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Hidden symmetries in minimal five-dimensional supergravity

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Abstract

We study the hidden symmetries arising in the dimensional reduction of $d=5$, $\mathcal{N}=2$ supergravity to three dimensions. Extending previous partial results for the bosonic part, we give a derivation that includes fermionic terms, shedding light on the appearance of the local hidden symmetry $SO(4)$ in the reduction.

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1. Introduction

Since their role in the greater scheme of M-theory was postulated [1], there has been intense renewed interest in the hidden symmetries of supergravity [2–4]. While most work has, understandably, centered on the maximal eleven-dimensional theory and its various dimensional reductions, with recent work ranging from the gauging of subgroups of the global exceptional groups [5,6] to the identification of new vacua [7] to possibilities for enlarging the hidden symmetries even further [8], there have also been results of a more general scope; notably the work showing in the most systematic manner yet how the hidden symmetries arise in successive dimensional reduction [9] has been generalized to other dimensional reductions to three dimensions [10].

The present work, whose results form part of the thesis [11], is concerned with the hidden symmetry arising from the reduction to three dimensions of (minimal) five-dimensional supergravity [4,12]. Since its inception, that model, in many ways a “little brother” to the eleven-dimensional theory, has been used time and again to learn more about its higher-dimensional kin. Recent examples include toy models of the M5 brane [13], cosmological models [14], and methods developed for the study of the U-dualities of M-theory [15]. Concerning the hidden symmetry in question, there are so far only partial results, namely, a construction [16] of the bosonic part of the model using a decomposition of $G_{2(+2)}$ with respect to $SL(3,\mathbb{R})$ and a corresponding construction by Cremmer et al. as part of the aforementioned more general study of reductions to three dimensions [10]. The fact that the hidden symmetry is $G_2/SO(4)$ already follows from results on four-dimensional gravity coupled to appropriate

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vector and hypermultiplets [17]. What has been missing, so far, is a complete analysis, notably one that includes the fermionic sector. The latter is interesting not only as another data point in an area for which, in contrast to the bosons, no systematic scheme yet exists, namely the relationship between the local extended symmetry and the dimensional reduction of the spinors, but also for another reason: in maximal supergravity, hidden symmetries have been successfully "lifted" to eleven dimensions [18], and recent work has uncovered tantalizing hints of "exceptional geometric structures" associated with these liftings [19]. The fermionic sector plays a crucial part in this type of lifting, and the results of this Letter are thus a prerequisite for a search for such "exceptional geometry" in five-dimensional gravity. The present Letter contains a construction of the $\mathfrak{g}_{2(+2)}/\mathfrak{so}(4)$ target model in three dimensions and then proceeds to a dimensional reduction of five-dimensional supergravity, including the fermionic sector, in which the emergence of the hidden symmetry is shown.

2. The $\mathfrak{g}_{2(+2)}$-model in $(2+1)$ dimensions

In this part, we construct the $\mathfrak{g}_{2(+2)}/\mathfrak{so}(4)$-supergravity in $2+1$ space–time dimensions. For the sigma-model part we use the conventions[1] (as well as some general formulae) of Marcus and Schwarz [20]. The basic fields of our model will be, firstly, scalars φ_i parametrizing the coset; as detailed in [20], they occur in the Lagrangian in the form of a Lie-algebra valued field P_μ and composite connection coefficients Q_μ; secondly, a dreibein field $e_\mu{}^\alpha$; finally, fermionic superpartners of these bosonic fields: a spin-1/2 field χ and a spin-3/2 (gravitino) field Ψ, respectively. To match degrees of freedom, we need an $\mathcal{N}=4$ extended supersymmetry so that, in addition to Lorentz symmetry, the gravitino transforms non-trivially under an additional R-symmetry.

