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STOCHASTIC INTEGRAL EQUATIONS APPLIED TO TELECOMMUNICATIONS TRAFFIC WITHOUT DELAY

P. LE GALL

Teletraffic and Operational Research Department, SOCOTEL, Paris, France

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Abstract. Modern telecommunication techniques cast the problem of traffic handling in the framework of fairly general networks, as applied to traffic without delay but with virtually arbitrary service-time distributions. In this paper we use stochastic integral equations to deal with the case involving the most general input process and lost calls. For this purpose, Fortet's equation, unsolved so far in the general case, is solved to analyze the single trunk group model. The stationary case is then treated as a special case. Finally we study networks which satisfy a certain assumption of symmetry. The same general stochastic assumptions are maintained throughout the paper.

telecommunications traffic	stationary case
general input	recurrent arrivals
lost calls	symmetrical networks
Fortet's equation	

1. Introduction

In this paper we present an interesting application of stochastic integral equations to the study of traffic handling in fairly general telecommunication networks; more specifically, to traffic without delay but with arbitrary service time distribution. We establish general theorems dropping the usual Markovian assumptions. In [3] a full description of telecommunication networks is given. Here we briefly characterize telecommunication traffic in the case of the lost-calls model.

The calls occur according to an *input process*, the random arrival epochs constituting a point process on a line. The telecommunication network consists of a set of allowable paths connecting two arbitrary subscribers. Hereafter, the input process corresponding to a large number of subscribers is assumed to be *independent of the subscriber calls in* *progress.* The setting-up of the call is assumed to be quasi-instantaneous. The call holds one of the possible free paths connecting the caller to the called subscriber. The *holding time* of this path is equal to the duration of the call, and is a random variable. Here the hunting rule of the free paths may also be of some importance. We assume that a call which cannot be served at its arrival time is refused and leaves the system ("lost-calls" model). We shall first consider the simple case of a single trunk group.

Let $N(t)$ be the random number of arrivals in the time interval $(0, t]$; we assume that $\{N(t), t \geqslant 0\}$ is a point process (on a line) for which the arrivals occur successively and the simultaneous occurrence of two or more arrivals is impossible. For such a process $N(t)$ we may formulate the following theorem.

Theorem 1.1. *If $h(u, t)$ is an algebraic function of real variables u and t, then for a point process $N(t)$ with nonsimultaneous arrivals we have*

$$\exp\left\{\int_0^t \log\left[1 + h(u, t)\right] \mathrm{d}N(u)\right\} =$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^t h(u_1, t)\, \mathrm{d}N(u_1) \ldots \int_0^t h(u_n, t)\, \mathrm{d}N(u_n)\,, \quad (1)$$

with $\mathrm{d}N(u_i)\, \mathrm{d}N(u_j) = 0$ *if* $u_i = u_j$.

Proof. We note that by our assumption the number of arrivals in an infinitesimal interval $(t, t + \mathrm{d}t]$ is 0 or 1. Let

$$0 = u_0 < u_1 < \ldots < u_m = t$$

be a subdivision of the interval $(0, t]$ such that $\Delta u_i = u_{i+1} - u_i$ is sufficiently small to ensure that

$$\Delta N(u_i) = N(u_{i+1}) - N(u_i) = 0 \text{ or } 1\,, \quad i = 1, 2, \ldots, m-1\,.$$

We can write

$$\exp\{\log\left[1 + h(u_i, t)\right] \Delta N(u_i)\} = \left[1 + h(u_i, t)\right]^{\Delta N(u_i)}$$

$$= 1 + h(u_i, t)\, \Delta N(u_i)\,, \quad 1 \leqslant i \leqslant m - 1\,.$$

Hence

$$\exp\left\{\sum_{i=1}^{m-1} \log\left[1+h(u_i,t)\right] \Delta N(u_i)\right\} =$$

$$= \prod_{i=1}^{m-1} \left[1+h(u_i,t)\,\Delta N(u_i)\right]$$

$$= 1 + \sum_{n=1}^{m-1} \sum_{1\leqslant i_1<\ldots<i_n\leqslant m-1} h(u_{i_1},t)\,\Delta N(u_{i_n})\ldots h(u_{i_n},t)\,\Delta N(u_{i_n}) \,.$$

This leads to the relation

$$\exp\left\{\int_0^t \log\left[1+h(u,t)\right] \mathrm{d}N(u)\right\} =$$

$$= 1 + \sum_{n=1}^{\infty} \int_0^t h(u_1,t)\,\mathrm{d}N(u_1) \int_0^{u1} h(u_2,t)\,\mathrm{d}N(u_2)\ldots$$

$$\times \int_0^{u_{n-1}} h(u_n,t)\,\mathrm{d}N(u_n)$$

with $\mathrm{d}N(u_i)\,\mathrm{d}N(u_j) = 0$ if $u_i = u_j$. This relation being equivalent to the formula (1), the theorem is proved. □

In particular, if we choose $h(u,t) \equiv \mathrm{e}^z - 1$, we deduce the following corollary:

Corollary 1.2. *For z such that* $|\mathrm{e}^z - 1| < 1$, *we have*

$$\exp\left[z\,N(t)\right] = 1 + \sum_{n=1}^{\infty} \frac{(\mathrm{e}^z-1)^n}{n!} \int_0^t \mathrm{d}N(u_1) \ldots \int_0^t \mathrm{d}N(u_n)\,, \tag{2}$$

with $\mathrm{d}N(u_i)\,\mathrm{d}N(u_j) = 0$ *if* $u_i = u_j$.

