Available online at www.sciencedirect.com
ScienceDirect
Linear Algebra and its Applications 422 (2007) 808–823
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

The Chevalley group $G_2(2)$ of order 12096 and the octonionic root system of E_7

Mehmet Koca [a], Ramazan Koç [b,*], Nazife Ö. Koca [c]

[a] Department of Physics, College of Science, Sultan Qaboos University, P.O. Box 36, Al-Khowd 123, Muscat, Oman
[b] Department of Physics, Faculty of Engineering, Gaziantep University, 27310 Gaziantep, Turkey
[c] Higher College of Technology, Al-Khuwair, Muscat, Oman

Received 20 March 2006; accepted 17 December 2006
Available online 3 January 2007
Submitted by H. Schneider

Abstract

The octonionic root system of the exceptional Lie algebra E_8 has been constructed from the quaternionic roots of F_4 using the Cayley–Dickson doubling procedure where the roots of E_7 correspond to the imaginary octonions. It is proven that the automorphism group of the octonionic root system of E_7 is the adjoint Chevalley group $G_2(2)$ of order 12096. One of the four maximal subgroups of $G_2(2)$ of order 192 preserves the quaternion subalgebra of the E_7 root system. The other three maximal subgroups of orders 432; 192 and 336 are the automorphism groups of the root systems of the maximal Lie algebras $E_6 \times U(1)$, $SU(2) \times SO(12)$ and $SU(8)$ respectively. The 7-dimensional manifolds built with the use of these discrete groups could be of potential interest for the compactification of the M-theory in 11-dimension.

PACS: 02.20.Bb; 21.60.Fw; 31.15.Hz

Keywords: Group structure; Quaternions; Subgroup structure; M-Theory

1. Introduction

The Chevalley groups are the automorphism groups of the Lie algebras defined over the finite fields [1]. The group $G_2(2)$ is the automorphism group of the Lie algebra g_2 defined over the

* Corresponding author. Tel.: +90 342 360 1200; fax: +90 342 360 1013.
E-mail address: koc@gantep.edu.tr (R. Koç).

0024-3795/$ - see front matter
doi:10.1016/j.laa.2006.12.011

finite field F_2 which is one of the finite subgroups of the Lie group G_2 [2]. Here we prove that it is the automorphism group of the octonionic root system of the exceptional Lie group E_7.

A few words are in order as to how the exceptional Lie groups are used in physics. The exceptional Lie groups are fascinating symmetries arising as groups of invariants of many physical models suggested for fundamental interactions. In the sequel of grand unified theories (GUT's) after $SU(5) \approx E_4$ [3], $SO(10) \approx E_5$ [4] the exceptional group E_6 [5] has been suggested as the largest GUT for a single family of quarks and leptons. The 11-dimensional supergravity theory admits an invariance of the non-compact version of $E_7[E_{7(-7)}]$ with a compact subgroup $SU(8)$ as a global symmetry [6]. The largest exceptional group E_8, originally proposed as a grand unified theory [7] allowing a three family interaction of E_6 has naturally appeared in the heterotic string theory as the $E_8 \times E_8$ gauge symmetry [8].

The infinite tower of the spin representations of $SO(9)$, the little group of the 11-dimensional M-theory, seems to be unified in the representations of the exceptional group F_4 [9]. Moreover, it has been recently shown that the root system of F_4 can be represented with discrete quaternions whose automorphism group is the direct product of two binary octahedral groups of order $48 \times 48 = 2304$ [10].