Before the actual construction, we need to assign the different fields to their proper representations with respect to the internal symmetries involved, notably the local $\mathfrak{so}(4)$ symmetry of our model. In parallel with the $\mathfrak{e}_{8(+8)}$-case, one might think that the $\mathfrak{so}(4)$ R-symmetry would have to be identified with the local $\mathfrak{so}(4)$ symmetry from the coset construction; however, the situation is more complicated. If we assign the supersymmetry parameter ϵ (and hence the gravitino Ψ) to the vector representation of the R-symmetry $\mathfrak{so}(4)$, then from a general analysis of the supercharges as in [21] we must conclude that regarding the representations possible for the (massless) matter fields, one chiral subgroup, henceforth denoted $\mathfrak{so}(3)_F$, acts only on fermionic degrees on freedom, the other, $\mathfrak{so}(3)_B$, only on bosons. On the other hand, from the group theory literature [22] we know that the proper coset decomposition for the **14** representation of the $\mathfrak{g}_2$ is $\mathbf{14}=(\mathbf{1},\mathbf{3})+(\mathbf{3},\mathbf{1})+(\mathbf{4},\mathbf{2})$ (with the usual convention of denoting representations by their dimensions, and with the tuples on the right-hand side referring to the two chiral $\mathfrak{so}(3)$ components), so, in particular, the coset scalars transform non-trivially under both of the local $\mathfrak{so}(3)$. To resolve the problem, we need to introduce a third algebra, which we shall call $\mathfrak{so}(3)_2$. This symmetry included, there is indeed an assignment to representations of $\mathfrak{so}(3)_F\times\mathfrak{so}(3)_B\times\mathfrak{so}(3)_2$ that is consistent with the above requirements as well as with the form of the supersymmetry transformations (schematically, $\delta_S\Psi\sim D\epsilon$; $\delta_S\chi\sim\varphi\epsilon$; $\delta_S\varphi\sim\bar{\epsilon}\chi$), namely, with Ψ/ϵ in the representation $(\mathbf{2}_F,\mathbf{2}_B,\mathbf{1}_2)$, the φ transforming as $(\mathbf{1}_F,\mathbf{2}_B,\mathbf{4}_2)$, and the χ as $(\mathbf{2}_F,\mathbf{1}_B,\mathbf{4}_2)$.

Next, we need the algebra for the coset decomposition. We denote the indices of the two chiral components $\mathfrak{so}(3)_B$ and $\mathfrak{so}(3)_2$ of the maximal compact $\mathfrak{g}_2$ subalgebra $\mathfrak{so}(4)$ by $\bar{a},\bar{b},\ldots$ and $\dot{a},\dot{b},\ldots$, respectively. Decomposed with respect to representations of that subalgebra, an algebra element can be written as a contraction of coefficients with generators E, with the part in $\mathfrak{so}(3)_2$ given as $M^{\dot{a}}{}_{\dot{b}}E^{\dot{b}}{}_{\dot{a}}$, the $\mathfrak{so}(3)_B$ as $N^{\bar{a}}{}_{\bar{b}}E^{\bar{a}}{}_{\bar{b}}$ and the non-compact part as $Y^{\bar{a}\dot{a}\dot{b}\dot{c}}E_{\bar{a}\dot{a}\dot{b}\dot{c}}$. The commutators can be found in the usual way, by decomposing tensor products and imposing Jacobi identities. They are the usual matrix commutators for the $M^{\dot{a}}{}_{\dot{b}}$ and $N^{\bar{a}}{}_{\bar{b}}$ among themselves plus $[M,N]=0$

[1] As for space–time conventions, $\mu,\nu,\ldots$ are curved and $\alpha,\beta,\ldots$ flat space–time indices; our metric is "mostly plus"; our gamma matrices are real with $\gamma^0\gamma^1\gamma^2=+1$.

as well as

$$
\begin{aligned}
&[M,Y]^{\bar a\dot a\dot b\dot c} = 3Y^{\bar a\dot d(\dot a\dot b}M^{\dot c)}{}_{\dot d}, &&[Y,Y']^{\dot a}{}_{\dot b} = \big(Y'^{\bar a\dot a\dot c\dot d}Y_{\bar a\dot b\dot c\dot d} - Y^{\bar a\dot a\dot c\dot d}Y'_{\bar a\dot b\dot c\dot d}\big),\\
&[N,Y]^{\bar a\dot a\dot b\dot c} = N^{\bar a}{}_{\bar c}Y^{\bar c\dot a\dot b\dot c}, &&[Y,Y']^{\bar a}{}_{\bar b} = \big(Y'^{\bar a\dot a\dot b\dot c}Y_{\bar b\dot a\dot b\dot c} - Y^{\bar a\dot a\dot b\dot c}Y'_{\bar b\dot a\dot b\dot c}\big),
\end{aligned} \tag{1}
$$

where group indices are lowered by contraction with the rightmost index of totally antisymmetric $\varepsilon_{\dot a\dot b}$ or $\varepsilon_{\bar a\bar b}$ with $\varepsilon_{12} = +1$. Examination of the Killing form $24\,\mathrm{Tr}(MM') + 8\,\mathrm{Tr}(NN') - 16Y_{\bar a\dot a\dot b\dot c}Y'^{\bar a\dot a\dot b\dot c}$ shows that this defines the maximally non-compact $\mathfrak{g}_{2(+2)}$ if the generators are real and the coefficients satisfy symplectic reality conditions $(M^{\dot a}{}_{\dot b})^* = (M^*)_{\dot a}{}^{\dot b} = -M_{\dot a\dot d}\varepsilon^{\dot d\dot b}$, $(N^{\bar a}{}_{\bar b})^* = -N_{\bar a\bar d}\varepsilon^{\bar d\bar b}$ and $(Y^{\bar a\dot a\dot b\dot c})^* = -Y_{\bar a\dot a\dot b\dot c}$ (adopting the convention by which conjugation automatically shifts index positions).