Now for every finite non-negative real number y and z as above, we have the relation

$$\mathrm{e}^{zy} = \left[1+(\mathrm{e}^z-1)\right]^y = 1 + \sum_{n=1}^{\infty} \frac{(\mathrm{e}^z-1)^n}{n!} \left[y(y-1)\ldots(y-n+1)\right] .$$

Consequently, for every stochastic function $N(t)$ of the real variable t, taking only real, finite and non-negative values, we have

$$\exp[z\,N(t)] = 1 + \sum_{n=1}^{\infty} \frac{(e^z - 1)^n}{n!}\,[N(t)\,(N(t)-1)\ldots(N(t)-n+1)]\;. \qquad (3)$$

Comparing this with (2), we deduce the important stochastic relation for point processes with nonsimultaneous arrivals:

$$N(t)\,[N(t)-1]\ldots[N(t)-n+1] = \int_0^t \mathrm{d}N(t_1)\ldots\int_0^t \mathrm{d}N(t_n) \qquad (4)$$

with $\mathrm{d}N(t_i)\,\mathrm{d}N(t_j) = 0$ if $t_i = t_j$.

The general input process $N(t)$ described above is offered to a group of L *circuits*. When the group is not congested, the call arriving at time u is served without delay, its *holding time* being a random variable T_u. For the moment T_u may depend on u, on the input process and on the durations of the other communications. On the other hand, when the group is congested at time u, this call is rejected and is lost.

Let us define the stochastic function $R(u, t)$ and the algebraic function $V(y)$ by

$$R(u,t) = \begin{cases} 1 & \text{if } u \leqslant t < u + T_u\,, \\ 0 & \text{if } t < u \text{ or } t \geqslant u + T_u\,, \end{cases} \qquad (5)$$

$$V(y) = \begin{cases} 1 & \text{if } y = 1, 2, \ldots, L-1\,, \\ 0 & \text{otherwise}\,. \end{cases} \qquad (6)$$

The behavior of $V(y)$ outside the positive integers is unimportant. Let $Y(t)$ be the stochastic function of t representing the *number of occupied circuits* at the time t. $Y(t)$ is the sum of the calls arriving in the interval $(0, t]$, which are being served and which are in progress at time t. For simplicity we assume the system to be empty at the epoch 0. The function $Y(t)$ satisfies the stochastic integral equation

$$Y(t) = \int_0^t V(Y(u))\,R(u,t)\,\mathrm{d}N(u)\,, \qquad (7)$$

and is in fact completely determined by (7), as was shown by Fortet [1], who solved it only for $L = 1$ (one circuit). The inherent difficulty is the *nonlinearity* of this integral equation. In the next section we give the solution of (7) for arbitrary positive integer L [2].

2. Solution of the stochastic integral equation for $L \geqslant 1$

We begin by introducing the stochastic function of t,

$$X(\nu,t) = \frac{1}{\nu!} Y(t)\,[Y(t)-1] \dots [Y(t)-\nu+1] \,. \tag{8}$$

Observe that the integrand in (7) defines an input process: it is related to the arrivals of calls served during the interval $(0, t]$ which are still in progress at time t. We may therefore apply the stochastic relation (4), and write (8) as

$$X(\nu, t) = \frac{1}{\nu!} \int_0^t V(Y(u_1))\, R(u_1, t)\, \mathrm{d}N(u_1) \dots \int_0^t V(Y(u_\nu))\, R(u_\nu, t)\, \mathrm{d}N(u_\nu) \,. \tag{9}$$

The function $W_L(t)$ defined by

$$W_L(t) = V(Y(t)) = 1 - X(L, t) \tag{10}$$

is a stochastic function, which assumes the value 1 when the group of circuits is not congested at the epoch t, and the value 0 otherwise. From (7), (9) and (10) it follows that

$$W_L(t) = 1 - \int_0^t W_L(u_1) R(u_1, t)\, \mathrm{d}N(u_1) \dots \int_0^t W_L(u_L) R(u_L, t)\, \mathrm{d}N(u_L) \,, \tag{11}$$

with $\mathrm{d}N(u_i)\, \mathrm{d}N(u_j) = 0$ if $u_i = u_j$.

This new multiple integral equation of order L is *multilinear*. We can obtain a series expansion of $W_L(t)$ by means of the method of successive approximations. With this knowledge of $V(Y(t))$, $Y(t)$ can be obtained from (7).