The smallest exceptional group G_2, the automorphism group of octonion algebra, turned out to be the best candidate as a holonomy group of the 7-dimensional manifold for the compactification of M-theory [11]. For a "topological M-theory" [12] one may need a crystallographic structure in 7 dimensions. In this context the root lattices of the Lie algebras of rank 7 may play some role, such as those of $SU(8)$, E_7 and the other root lattices of rank-7 Lie algebras. The $SU(8)$ is a maximal subgroup of E_7 therefore it is tempting to study the E_7 root lattice. Here a miracle happens! The root system of E_7 can be described by the imaginary discrete octonions [13]. The Weyl group $W(E_7)$ is isomorphic to a finite subgroup of O(7) which is the direct product $Z_2 \times SO_7(2)$ where the latter group is the adjoint Chevalley group of order $2^9 \cdot 3^4 \cdot 5 \cdot 7$ [14]. However, the Weyl group $W(E_7)$ does not preserve the octonion algebra. When one imposes the invariance of the octonion algebra on the transformations of the E_7 roots one obtains a finite subgroup of G_2, as expected, the adjoint Chevalley group $G_2(2)$ of order 12096 [2,15,16].

In what follows we discuss the mathematical structure of the adjoint Chevalley group $G_2(2)$ using the 126 non-zero octonionic roots of E_7 without referring to its matrix representation [2,16]. First we construct the root system of E_8 from the quaternionic roots of F_4 *àla* Cayley–Dickson doubling procedure. The roots of E_7 are represented by imaginary octonions which can be constructed by doubling the quaternionic roots of $SP(3)$ and F_4. First we determine a group of order 192 which preserves the quaternion subalgebra in the root system of E_7 and prove that this is a maximal subgroup $G_2(2)$ which can be embedded in the larger group 63 different ways. We also determine three other maximal subgroups of orders 432;192;336 respectively corresponding to the automorphism groups of the octonionic root systems of the maximal Lie algebras E_6, $SU(2) \times SO(12)$, $SU(8)$ and study them in some depth. The root system of $SU(8)$ has a fascinating geometrical structure where the roots can be decomposed as 7 hyperoctahedra in 4 dimensions which are permuted to each other by one of the generators of the Klein's group $PSL_2(7)$.

The paper is organized as follows. In Section 2 we construct the octonionic roots of E_8 [13,17] using the two sets of quaternionic roots of F_4 which follows the magic square structure [18] where imaginary octonions represent the roots of E_7. We build up a maximal subgroup of $G_2(2)$ of order 192 which preserves the quaternionic decomposition of the octonionic roots of E_7. It is a finite subgroup of $SO(4)$. Section 3 is devoted to a discussion on the embeddings of the group of order

192 in the $G_2(2)$. In Section 4 we study the maximal subgroups of $G_2(2)$ and their relevance to the root systems of the maximal Lie algebras of E_7.

2. Octonionic root system of E_8

In this section we give a brief review of what has been obtained in various references and prove a lemma that the group $G_2(2)$ has a maximal subgroup of order 192 which preserves the quaternionic subalgebra of the octonionic roots of E_7.

In Ref. [13] we have shown that the octonionic root system of E_8 can be constructed by doubling two sets of quaternionic root system of F_4 [11] via Cayley–Dickson procedure. Symbolically we can write,

$$(F_4, F_4) = E_8, \tag{1}$$

where the short roots of F_4 match with the short roots of the second set of F_4 roots and the long roots match with the zero roots. Actually (1) follows from the magic square given by Table 1. The quaternionic scaled roots of F_4 can be given by

$$F_4 : T \oplus \frac{T'}{\sqrt{2}}, \tag{2}$$

where $T \oplus T'$ is the set of elements of the binary octahedral group. Compactly,

$$\begin{aligned} T &= \{V_0 \oplus V_+ \oplus V_-\}, \\ T' &= \{V_1 \oplus V_2 \oplus V_3\}. \end{aligned} \tag{3}$$

More explicitly, the sets of quaternions V_0, V_+ and V_-, read

$$\begin{aligned} V_0 &= \{\pm 1, \pm e_1, \pm e_2, \pm e_3\}, \\ V_+ &= \frac{1}{2}\{\pm 1 \pm e_1 \pm e_2 \pm e_3\}, \text{ even number of } (-) \text{ signs}, \\ V_- = \overline{V}_+ &= \frac{1}{2}\{\pm 1 \pm e_1 \pm e_2 \pm e_3\}, \text{ odd number of } (-) \text{ signs}, \end{aligned} \tag{4}$$