Next, for the realization of the different $\mathfrak{so}(3)$-representations. Denoting the fundamental indices of $\mathfrak{so}(3)_F$ by $i, j, \ldots$, the assignment of representations leads to an index structure $\phi^{\bar a\dot a\dot b\dot c}$ for the scalars, $\Psi^{i\bar a}$ for gravitino and supersymmetry parameter, and $\chi^{i\dot a\dot b\dot c}$ for the matter fermions. The action of infinitesimal $\mathfrak{so}(3)$-transformations is fixed by linearity and by demanding for each two such transformations X, Y that $[\delta_X, \delta_Y] = \delta_{[Y,X]}$. To ensure consistency, both the $\mathfrak{so}(3)_F$ coefficients and the fields inherit the symplectic reality condition, $(\varphi^*)_{\bar a\,\dot a\dot b\dot c} = -\varphi_{\bar a\,\dot a\dot b\dot c}$, $(\chi^*)_{i\,\dot a\dot b\dot c} = -\chi_{i\,\dot a\dot b\dot c}$, $(\Psi^*)_{i\bar a} = -\Psi_{i\bar a}$. On the fermionic side, this makes fully contracted products of (anticommuting) spinors symmetric, with Clifford conjugation as their adjoint, e.g., $(\bar\chi_{i\,\dot a\dot b\dot c}\gamma^{\mu_1\cdots\mu_m}\zeta^{i\,\dot a\dot b\dot c}) = (-)^{m(m+1)/2}(\bar\zeta_{i\,\dot a\dot b\dot c}\gamma^{\mu_1\cdots\mu_m}\chi^{i\,\dot a\dot b\dot c})$. Introducing connection coefficients and $Q_\mu{}^{\bar a}{}_{\bar b}$ and $Q_\mu{}^{\dot a}{}_{\dot b}$ for the $\mathfrak{so}(3)_B$ and $\mathfrak{so}(3)_2$, respectively, we can define the action of a derivative covariant under these local symmetries, $(D_\mu(Q)\Psi)^{i\bar a} = \partial_\mu\Psi^{i\bar a} + Q_\mu{}^{\bar a}{}_{\bar b}\Psi^{i\bar b}$ and corresponding expressions for the other fields; replacing ∂_μ by the Lorentz-covariant $D_\mu(\omega) = \partial_\mu + \frac{1}{4}\gamma^{\alpha\beta}\omega_{\mu\alpha\beta}$, we obtain a derivative $D_\mu(\omega, Q)$ that is gauge- as well as Lorentz-covariant.

After these preparations, we can find the Lagrangian. We restrict ourselves to the terms that are necessary for the comparison with the dimensional-reduced theory, omitting quartic or higher fermionic terms in the Lagrangian. Starting with the fields' standard kinetic terms and supersymmetry variations, supersymmetry demands the inclusion of a Noether term and fixes ambiguities of the relative constants, with the resulting Lagrangian

$$
\begin{aligned}
\mathcal{L} = e\bigg\{&-\frac{1}{4\kappa^2}\mathcal{R} - \frac{\mathrm{i}}{2}\big(\bar\Psi_{\mu i\bar a}\gamma^{\mu\nu\rho}D_\nu(\omega,Q)\Psi_\rho^{i\bar a}\big) - \frac{1}{2\kappa^2}g^{\mu\nu}(P_\mu)_{\bar a\dot a\dot b\dot c}(P_\nu)^{\bar a\dot a\dot b\dot c}\\
&- \frac{\mathrm{i}}{4}\big(\bar\chi_{i\dot a\dot b\dot c}\gamma^\mu D_\mu(\omega,Q)\chi^{i\dot a\dot b\dot c}\big) + \frac{1}{\sqrt2}\big(\bar\chi_{i\dot a\dot b\dot c}\gamma^\rho\gamma^\mu\Psi_\rho^{i\bar b}\big)(P_\mu)^{\bar a\dot a\dot b\dot c}\varepsilon_{\bar a\bar b}\bigg\},
\end{aligned} \tag{2}
$$