For $L = 1$, the solution, already provided by Fortet [1], is given by

$$W_1(t) = 1 - \sum_{n=1}^{\infty} (-1)^{n+1} \int_0^t R(u_1, t)\, \mathrm{d}N(u_1) \int_0^{u_1} \dots$$

$$\times \int_0^{u_{n-1}} R(u_n, u_{n-1})\, \mathrm{d}N(u_n) \,. \tag{12}$$

For $L > 1$, since direct calculations are very intricate, it is better to proceed by induction. The functions $W_L(t)$, $L = 1, 2, \ldots$, can be linked recursively as follows. If in the group of L circuits, the circuits are hunted in order from the first one, then $W_L(t)$ remains unchanged. But now the last circuit receives the arrival process

$$[1 - W_{L=1}(t)]\, \mathrm{d}N(t)\,.$$

The solution for $L = 1$ applies, and (12) provides the expression $W(t)$ for this circuit. From the relation

$$[1 - W_{L=1}(t)]\,[1 - W(t)] = 1 - W_L(t)\,,$$

we deduce the recurrence formula for $[1 - W_L(t)]$ given by the following theorem.

Theorem 2.1. *The solution of Fortet's stochastic integral equation* (7) *is*

$$Y(t) = \int_0^t W_L(u)\, R(u, t)\, \mathrm{d}N(u)\,, \tag{13}$$

where $W_L(t)$ may be deduced by the recurrence formula

$$1 - W_L(t) = [1 - W_{L-1}(t)] \sum_{n=1}^{\infty} (-1)^{n+1} \tag{14}$$

$$\times \int_0^t R(u_1, t)\, [1 - W_{L-1}(u_1)]\, \mathrm{d}N(u_1) \int_0^{u_1} \ldots$$

$$\times \int_0^{u_{n-1}} R(u_n, u_{n-1})\, [1 - W_{L-1}(u_n)]\, \mathrm{d}N(u_n)\,.$$

We now derive other relations which will be found useful. For the *process* $\mathrm{d}N_0(t)$ *of served calls*, let us set

$$\mathrm{d}N_0(t) = W_L(t)\, \mathrm{d}N(t)\,. \tag{15}$$

Note that $\mathrm{d}N_0(t)$ can be considered as the solution of the following stochastic integral equation deduced from (11),

$$\mathrm{d}N_0(t) = \mathrm{d}N(t) - \frac{1}{L!}\int_0^t R(u_1, t)\,\mathrm{d}N_0(u_1) \ldots$$

$$\times \int_0^t R(u_L, t)\,\mathrm{d}N_0(u_L)\,\mathrm{d}N(t)\,. \qquad (16)$$

A more general relation is obtained in the following way. Suppose that, at time t, $Y(t)$ assumes the value j. From (8) we get

$$X(\nu - 1, t) = \binom{j}{\nu - 1}, \qquad X(L, t) = \begin{cases} 0 & \text{if } j < L, \\ 1 & \text{if } j = L. \end{cases} \qquad (17)$$

Moreover,

$$\mathrm{d}N_0(t)\,X(\nu - 1, t) = \begin{cases} \mathrm{d}N(t)\,X(\nu - 1, t) & \text{if } j < L, \\ 0 & \text{if } j = L. \end{cases}$$

Consequently we have the following stochastic relation, which is more general than (16).

Corollary 2.2. *We have the stochastic relation*

$$\mathrm{d}N_0(t)\,X(\nu - 1, t) = \mathrm{d}N(t)\,\left[X(\nu - 1, t) - \binom{L}{\nu - 1} X(L, t)\right] \quad (18)$$

with $X(0, t) \equiv 1$, $X(\nu, t)$ *being defined by* (8).

We conclude this section with a brief review of relations involving first moments. Set

$$\begin{aligned} &\mathrm{P}[Y(t) = j] = P(j, t)\,, \\ &\mathrm{P}[\mathrm{d}N(t)\,Y(t) = j] = Q(j, t)\,\rho(t)\,\mathrm{d}t\,, \end{aligned} \qquad (19)$$

where $Q(j, t)$ is the conditional probability that when a call arrives at the time t it finds j occupied circuits, and $\rho(t)$ is the *density of arrivals.*

It may be convenient to introduce the binomial moments

$$\begin{aligned} &\mathrm{E}\{X(\nu, t)\} = S(\nu, t) = \sum_{j=\nu}^{L} \binom{j}{\nu} P(j, t)\,, \\ &\mathrm{E}\{\mathrm{d}N(t)\,X(\nu, t)\} = T(\nu, t)\,\mathrm{d}t = B(\nu, t)\,\rho(t)\,\mathrm{d}t\,, \end{aligned} \qquad (20)$$

with

$$B(\nu, t) = \sum_{j=\nu}^{L} \binom{j}{\nu} Q(j, t) .$$

Here $S(L, t) = P(L, t)$ is the probability that at time t all circuits are occupied, and $B(L, t) = Q(L, t)$ is the conditional probability that at the instant t the group of circuits is congested, when a call arrives.

