leq d+1})$ agree in degree $\leq d+1$, R/I has such a socle element, a contradiction, which completes the proof. □

The following example shows a case where $j = 1$ and $h_d = 2d + 3$ in Theorem 4.1.

Example 4.6. Let I be the lex-segment ideal in $R = k[x_1, x_2, x_3]$ with Hilbert function

$$\mathbf{H} \;:\; 1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad 18 \quad 17 \quad 17 \quad 0 \quad \rightarrow.$$

Note that $h_7 = 17 = 2\times 7+3 = 2d+3$, which satisfies the condition in Theorem 4.1, and $j = h_6 - h_7 = 18-17 = 1$. Hence, any Artinian algebra having Hilbert function $\mathbf{H}$ cannot be level.

Inverse systems can also be used to produce new level algebras from known level algebras. This method is based on the idea of *Macaulay's Inverse Systems* (see [14,26] for details). We want to recall some results from [25]. Actually, Iarrobino shows an even stronger result and the application to level algebras is:

Theorem 4.7 (*Theorem 4.8A, [25]*). *Let $R = k[x_1, \ldots, x_r]$ and $\mathbf{H}' = (h_0, h_1, \ldots, h_e)$ be the h-vector of a level algebra $A = R/\mathrm{Ann}(M)$. Then, if F is a generic form of degree e, the level algebra $R/\mathrm{Ann}(\langle M, F\rangle)$ has h-vector $\mathbf{H} = (H_0, H_1, \ldots, H_e)$, where, for $i = 1, \ldots, e$,*

$$H_i = \min\left\{h_i + \binom{(r-1)+(e-i)}{(e-i)}, \binom{(r-1)+i}{i}\right\}.$$

The following example is another case of a level O-sequence of codimension 3 of type in Eq. (1.1) satisfying $h_d = 2d + 4$.

Example 4.8. Consider a level O-sequence $(1, 3, 5, 7, 9, 11, 13)$ of codimension 3. By Theorem 4.7, we obtain the following level O-sequence:

$$(1, 3, 6, 10, 15, 14, 14).$$

Then $14 = 2 \times 5 + 4$, which shows there exists a level O-sequence of codimension 3 of type in Eq. (1.1) when $h_d = 2d + 4$.

In general, we can construct a level O-sequence of codimension 3 of type in Eq. (1.1) satisfying $h_d = 2d + 4$ for every $d \geq 5$ as follows.

Proposition 4.9. *There exists a level O-sequence of codimension 3 of type in Eq. (1.1) satisfying $h_d = 2d + 4$ for every $d \geq 5$.*

Proof. Note that, from Example 4.8, this proposition holds for $d = 5$.

Now assume $d \geq 6$. Consider a level O-sequence $h = (1, 3, 5, 7, \ldots, \overset{d\text{-th}}{2d+1}, \overset{(d+1)\text{-st}}{2d+3})$ where $d \geq 6$. Since

$$\left(h_i + \binom{d+3-i}{d+1-i}\right) - \binom{i+2}{i} = \left(2i+1+\frac{(d+3-i)(d+2-i)}{2}\right) - \frac{(i+1)(i+2)}{2}$$
$$= \frac{(2+d)(3+d-2i)}{2} \geq 0,$$

for every $i = 0, 1, \ldots, d-3$, we have

$$\begin{aligned} H_i &= \min\left\{h_i + \binom{d+3-i}{d+1-i}, \binom{i+2}{i}\right\} \\ &= \min\left\{2i+1+\frac{(d+3-i)(d+2-i)}{2}, \frac{(i+1)(i+2)}{2}\right\} \\ &= \frac{(i+1)(i+2)}{2}. \end{aligned}$$

Hence, by Theorem 4.7, we obtain a level O-sequence $\mathbf{H} = (H_0, H_1, \ldots, H_d, H_{d+1})$ as follows:

$$\begin{aligned} H_0 &= 1, \\ H_1 &= 3, \\ &\vdots \\ H_i &= \frac{(i+1)(i+2)}{2}, \\ &\vdots \\ H_{d-2} &= \min\left\{h_{d-2} + \binom{5}{3}, \binom{d}{d-2}\right\} = \min\left\{2d+7, \frac{(d-1)d}{2}\right\} = 2d+7, \\ H_{d-1} &= \min\left\{h_{d-1} + \binom{4}{2}, \binom{d+1}{d-1}\right\} = \min\left\{2d+5, \frac{d(d+1)}{2}\right\} = 2d+5, \\ H_d &= \min\left\{h_d + \binom{3}{1}, \binom{d+2}{d}\right\} = \min\left\{2d+4, \frac{(d+1)(d+2)}{2}\right\} = 2d+4, \\ H_{d+1} &= \min\left\{h_{d+1} + \binom{2}{0}, \binom{d+3}{d+1}\right\} = \min\left\{2d+4, \frac{(d+2)(d+3)}{2}\right\} = 2d+4, \end{aligned}$$