invariant under the supersymmetry variations

$$
\begin{aligned}
&\delta_S e_\mu{}^\alpha = \mathrm{i}\kappa^2\big(\bar\epsilon_{i\bar a}\gamma^\alpha\Psi_\mu^{i\bar a}\big),\\
&\delta_S\Psi_\mu{}^{i\bar a} = -\big(D_\mu(\omega,Q)\epsilon\big)^{i\bar a} - (\Sigma_S)^{\bar a}{}_{\bar b}\Psi^{i\bar b},\\
&\delta_S\chi^{i\dot a\dot b\dot c} = \sqrt2\,\mathrm{i}\gamma^\mu(P_\mu)^{\bar a\dot a\dot b\dot c}\varepsilon_{\bar a\bar b}\epsilon^{i\bar b} - 3\chi^{i\dot d(\dot a\dot b}(\Sigma_S)^{\dot c)}{}_{\dot d},\\
&\delta_S(P_\mu)^{\bar a\dot a\dot b\dot c} = -\frac{\kappa^2}{\sqrt2}\varepsilon^{\bar a\bar b}D_\mu(Q)\big(\bar\epsilon_{i\bar b}\chi^{i\dot a\dot b\dot c}\big) - 3(P_\mu)^{\bar a\dot d(\dot a\dot b}(\Sigma_S)^{\dot c)}{}_{\dot d} - (\Sigma_S)^{\bar a}{}_{\bar b}(P_\mu)^{\bar b\dot a\dot b\dot c},\\
&\delta_S(Q_\mu)^{\dot a}{}_{\dot b} = D_\mu(Q)\Sigma_S{}^{\dot a}{}_{\dot b} - \frac{\kappa^2}{\sqrt2}(P_\mu)_{\bar a\dot f\dot c\dot d}\big(2\delta^{\dot a}_{\dot e}\delta^{\dot f}_{\dot b} - \delta^{\dot a}_{\dot b}\delta^{\dot f}_{\dot e}\big)\varepsilon^{\bar a\bar b}\big(\bar\epsilon_{i\bar b}\chi^{i\dot e\dot c\dot d}\big),\\
&\delta_S(Q_\mu)^{\bar a}{}_{\bar b} = D_\mu(Q)\Sigma_S{}^{\bar a}{}_{\bar b} - \frac{\kappa^2}{\sqrt2}(P_\mu)_{\bar e\dot a\dot b\dot c}\big(2\delta^{\bar e}_{\bar b}\varepsilon^{\bar a\bar d} - \delta^{\bar a}_{\bar b}\varepsilon^{\bar e\bar d}\big)\big(\bar\epsilon_{i\bar d}\chi^{i\dot a\dot b\dot c}\big),
\end{aligned} \tag{3}
$$

where Σ_S is the highly non-linear expression $\Sigma_S = -\tanh((1/2)\,\mathrm{ad}_\varphi)\varphi$ that is a consequence of describing the variation of group-valued objects in terms of algebra-valued objects [20, Section 2]. The coset-specific properties come into play in that the variation of the Rarita–Schwinger term contains $[D_\mu(Q), D_\nu(Q)]$, to be cancelled by a term proportional to $[P_\mu, P_\nu]$ arising from the variation of the Noether term with respect to χ.

3. The dimensional reduction from five to three dimensions

The stage is now set for the dimensional reduction from which we mean to recover the model found in the previous section. Starting point is the minimal $\mathcal{N}=2$ supergravity in five dimensions [4,12]. We choose a "mostly minus" metric with the gravitini symplectic-Majorana spinors, denote curved indices by $M, N, P, \ldots$ and flat indices by $A, B, C, \ldots$ and define tensorial epsilon symbols, obtained from an all-flat $\epsilon_{12345}=+1$ by applications of vielbein and metric. The fields in question are vielbeins $E_M{}^A$, gravitini Ψ_M^i and a one-form field A_M; the Lagrangian is

$$\mathcal{L}_{5|2} = -\frac{1}{4\kappa^2}E\mathcal{R} - \frac{1}{4}EF^2 - \epsilon_3\frac{1}{6\sqrt{3}}\big(E\varepsilon^{MNPQR}\big)\kappa A_M F_{NP}F_{QR} + \frac{\mathrm{i}}{2}E\big(\bar{\Psi}_{Mi}\Gamma^{MNP}D_N(\omega)\Psi_P^i\big) + \frac{\sqrt{3}}{8}\mathrm{i}\kappa E F_{MN}\big[2\big(\bar{\Psi}_i^M\Psi^{Ni}\big) + \big(\bar{\Psi}_{Pi}\Gamma^{MNPQ}\Psi_Q^i\big)\big]$$

(with quartic fermionic terms omitted and with $F_{MN} := 2\partial_{[M}A_{N]}$ the field strength), and it is invariant under supersymmetry variations

$$\delta E_M{}^A = \mathrm{i}\kappa^2\big(\bar{\epsilon}_i\Gamma^A\Psi_M^i\big), \qquad \delta\Psi^i = \hat{D}(\omega, F)\epsilon^i \quad \text{and} \quad \delta A = \frac{\sqrt{3}}{2}\mathrm{i}\kappa\big(\bar{\epsilon}_i\Psi^i\big), \tag{4}$$

where the supercovariant derivative $\hat{D}$ is defined by

$$\hat{D}_M(\omega, F)\epsilon := D_M(\omega)\epsilon + \frac{1}{4\sqrt{3}}\kappa(\Gamma_{ABC} + 4\eta_{BC}\Gamma_A)F^{AB}E_M{}^C\epsilon. \tag{5}$$