Relations (19) imply the (well-known) relations

$$\begin{aligned} P(j, t) &= \sum_{\nu=j}^{L} \binom{\nu}{j} (-1)^{\nu-j} S(\nu, t) , \\ Q(j, t) &= \sum_{\nu=j}^{L} \binom{\nu}{j} (-1)^{\nu-j} B(\nu, t) . \end{aligned} \tag{21}$$

Finally, if we set

$$\begin{aligned} H_\nu(t, \theta)\, \mathrm{d}\theta = \int_0^\theta \dots \int_0^\theta [(\nu-1)!]^{-1}\, \mathbf{E}\{&R(u_1, t)\, \mathrm{d}N_0(u_1) \dots \\ &\times R(u_{\nu-1}, t)\, \mathrm{d}N_0(u_{\nu-1})\, \mathrm{d}N_0(\theta)\} \end{aligned} \tag{22}$$

for $t \geqslant \theta$, then the stochastic relation (18) yields the equation

$$H_\nu(t, t) = T(\nu-1, t) - \binom{L}{\nu-1} T(L, t) , \tag{23}$$

where $T(0, t) = \rho(t)$.

3. The stationary limit process

We now turn to the study of stationary limit processes. We shall assume that the call durations are random variables which are mutually independent, independent of the input process, and have the same arbitrary distribution function $F(t)$. Thus (22) can now be written

$$H_\nu(t,\theta)\,\mathrm{d}\theta = \frac{1}{(\nu-1)!}\int_0^\theta [1-F(t-u_1)] \ldots$$

$$\times \int_0^\theta [1-F(t-u_{\nu-1})]\ \mathrm{E}\{\mathrm{d}N_0(u_1)\ldots \mathrm{d}N_0(u_{\nu-1})\,\mathrm{d}N_0(\theta)\}\ . \tag{24}$$

From (9) and (20) we obtain

$$S(\nu,t) = \int_0^t [1-F(t-\theta)]\ H_\nu(t,\theta)\,\mathrm{d}\theta\ . \tag{25}$$

Assume now that we have a stationary limit process, which means for all practical purposes that $[N(t)/t]$ and $[(N_0(t)/t)\,\mathrm{d}N(t)]$ tend almost certainly to limit stochastic variables. Consequently, we have the limits

$$\begin{aligned}
&\lim_{t\to\infty} P(j,t) = P(j)\,, \qquad \lim_{t\to\infty} Q(j,t) = Q(j)\,,\\
&\lim_{t\to\infty} S(\nu,t) = S(\nu)\,, \qquad \lim_{t\to\infty} B(\nu,t) = B(\nu)\,,\\
&\lim_{t\to\infty} T(\nu,t) = \rho\,B(\nu)\,.
\end{aligned} \tag{26}$$

The density ρ is called the density of arrivals at any instant of the limit process.

We seek to evaluate the limit (as $t\to\infty$) of the expression (25) for $S(\nu,t)$. If we take as time unit the mean duration of a call, then we get

$$\int_0^\infty [1-F(t)]\,\mathrm{d}t = 1\ . \tag{27}$$

By Theorem A.3, for a fixed value of $(t-\theta)$ the expression (24) of $H_\nu(t,\theta)$ converges (as $t\to\infty$), with

$$H_\nu(\theta,\theta)\left[\int_{t-\theta}^\infty [1-F(u)]\,\mathrm{d}u\right]^{\nu-1}\ .$$

Now applying Theorem A.1 to the integral (25), we deduce that the limit $S(\nu)$ is given by

$$S(\nu) = \left[\lim_{t\to\infty} \left\{ \frac{1}{t} \int_0^t H_\nu(\theta,\theta)\,\mathrm{d}\theta \right\} \right] \int_0^\infty G(u)\,\mathrm{d}u\ , \tag{28}$$

where

$$G(u) = [1-F(u)] \left[\int_u^\infty [1-F(v)]\,\mathrm{d}v \right]^{\nu-1} . \tag{29}$$

Integration by parts gives

$$\int_0^\infty G(u)\,\mathrm{d}u = \frac{1}{\nu}\ , \tag{30}$$

so that

$$\nu S(\nu) = \lim_{t\to\infty} \left\{ \frac{1}{t} \int_0^t H_\nu(\theta,\theta)\,\mathrm{d}\theta \right\} . \tag{31}$$

The relations (23), (26) and (31) finally give the important equation

$$B(0) = 1\ , \qquad \nu S(\nu) = \rho \left[B(\nu-1) - \binom{L}{\nu-1} B(L) \right], \quad \nu = 1, 2, ..., L\ . \tag{32}$$

Recall that $S(\nu)$ *and* $P(j)$ *refer to an arbitrary instant and* $B(\nu)$ *and* $Q(j)$ *to the arrival of any call.* Then (26) allows us to write (32) in the more expressive form

$$jP(j) = \rho\, Q(j-1)\ , \quad j = 1, 2, ..., L\ . \tag{33}$$

Unfortunately, we lack a relationship between $S(\nu)$ and $B(\nu)$, primarily because any such relationship is obtained easily only in the case of recurrent arrival processes.