as we desired. □

Remark 4.10. As with the proof of Proposition 4.9, we can construct a level O-sequence of codimension 3 of type in Eq. (1.1) satisfying

$$2d + (k+1) = H_{d-1} > H_d = H_{d+1} = 2d + k, \quad \left(5 \le k \le \frac{d^2 - 3d + 2}{2}\right).$$

For example, if we use

$$h = (1, 3, 6, \ldots, \overset{(d-1)\text{-st}}{2d + (k-5)}, \overset{d\text{-th}}{2d + (k-3)}, \overset{(d+1)\text{-st}}{2d + (k-1)}),$$

then we construct a level O-sequence of codimension 3 of type in Eq. (1.1) satisfying

$$\begin{aligned} H_{d-1} &= \min\left\{h_{d-1} + \binom{4}{2}, \binom{d+1}{d-1}\right\} = \min\left\{2d + (k+1), \frac{d(d+1)}{2}\right\} = 2d + (k+1), \\ &\quad \left(\because k \le \frac{d^2 - 3d - 2}{2}\right), \\ H_d &= \min\left\{h_d + \binom{3}{1}, \binom{d+2}{d}\right\} = \min\left\{2d+k, \frac{(d+1)(d+2)}{2}\right\} = 2d+k, \\ H_{d+1} &= \min\left\{h_{d+1} + \binom{2}{0}, \binom{d+3}{d+1}\right\} = \min\left\{2d+k, \frac{(d+2)(d+3)}{2}\right\} = 2d+k, \end{aligned}$$

as we desired.

Using Theorem 4.1, we know that some non-unimodal O-sequence of codimension 3 cannot be level as follows.

Corollary 4.11. *Let* $\mathbf{H} = \{h_i\}_{i\geq 0}$ *be an O-sequence with* $h_1 = 3$. *If*

$$h_{d-1} > h_d, \qquad h_d \leq 2d + 3, \quad \text{and} \quad h_{d+1} \geq h_d$$

for some degree d, *then* $\mathbf{H}$ *is not level.*

Proof. Note that, by the proof of Theorem 4.1, any graded ring with Hilbert function

$$\mathbf{H}' \ : \ h_0 \quad h_1 \quad \cdots \quad h_{d-1} \quad h_d \quad h_d \quad \rightarrow$$

has a socle element in degree $d - 1$.

Now let $A = \bigoplus_{i\geq 0} A_i$ be a graded ring with Hilbert function $\mathbf{H}$. If $A_{d+1} = \langle f_1, f_2, \ldots, f_{h_{d+1}}\rangle$ and $I = (f_{h_d+1}, \ldots, f_{h_{d+1}}) \bigoplus_{j\geq d+2} A_j$, then a graded ring $B = A/I$ has Hilbert function

$$h_0 \quad h_1 \quad \cdots \quad h_{d-1} \quad h_d \quad h_d.$$

Hence B has a socle element in degree $d - 1$ or d by Theorem 4.1. Since $A_i = B_i$ for every $i \leq d$, A also has the same socle element in degree $d - 1$ or d as B, and thus $\mathbf{H}$ is not level as we desired. □

The following is an example of a non-level and non-unimodal O-sequence of codimension 3 satisfying the condition of Corollary 4.11.

Example 4.12. Consider an O-sequence

$$\mathbf{H} \ : \ 1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 20 \quad 18 \quad 17 \quad h_8 \quad \cdots.$$

There are only three possible O-sequences such that $h_8 \geq h_7 = 17$ since $h_8 \leq h_7^{\langle 7\rangle} = 17^{\langle 7\rangle} = 19$. By Theorem 4.1, $\mathbf{H}$ is not level if $h_8 = h_7 = 17$. Neither can the other two non-unimodal O-sequences, by Corollary 4.11,

$$1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 20 \quad 18 \quad 17 \quad 18 \quad \cdots \quad \text{and}$$
$$1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 20 \quad 18 \quad 17 \quad 19 \quad \cdots$$

be level.

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