The initial steps of the dimensional reduction to three dimensions are fairly generic [2,4,23]: exploiting part of the Lorentz gauge freedom, certain vielbein components can be gauged to zero, allowing the decomposition

$$E_M{}^A = \begin{pmatrix} \Delta^{-1}e'_\mu{}^\alpha & B_\mu{}^m e_m{}^a \\ 0 & e_m{}^a \end{pmatrix}, \tag{6}$$

where $\Delta = \det e_m{}^a$ is a Weyl scaling factor. Here and in the following, curved indices are decomposed in the manner $M = (\mu, m)$, with μ a three-dimensional space–time index and m an index in the two-dimensional internal space; flat indices are split analogously as $A = (\alpha, a)$. We adopt the convention of splitting fields such as F or the gravitino starting from their (all-lowered) flat-indexed form; curved-index versions are then obtained by application of the component vielbeins $e'_\mu{}^\alpha$ and $e_m{}^a$.

The dimensional split of the anholonomy coefficients leads to non-zero expressions

$$\Omega_{\alpha\beta}{}^\gamma = \Delta\big[\Omega'_{\alpha\beta}{}^\gamma + 2e'_{[\alpha}{}^\mu\delta^\gamma_{\beta]}(\partial_\mu \ln\Delta)\big], \qquad \Omega_{\alpha\beta}{}^c = \Delta^2\Omega'_{\alpha\beta}{}^c, \qquad \Omega_{a\beta}{}^c = \Delta\Omega'_{a\beta}{}^c, \tag{7}$$

using Δ-independent primed coefficients defined as

$$\Omega'_{\alpha\beta}{}^\gamma := -2e'_{[\alpha}{}^\mu e'_{\beta]}{}^\nu\partial_\mu\big(e'_\nu{}^\gamma\big), \qquad \Omega'_{\alpha\beta}{}^c := -e'_\alpha{}^\mu e'_\beta{}^\nu G^n_{\mu\nu}e_n{}^c, \qquad \Omega'_{a\beta}{}^c := e'_\beta{}^\nu e_a{}^m\big(\partial_\nu e_m{}^c\big), \tag{8}$$

where $G^n_{\mu\nu} := \partial_\mu B_\nu{}^n - \partial_\nu B_\mu{}^n$ is the field strength of the Kaluza–Klein vector field.

On the fermionic side, the split is dimension-specific. The five-dimensional gamma matrices Γ^A are split into $\Gamma^\alpha = \gamma^\alpha\hat{\Gamma}^v$ and $\Gamma^a = \hat{\Gamma}^a$, with γ^α the three-dimensional matrices and $\hat{\Gamma}^a$ those of the two-dimensional internal space with signature $(-,-)$; for concreteness, we choose $\gamma^0 = i\epsilon_3\sigma_2$, $\gamma^1 = \sigma_1$, $\gamma^2 = \sigma_3$, $\hat{\Gamma}^1 = i\sigma_1$, $\hat{\Gamma}^2 = i\sigma_3$ and use the abbreviation $\hat{\Gamma}^v := \hat{\Gamma}^1\hat{\Gamma}^2$. The internal matrices $\hat{\Gamma}^a$ are $\mathfrak{so}(2)$-gamma matrices, but it is straightforward to use them to generate $\mathfrak{so}(3)$: the generators $\hat{\Gamma}^r$, $r = 1, 2, 3$, where $\hat{\Gamma}^r = (\hat{\Gamma}^a, \hat{\Gamma}^v)$, satisfy $\hat{\Gamma}^r\hat{\Gamma}^s = \varepsilon^{rst}\hat{\Gamma}_t - \delta^{rs}$, with $\varepsilon^{123} = -1$, the $\mathfrak{su}(2)$ algebra. In addition, the reality condition $((\hat{\Gamma}^r)^{\bar{a}}{}_{\bar{b}})^* = -\varepsilon_{\bar{a}\bar{c}}(\hat{\Gamma}^r)^{\bar{c}}{}_{\bar{d}}\varepsilon^{\bar{d}\bar{b}}$ (with $SO(2)$-spinor indices $\bar{a}, \bar{b}, \ldots$) which these matrices inherit is the same that is needed for the $\mathfrak{so}(3)$-generators in the $\mathfrak{g}_{2(+2)}$ decomposition introduced above. From Fierz identities for the $\hat{\Gamma}$'s Clifford algebra, one can derive useful relations such as