4. Recurrent arrivals

We note that until now this case has been solved only for exponential holding times. We shall investigate in this section the case of an arbitrary holding-time distribution, still assuming stationarity, the existence of which is not yet proved.

For $u_1 < ... < u_\nu < t$ we can write

$$\mathrm{E}\{\mathrm{d}N_0(u_1) ... \mathrm{d}N_0(u_\nu)\,\mathrm{d}N(t)\} = \mathrm{E}\{\mathrm{d}N_0(u_1) ... \mathrm{d}N_0(u_\nu)\}\ \rho_0(t-u_\nu)\,\mathrm{d}t\,, \tag{34}$$

where $\rho_0(t)$ is the density of arrivals at time t given that a call occurs at time 0. From (20) and (22) we deduce that

$$T(\nu, t) = \int_0^t [1 - F(t-\theta)]\,\rho_0(t-\theta)\,H_\nu(t,\theta)\,\mathrm{d}\theta\,. \tag{35}$$

Reasoning as in the preceding section, Theorems A.1 and A.3 give us in the limit,

$$\lim_{t\to\infty} T(\nu,t) = \rho\,B(\nu) =$$

$$= \left[\lim_{t\to\infty} \left\{\frac{1}{t}\int_0^t H_\nu(\theta,\theta)\,\mathrm{d}\theta\right\}\right] \int_0^\infty G(u)\,\mathrm{d}u\,, \tag{36}$$

where

$$G(u) = [1-F(u)]\left[\int_u^\infty [1-F(v)]\,\mathrm{d}v\right]^{\nu-1} \rho_0(u)\,.$$

Finally, the relation (31) implies the second important relation

$$S(\nu) = \frac{\rho}{\nu}\frac{B(\nu)}{\gamma(\nu)}, \qquad \nu = 1, 2, ..., L\,, \tag{37}$$

where

$$\gamma(0) = 1\,, \qquad \gamma(\nu) = \int_0^\infty [1-F(u)]\left[\int_u^\infty [1-F(v)]\,\mathrm{d}v\right]^{\nu-1} \rho_0(u)\,\mathrm{d}u\,. \tag{38}$$

If we apply the expression (37) to the relation (32), we get the general equation for the stochastic process

$$B(0) = 1\,, \qquad \frac{B(\nu)}{\gamma(\nu)} = B(\nu-1) - \binom{L}{\nu-1} B(L)\,, \qquad \nu = 1, 2, ..., L\,. \tag{39}$$

The rest of the calculations are identical with those in the case of exponential holding times. We find the general expression

$$B(\nu) = B_\infty(\nu) \sum_{\lambda=\nu}^{L} \binom{L}{\lambda} \frac{1}{B_\infty(\lambda)} \left[\sum_{\lambda=0}^{L} \binom{L}{\lambda} \frac{1}{B_\infty(\lambda)} \right]^{-1}, \tag{40}$$

with the special case of a group of unlimited size corresponding to

$$B_\infty(\nu) = \prod_{i=0}^{\nu} \gamma(\nu) . \tag{41}$$

The call congestion[1] is

$$B(L) = \left[\sum_{\lambda=0}^{L} \binom{L}{\lambda} \frac{1}{B_\infty(\lambda)} \right]^{-1} . \tag{42}$$

The time congestion[2] $S(L)$ can be deduced from eq. (37).

In the case of Poisson arrivals, (38) becomes

$$\gamma(\nu) = \rho/\nu . \tag{43}$$

Observe that $F(t)$ does not appear in the expression (43). On the other hand, the expression (38) shows that $F(t)$ does appear when the arrival process is no longer Poisson.

The expression (38) means that, in the case of the stationary limit process, everything proceeds as though the arrivals occurred at mutually arbitrary instants, at least from the point of view of the ages of the occupancies at these instants.

5. Networks

The connecting networks in exchanges satisfy a certain *assumption of symmetry* which results from connecting the successive selectors by means of a regular trunking diagram and from the *random hunting rule* concerning the search for free paths. Then the state probabilities depend only on the number of calls in progress in the various traffic streams. In this section we describe the laws for handling the most general traffic,

[1] The call congestion is the conditional probability that when a call arrives it finds the group of circuits congested. Alternatively, in the case of stationary limit processes it is the ratio of the number of calls lost to the number of calls offered, during a long period.

[2] The time congestion is the ratio of time during which the congestion occurs to the total time of a long observation. Alternatively, in the case of stationary limit processes, it is the probability that at any time the congestion occurs.

taking into account the stochastic dependence between various parts of the network, while maintaining a symmetry assumption which is particularly useful in the case of connecting networks in the exchanges. We shall see that it is then possible to dissociate the geometric and combinatorial characteristics of the network from the laws of traffic handling in a simple group, whether stationary or not. In other words, the stochastic aspect is as already discussed in the preceding sections.

Suppose we have x traffic streams, where x is a very large number for real networks. Stream i receives the arrival process $\mathrm{d}N_i(t)$. The various processes $\mathrm{d}N_i(t)$ are assumed to be mutually independent. Each traffic stream may follow one or more possible paths, all of which may, if necessary, be tried when a new call is to be placed in the respective traffic stream. In general, these networks serve outgoing trunkgroups for which all cases of occupation are possible. Each one of these outgoing trunkgroups serves one or more traffic streams, which are independent of those served by the other groups. We therefore impose virtually no restrictions if we make the following assumptions.

Assumption 1 (independence between outgoing trunkgroups).
(a) Each outgoing trunkgroup carries a traffic corresponding to one or more arrival processes $\mathrm{d}N_i(t)$ which are independent of those served by the other outgoing trunkgroups.

(b) It is possible to find states of internal occupation causing all outgoing trunkgroups to be congested simultaneously.

Now it remains to define carefully the assumption of symmetry. A great simplification in the calculations results from the equality of the probabilities of possible states for a given number of calls going on in the considered traffic stream. Under these conditions, this traffic stream can be represented by a simple stochastic function $Y(t)$ giving the number of calls existing at the epoch t as in Section 1 in the case of a simple trunkgroup. But this equality between states also assumes an equal chance of reaching the various higher states that are accessible from the considered state by letting a new call arrive. To this end, the cabling of the circuits should be symmetric and the selection of free circuits should be made at random.

Assumption 2 (symmetry). (a) All occupation states $(j_1 \dots j_x)$ that correspond to a given number j_i of calls going on in the traffic stream i $(i = 1, \dots, x)$ are assumed to have an equal chance of occurring.

(b) The assumption formulated in (a) puts conditions on the cabling of the circuits as well as on the rule of hunting for a free path, a new call having to be placed at random on one of the free paths.

For simplicity, we also assume the network to be empty at the epoch 0. Let us denote by $R_0(j_1 ... j_x)$ the number of upward "trajectories" that are strictly permitted for placing the $(j_1 ... j_x)$ calls arriving in the interval $(0, t]$ in a determined order. Indeed, there is good reason for remarking that the order is important, for the paths may or may not exist, according to the respective sequences of call arrivals of the different traffic streams. The order within a traffic stream is, however, of very little importance. We also observe that, if several paths exist for one and the same order, they do not count for more than one.

More precisely, if for a given number $j_1 + ... + j_x$ we select at random each call with a probability $1/N_i$ for the i^{th} traffic stream, then the probability of selecting the distribution $(j_1, ..., j_x)$ is

$$R_0(j_1, ..., j_x) \prod_{i=1}^{x} \left(\frac{1}{N_i}\right)^{j_i} = R(j_1, ..., j_x) \left[\frac{(j_1 + ... + j_x)!}{j_1! ... j_x!} \prod_{i=1}^{x} \left(\frac{1}{N_i}\right)^{j_i}\right], \tag{44}$$

where

$$\sum_{i=1}^{x} \frac{1}{N_i} = 1 \ , \qquad R(j_1, ..., j_x) = \frac{R_0(j_1, ..., j_x)}{(j_1 + ... + j_x)!/j_1! ... j_x!} .$$

Here $R(j_1, ..., j_x)$ is the reduction factor of the number of possible states. If we decide to place the calls according to the impressed distribution $(j_1, ..., j_x)$, the paths being chosen at random according to this distribution, then $R(j_1, ..., j_x)$ is the probability of placing all these calls.

Now we consider the simple assumption of Poisson arrivals under stationarity. If the system is in the state $(j_1, ..., j_x)$ and a new call arrives in the traffic stream i, then, according to Assumption 2, the probability of placing it is

$$\frac{R(j_1, ..., j_{i-1}, j_i + 1, j_{i+1}, ..., j_x)}{R(j_1, ..., j_x)} .$$

We assume a traffic a_i offered to the traffic stream i. The general equations for statistical equilibrium (33) may now be written

$$(j_1 + 1) P(j_1, ..., j_{i-1}, j_i + 1, j_{i+1}, ..., j_x) =$$

$$= a_i \frac{R(j_1, ..., j_{i-1}, j_i + 1, j_{i+1}, ..., j_x)}{R(j_1, ..., j_x)} P(j_1, ..., j_x) . \tag{45}$$

After a few successive calculations we get

$$P(j_1, ..., j_x) = P(0, ..., 0)\, R(j_1, ..., j_x) \left(\prod_{i=1}^{x} \frac{a_i^{j_i}}{j_i!} \right). \tag{46}$$