$$\begin{aligned}
&(\hat{\Gamma}^r)^{\bar{a}}{}_{\bar{b}}(\hat{\Gamma}_r)^{\bar{c}}{}_{\bar{d}} = 2\delta^{\bar{a}}_{\bar{d}}\delta^{\bar{c}}_{\bar{b}} - \delta^{\bar{a}}_{\bar{b}}\delta^{\bar{c}}_{\bar{d}}, \qquad \varepsilon_{rst}(\hat{\Gamma}^s)^{\bar{a}}{}_{\bar{b}}(\hat{\Gamma}^t)^{\bar{c}}{}_{\bar{d}} = \delta^{\bar{a}}_{\bar{d}}(\hat{\Gamma}_r)^{\bar{c}}{}_{\bar{b}} - \delta^{\bar{c}}_{\bar{b}}(\hat{\Gamma}_r)^{\bar{a}}{}_{\bar{d}},\\
&(\hat{\Gamma}^a)^{\bar{e}}{}_{[\bar{b}}(\hat{\Gamma}^v)^{\bar{d}}{}_{\bar{g}]} = (\hat{\Gamma}^v\hat{\Gamma}^a)^{\bar{e}}{}_{[\bar{b}}\delta^{\bar{d}}_{\bar{g}]},
\end{aligned} \tag{9}$$

to be exploited later on. Let us note one consequence of these relations, namely that, using a flat *Spin*(2)-invariant metric $\delta_{\bar{a}\bar{b}}$ to lower indices, one can derive

$$(\hat{\Gamma}^a)_{\bar{a}\bar{b}}(\hat{\Gamma}_a)_{\bar{c}\bar{d}} + (\hat{\Gamma}^a)_{\bar{d}\bar{b}}(\hat{\Gamma}_a)_{\bar{c}\bar{a}} = 2\delta_{\bar{a}\bar{d}}\delta_{\bar{b}\bar{c}}. \tag{10}$$

This is a Clifford relation, but with contraction over vector instead of spinorial indices, corresponding to identical relations for $SO(8)$ which are associated with the triality property of that group and are used in the construction of the $E_{8(+8)}/SO(16)$ model.

As for the five-dimensional gravitino, and adopting the convention to suppress (five- and three-dimensional) space–time spinor indices, the first decomposition is of Ψ^i_A into a three-dimensional spin-3/2 fermion $\Psi^{i\bar{a}}_\alpha$ and a spin-1/2 fermion $\Psi^{i\bar{a}}_a$. $\bar{a}$ can be promoted to an $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$-index: as far as internal spinor indices are concerned, the spinor product's adjoint is Hermitian conjugation, with an invariance group $U(2)$. The symplectic reality condition restricts this to $SU(2)$. However, there are problems: this decomposition leads both to a non-standard form for the threebein's supersymmetry variations and to mixed first-derivative terms between spin-3/2 and spin-1/2 degrees of freedom. The remedy is a redefinition $\Psi'^{i\bar{a}}_\mu = \Delta^{-1/2}[\gamma_\beta(\hat{\Gamma}^v\hat{\Gamma}^a)^{\bar{a}}{}_{\bar{b}}\Psi^{i\bar{b}}_a - \Psi^{i\bar{a}}_\beta]e'_\mu{}^\beta$; with this field as the three-dimensional gravitino and $e'_\mu{}^\beta$ as vielbein, the dimensional reduction reproduces both the gravitino kinetic term (with the mixed terms now absent) and the vielbein supersymmetry variation given in (2) and (3), respectively. From this, it is tempting to identify the gravitino's index $\bar{a}$ as an index of $\mathfrak{so}(3)_B$. However, matters are more complicated, which can be seen by considering the spin-1/2 fields. To start with, they have the index structure $\Psi^{i\bar{a}}_a$, where $\bar{a}$ can be promoted to an $\mathfrak{so}(3)$ index, as before. However, from the construction in Section 2 we know that the matter fermions transform non-trivially only with respect to $\mathfrak{so}(3)_2$ and $\mathfrak{so}(3)_F$. This apparent problem can be resolved once it is realized that in similar situations, notably in the E_8 case, dimensional reduction leads to models in which the enhanced local symmetry is gauge-fixed, with expressions, e.g., for the P_μ and Q_μ that are not explicitly covariant under that symmetry. Our case is analogous in that, apparently, the dimensionally reduced model "sees" only the diagonal $\mathfrak{so}(3)$ subgroup of $\mathfrak{so}(3)_B \times \mathfrak{so}(3)_F$. To restore the enhanced symmetry, that diagonal group needs to be disentangled into its $\mathfrak{so}(3)_B \times \mathfrak{so}(3)_F$ parts, using the model developed in Section 2 as a guide. This promotes the $\bar{a}$ index of $\Psi'^{i\bar{a}}_\mu$ to an $\mathfrak{so}(3)_B$ index, and the internal spinor index of the spin-1/2 fields to an $\mathfrak{so}(3)_2$ index. For the latter, the kinetic term should have the same simple form as shown in (2). This can be achieved by exploiting the freedom to redefine $\Psi^{i\dot{a}}_a \to \Psi^{i\dot{a}}_a + (\hat{\Gamma}_a\hat{\Gamma}^c)^{\dot{a}}{}_{\dot{c}}\Psi^{i\dot{c}}_c$. All in all, new matter fermion fields defined as