We extend the relation (46) to the case of a process of general arrivals $\mathrm{d}N_i(t)$, the holding-time distribution $F(t)$ being assumed arbitrary and identical for all traffic streams. According to Assumption 2, it is sufficient to characterize the stochastic evolution of the traffic stream i by the stochastic function $Y_i(t)$ which represents the number of existing calls at the epoch t in this traffic stream. The evolution of the total system is thus represented by the random vector

$$Z(t) = \{Y_1(t), ..., Y_x(t)\}\,. \tag{47}$$

Consider first the special case of call occupancies that do not terminate or that have durations longer than t. Using Assumption 1 and eq. (7),

$$Y_i(t) = \int_0^t V_i(Y_1(u), ..., Y_x(u))\, \mathrm{d}N_i(u) = \int_0^t V_i(Z(u))\, \mathrm{d}N_i(u)\,, \tag{48}$$

where V_i depends on the structure of the network. This case is in fact similar to that one considered for formula (44) provided that we replace the term $(1/N_i)^{j_i}$ by the probability $P_i(j_i, t)$ of having j_i calls in stream i. We thus have

$$P(j_1, ..., j_x; t) = \frac{P(0, ..., 0; t)}{\prod_{i=1}^{x} P_i(0, t)} R(j_1, ..., j_x) \prod_{i=1}^{x} P_i(j_i, t)\,. \tag{49}$$

When the call durations have distribution function $F(t)$, (48) gives, according to (7),

$$Y_i(t) = \int_0^t V_i(Y_1(u), ..., Y_x(u))\, R_i(u, t)\, \mathrm{d}N_i(u)$$

$$= \int_0^t V_i(Z(u))\, R_i(u, t)\, \mathrm{d}N_i(u)\,. \tag{50}$$

We note that for a given t, $R_i(u, t)\, \mathrm{d}N_i(u)$ is analogous to the arrival process made up of arrivals in the stream i giving occupancies not terminating at t. The formula (49) is thus still valid, where $P_i(j_i, t)$ refers to the case of a simple trunkgroup that receives the process $dN_i(t)$ and has the maximum capacity L_i according to Assumption 1. If the considered outgoing trunkgroup receives more than one traffic stream, then the process $dN_i(t)$ should be replaced by the sum of corresponding processes in (50). Quite generally, (50) applies equally to an arbitrary instant as well as to an instant of arrival in a certain traffic stream. We can now state the following theorem (which is very convenient to apply).

Theorem 5.1 (symmetrical networks). *We consider the case of a network in which x traffic streams pass (under stationarity or not). The network is assumed to be empty at epoch 0 and to comply with Assumptions 1 and 2. Then the probability of finding j_i calls going on in the traffic stream i ($i = 1, ..., x$) at the arbitrary instant t or at the epoch of an arrival is given by*

$$P(j_1, ..., j_x; t) = \frac{P(0, ..., 0; t)}{\Pi_{i=1}^{x} P_i(0, t)} R(j_1, ..., j_x) \prod_{i=1}^{x} P_i(j_i, t) , \qquad (51)$$

where $P_i(j_i, t)$ corresponds to the case where the outgoing trunk group is considered as isolated (carrying one or more traffic streams). The quantity $R(j_1, ..., j_x)$ is the reduction factor for the number of possible states defined by (44).

The following remarks will clarify the value of this theorem.

(a) The structural and combinatorial aspect of the network, represented by $R(j_1, ..., j_x)$ has been dissociated from the random aspect represented by $P_i(j_i, t)$ for a single trunkgroup.

(b) The theorem holds both under stationarity and under nonstationarity. Assuming the network empty at zero time is practically no restriction, for the effect of the initial conditions is smoothed down after some time.

(c) The theorem holds for the most general assumptions on the arrival processes. The theorem also holds for arbitrary distribution functions $F(t)$. It does not assume the independence of call durations and the input process.

(d) The calculation of $R(j_1, ..., j_x)$ can therefore be made by considering the Markovian case, or by a combinatorial–analytical study not involving probabilistic calculations.

(e) With regard to (14), Theorem 5.1 also shows that, even for the random aspect, it is possible to separate the effect of the arrival process from that of the holding-time distribution.

Moreover, with regard to the contents of Theorem 5.1 we can understand the importance of presenting in Section 2, in the most general form, the rules for traffic handling in a fully available group of circuits, now that the mathematical tools have led us in a simple way to the formulation of this theorem.

Since

$$\sum_{j_1,\ldots,j_x} P(j_1,\ldots,j_x;t) = 1\ ,$$

we can write (51), for an arbitrary instant t, as

$$P(j_1,\ldots,j_x;t) = R(j_1,\ldots,j_x) \prod_{i=1}^{x} P_i(j_i,t) \Big[\sum_{k_1,\ldots,k_x} R(k_1,\ldots,k_x) \prod_{i=1}^{x} P_i(k_i,t) \Big]^{-1} . \tag{52}$$

Thus this general formula (52) shows us that it is sufficient to know the nonrandom terms $R(j_1,\ldots,j_x)$ in order to deduce the random properties of the traffic flow in the network. In [2] we have described a simulation method allowing estimation of the quantities $R(j_1,\ldots,j_x)$. It has the following features:

(1) The suggested simulation makes it possible to study in one run, with the aid of formulae given, the effect of all cases of traffic values and traffic nature that can be envisaged, under stationarity as well as under nonstationarity. This latter case is particularly useful for studies of resistance against overloads due to traffic fluctuations.