$$\chi^{i\dot{a}\dot{b}\dot{c}} = \Delta^{-1/2}\Psi^{i\dot{e}}_c\big[\delta^c_d\delta^{(\dot{a}}_{\dot{e}} + (\hat{\Gamma}_d\hat{\Gamma}^c)^{(\dot{a}}{}_{\dot{e}}\big](\hat{\Gamma}^d)^{\dot{b}}{}_{\dot{d}}\varepsilon^{\dot{c})\dot{d}} \tag{11}$$

have the required properties. It can be checked directly that the fermions thus defined have inherited the correct symplectic reality condition. With these preparations, the terms quadratic in Ψ' that are obtained from the five-dimensional Rarita–Schwinger term combine into the three-dimensional enhanced Rarita–Schwinger term with gauge-fixed connection coefficients

$$Q_\nu{}^{\bar{a}}{}_{\bar{b}} := -\frac{1}{4}e'_\nu{}^\alpha\big\{(\hat{\Gamma}^{de})^{\bar{a}}{}_{\bar{b}}\Omega'_{\alpha de} + \epsilon_3\Delta(\hat{\Gamma}^v\hat{\Gamma}^e)^{\bar{a}}{}_{\bar{b}}\Omega'_{\alpha e} + 2\sqrt{3}\kappa\Delta^{-1}\big[\epsilon_3(\hat{\Gamma}^v)^{\bar{a}}{}_{\bar{b}}F_\alpha + (\hat{\Gamma}^d)^{\bar{a}}{}_{\bar{b}}F_{\alpha d}\big]\big\}, \tag{12}$$

while the corresponding connection coefficients in the kinetic term of the spin-1/2 fermions turn out to be

$$Q_\nu{}^{\dot{a}}{}_{\dot{d}} = -\frac{1}{4}e'_\nu{}^\beta\bigg[(\hat{\Gamma}^v)^{\dot{a}}{}_{\dot{d}}\bigg(\varepsilon^{cd}\Omega'_{\beta cd} + \frac{2}{\sqrt{3}}\epsilon_3\kappa\Delta^{-1}F_\beta\bigg) - (\hat{\Gamma}^c)^{\dot{a}}{}_{\dot{g}}\bigg(\Delta\epsilon_3(\hat{\Gamma}^v)^{\dot{g}}{}_{\dot{d}}\Omega'_{\beta c} - \frac{2}{\sqrt{3}}\kappa\Delta^{-1}\delta^{\dot{g}}_{\dot{d}}F_{\beta c}\bigg)\bigg]. \tag{13}$$

The mixed term containing both Ψ' and χ, compared with the Noether term in the three-dimensional target Lagrangian, yield

$$(P_\mu)^{\bar{a}\dot{d}\dot{e}\dot{f}} = -\frac{\mathrm{i}}{2\sqrt{2}}\varepsilon^{\bar{a}\dot{b}}e'_\mu{}^\alpha\bigg\{\frac{1}{2}\Delta\epsilon_3\Omega'_{\alpha c}\big[\delta_{\dot{b}}^{(\dot{d}}\big(\hat{\Gamma}^c\big)^{\dot{e}}{}_{\dot{g}}\varepsilon^{\dot{f})\dot{g}} + \big(\hat{\Gamma}^v\hat{\Gamma}^c\big)^{(\dot{d}}{}_{\dot{b}}\big(\hat{\Gamma}^v\big)^{\dot{e}}{}_{\dot{g}}\varepsilon^{\dot{f})\dot{g}}\big]$$
$$+\big[\big(\hat{\Gamma}^v\big)^{(\dot{d}}{}_{\dot{g}}\delta_{\dot{b}}^{\dot{e}}\varepsilon^{\dot{f})\dot{g}}\Omega'_{\alpha d}{}^d + \big(\hat{\Gamma}^v\hat{\Gamma}^c\big)^{(\dot{d}}{}_{\dot{b}}\big(\hat{\Gamma}^b\big)^{\dot{e}}{}_{\dot{g}}\varepsilon^{\dot{f})\dot{g}}\Omega'_{\alpha(bc)}\big]$$
$$+\sqrt{3}\kappa\Delta^{-1}\big[F_{\alpha c}\big(\hat{\Gamma}^c\big)^{(\dot{d}}{}_{\dot{g}}\big(\hat{\Gamma}^v\big)^{\dot{e}}{}_{\dot{b}}\varepsilon^{\dot{f})\dot{g}} + \epsilon_3F_\alpha\big(\hat{\Gamma}^v\big)^{(\dot{d}}{}_{\dot{b}}\big(\hat{\Gamma}^v\big)^{\dot{e}}{}_{\dot{g}}\varepsilon^{\dot{f})\dot{g}}\big]\bigg\}. \tag{14}$$

By their index structure, it can be checked directly that these objects transform under the proper representations of $\mathfrak{so}(3)_2 \times \mathfrak{so}(3)_B$; by using the reality conditions for the $\hat{\Gamma}$, that they satisfy the proper reality conditions.

To complete the match, we present three consistency checks. From (3), it follows that $Q_\nu{}^{\bar{a}}{}_{\bar{b}}$ must occur in the new gravitino's supersymmetry variations, while one should also be able to read off $(P_\mu)^{\bar{a}\dot{d}\dot{e}\dot{f}}$ from the matter fermion's supersymmetry variation. Both expressions agree with those derived above. The final check is the match of the dimensionally reduced bosonic Lagrangian with the sigma-model kinetic term in terms of the $(P_\mu)^{\bar{a}\dot{d}\dot{e}\dot{f}}$ of Eq. (14); the same check used in [16] for the $\mathfrak{g}_2$-construction in terms of $\mathfrak{sl}(3)$ representations. The sigma-model term is

$$-\frac{1}{2\kappa^2}g^{\mu\nu}(P_\mu)_{\bar{a}\dot{a}\dot{b}\dot{c}}(P_\nu)^{\bar{a}\dot{a}\dot{b}\dot{c}} = e'\bigg\{-\frac{1}{16\kappa^2}(\partial_\nu\bar{g}_{mn})(\partial^\nu\bar{g}^{mn}) + \frac{1}{4\kappa^2}(\partial_\nu\ln\Delta)(\partial^\nu\ln\Delta) + \frac{\Delta^2G^2}{16\kappa^2}$$
$$-\frac{1}{2}(\partial_\mu A_m)(\partial^\mu A_n)\bar{g}^{mn} - \frac{1}{4}\Delta^{-2}(F')^2\bigg\}, \tag{15}$$

while the reduction of the bosonic terms gives the three-dimensional Einstein–Hilbert term plus

$$e'\bigg\{\frac{\Delta^2G^2}{16\kappa^2} + \frac{1}{16\kappa^2}(\partial^\mu\bar{g}_{mn})(\partial_\mu\bar{g}^{mn}) - \frac{1}{4\kappa^2}(\partial^\mu\ln\Delta)(\partial_\mu\ln\Delta) - \frac{1}{4}\Delta^{-2}(F')^2$$
$$+\frac{1}{2}\bar{g}^{mn}(\partial_\mu A_m)(\partial_\nu A_n) - \epsilon_3\kappa\frac{1}{3\sqrt{3}}\varepsilon^{\mu\nu\rho}\varepsilon^{mn}A_m(\partial_\rho A_n)\big[3F'_{\mu\nu} + \Delta^2G^p_{\mu\nu}A_p\big]\bigg\}. \tag{16}$$

That there is no match comes as no surprise, as it is a general feature of hidden symmetries to become manifest only upon dualization of appropriate dimensionally-reduced p-forms. In this case, the objects that allow dualization are the Kaluza–Klein field strength $G^m_{\mu\nu}$, dual to two scalars ξ_m, and a composite "field strength" $\tilde{F}_{\mu\nu} := \Delta^{-2}F'_{\mu\nu} - G^m_{\mu\nu}A_m$, tailor-made to fulfil the Bianchi identity and dual to a scalar φ. After dualization, the original Lagrangian (16) plus the constraint terms becomes

$$-\frac{1}{2}\Delta^{-2}g'^{\rho\lambda}\bigg((\partial_\rho\varphi) - \frac{2}{\sqrt{3}}\epsilon_3\Delta^2\kappa\varepsilon^{nr}A_n(\partial_\rho A_r)\bigg)\bigg((\partial_\lambda\varphi) - \frac{2}{\sqrt{3}}\epsilon_3\Delta^2\kappa\varepsilon^{nr}A_n(\partial_\lambda A_r)\bigg)$$
$$+2\kappa^2\Delta^{-2}g'^{\rho\lambda}\bigg((\partial_\rho\varphi)A^m - (\partial_\rho\xi_p)\bar{g}^{mp} + \frac{2}{3\sqrt{3}}\epsilon_3\kappa A^m\varepsilon^{pr}A_p(\partial_\rho A_r)\bigg)$$
$$\times\bigg((\partial_\rho\varphi)A_m - (\partial_\rho\xi_m) + \frac{2}{3\sqrt{3}}\epsilon_3\kappa A_m\varepsilon^{pr}A_p(\partial_\rho A_r)\bigg), \tag{17}$$

which is the same as the P^2-Lagrangian (15) upon substitution of the dualized entities. This completes our cross-checks.

There are a number of possible directions for extending the present results. The possibility of exploring "exceptional geometries" has already been mentioned; another interesting question would be to what happens to the hidden symmetry if the R-symmetry or some subgroup thereof is gauged [24] (making contact with recent, more general, studies of the possible gaugings in three dimensions [5]) or to study the case of compactification on $AdS_3 \times S^2$, for which the spectrum has already been worked out in [25].

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