(2) A single run is sufficient to produce a table of permissible traffic values, in accordance with a list of specified probabilities of blocking. So this same simulation in one run can produce the complete traffic tables for use in operation and planning.

Appendix

The following three theorems, proved in [2], are useful in the study of stationary limit processes. We assume the existence of the limits.

Theorem 2 in [2] permits us to state:

Theorem A.1 (key theorem). *Let $Q(t_1, ..., t_m)$ be a nonnegative function of each positive variable t_i such that the integral*

$$\int_0^\infty ... \int_0^\infty Q(t_1, ..., t_m)\, dt_1 ... dt_m \tag{A.1}$$

exists. If the limit

$$\lim_{t\to\infty} \frac{1}{t^m} \int_0^t ... \int_0^t \mathbf{E}\{dN(t_1) ... dN(t_m)\} = a_m \tag{A.2}$$

exists, then as $t \to \infty$,

$$H_m(t) \equiv \int_0^t ... \int_0^t Q(t - v_1, ..., t - v_m)\, \mathbf{E}\{dN(v_1) ... dN(v_m)\} \tag{A.3}$$

$$\to a_m \int_0^\infty ... \int_0^\infty Q(t_1, ..., t_m)\, dt_1 ... dt_m\,, \tag{A.4}$$

whenever this limit exists.

Here **E** denotes the expectation operator. Applied to the relation (4) we get the m^{th} factorial moment of $N(t)$. Finally the expression (A.2) is the limit of the m^{th} factorial moment of $[N(t)/t]$. Note that in view of (2) the condition (A.2) is certainly satisfied if $[N(t)/t]$ tends "almost certainly" to a limit stochastic variable as $t \to \infty$. (In fact, convergence in distribution is sufficient.)

This theorem provides a generalization of Smith's "key theorem" for renewal processes. If we apply this theorem to the function $Q(t_1 + u_1, ..., t_m + u_m)$, where $(t_1, ..., t_m)$ are given and positive, we get

$$\int_0^\infty ... \int_0^\infty Q(t_1 + u_1, ..., t_m + u_m)\, du_1 ... du_m = \int_{t_1}^\infty ... \int_{t_m}^\infty Q(v_1, ..., v_m)\, dv_1 ... dv_m$$

Theorem A.1 now allows us to formulate the following theorem (cf. [2, Theorem 3]).

Theorem A.2. *Let $Q(t_1, ..., t_m)$ be a nonnegative function of each positive variable t_i for which the integral* (A.1) *exists. If the limits* (A.2) *and* (A.4) *exist, then for t_0 positive and fixed, the function*

$$H_m(t, t_0) \equiv \int_0^t \dots \int_0^t Q(t_0 + t - v_1, \dots, t_0 + t - v_m) \, \mathrm{E}\{\mathrm{d}N(v_1) \dots \mathrm{d}N(v_m)\}$$
$$\to H_m(\infty, t) = H_m(\infty) \, K(t_0) \tag{A.5}$$

as $t \to \infty$, *where* $H_m(\infty)$ *is the limit* (A.4) *and*

$$K(t_0) = \int_{t_0}^{\infty} \dots \int_{t_0}^{\infty} Q(u_1, \dots, u_m) \, \mathrm{d}u_1 \dots \mathrm{d}u_m \Bigg/ \int_0^{\infty} \dots \int_0^{\infty} Q(u_1, \dots, u_m) \, \mathrm{d}u_1 \dots \mathrm{d}u_m \, . \tag{A.6}$$

In the same way, in the case of an arrival in the interval $(t, t + \mathrm{d}t)$, [2, Theorem 3.A] permits us to state the following theorem.

Theorem A.3. *Let* $Q(t_1, \dots, t_m)$ *be a nonnegative function, which is moreover assumed symmetrical with respect to the positive variables* t_i *such that the integral* (A.1) *exists. If the limit*

$$\lim_{t \to \infty} \left\{ \frac{1}{t^{m-1}} \int_0^t \dots \int_0^t \mathrm{E}\{\mathrm{d}N(u_1) \dots \mathrm{d}N(u_{m-1}) \, \mathrm{d}N(t)\} \right\} = b_{m-1} \tag{A.7}$$

exists, then as $t \to \infty$, *for fixed positive* t_0,

$$G_{m-1}(t, t_0) \equiv$$
$$\equiv \int_0^t \dots \int_0^t Q(t_0 + t - v_1, \dots, t_0 + t - v_{m-1}, t_0)$$
$$\times \, \mathrm{E}\{\mathrm{d}N(v_1) \dots \mathrm{d}N(v_{m-1}) \, \mathrm{d}N(t)\} \tag{A.8}$$

$$\to G_{m-1}(\infty, t_0) = b_{m-1} \int_{t_0}^{\infty} \dots \int_{t_0}^{\infty} Q(u_1, \dots, u_{m-1}, t_0) \, \mathrm{d}u_1 \dots \mathrm{d}u_{m-1} \tag{A.9}$$

whenever this limit exists.

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