where $\overline{V}_+$ is the quaternionic conjugate of V_+, and the sets V_1, V_2 and V_3 are given by

$$\begin{aligned} V_1 &= \left\{\frac{1}{\sqrt{2}}(\pm 1 \pm e_1), \frac{1}{\sqrt{2}}(\pm e_2 \pm e_3)\right\}, \\ V_2 &= \left\{\frac{1}{\sqrt{2}}(\pm 1 \pm e_2), \frac{1}{\sqrt{2}}(\pm e_3 \pm e_1)\right\}, \\ V_3 &= \left\{\frac{1}{\sqrt{2}}(\pm 1 \pm e_3), \frac{1}{\sqrt{2}}(\pm e_1 \pm e_2)\right\}. \end{aligned} \tag{5}$$

Here e_i $(i = 1, 2, 3)$ are the imaginary quaternionic units.

Table 1
Magic square

	$SU(3)$	$SP(3)$	F_4
$SU(3)$	$SU(3) \times SU(3)$	$SU(6)$	E_6
$SP(3)$	$SU(6)$	$SO(12)$	E_7
F_4	E_6	E_7	E_8

The set T denotes quaternionic elements of the binary tetrahedral group which represents the root system of $SO(8)$ and $\frac{T'}{\sqrt{2}}$ represents the weights of the three 8-dimensional representations of $SO(8)$ or, equivalently, T and $\frac{T'}{\sqrt{2}}$ represent the long and short roots of F_4 respectively. The geometrical meaning of these vectors are also interesting [19]. Here each of the sets V_0, V_+, V_- represents a hyperoctahedron in 4-dimensional Euclidean space. The set T is also known as a polytope {3, 4, 3} called a 24-cell [20]. Its dual polytope is T' where V_i $(i = 1, 2, 3)$ are the duals of the octahedron in T. Any two of the sets V_0, V_+, V_- form a hypercube in 4-dimensions. Using the Cayley–Dickson doubling procedure one can construct the octonionic roots of E_8 as follows:

$$\begin{aligned}
&(T, 0) = T, \quad (0, T) = e_7 T,\\
&\left(\frac{V_1}{\sqrt{2}}, \frac{V_1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(V_1 + e_7 V_1),\\
&\left(\frac{V_2}{\sqrt{2}}, \frac{V_3}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(V_2 + e_7 V_3),\\
&\left(\frac{V_3}{\sqrt{2}}, \frac{V_2}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(V_3 + e_7 V_2),
\end{aligned} \tag{6}$$

where e_1, e_2 and e_7 are the basic imaginary units used to construct the other units of octonions $1, e_1, e_2, e_3 = e_1 e_2, e_4 = e_7 e_1, e_5 = e_7 e_2, e_6 = e_7 e_3$. They satisfy the algebraic identity

$$e_i e_j = -\delta_{ij} + \phi_{ijk} e_k, \quad (i, j, k = 1, 2, \ldots, 7),$$

where ϕ_{ijk} is totally anti-symmetric under the interchange of the indices i, j, k and takes the values +1 for the indices 123, 246, 435, 367, 651, 572, 714 [21]. The set of E_8 roots in (6) can also be compactly written as the sets of octonions

$$\pm 1, \frac{1}{2}(\pm 1 \pm e_a \pm e_b \pm e_c), \tag{7a}$$

$$\pm e_i \ (i = 1, 2, \ldots, 7), \frac{1}{2}(\pm e_d \pm e_f \pm e_g \pm e_h), \tag{7b}$$

where the indices take the forms $abc = 123$, 156, 147, 245, 267, 346, 357 and $dfgh = 1246$, 1257, 1345, 1367, 2356, 2347, 4567. When ± 1 represents the non-zero roots of $SU(2)$ the imaginary roots in (7b) which are orthogonal to ± 1 represent the roots of E_7. The decomposition of the roots in (7a)–(7b) represents the branching of E_8 under its maximal subalgebra $SU(2) \times E_7$ where the 112 roots in (7a) are the weights $(\underline{2}, \underline{56})$.

A subset of roots of F_4 consisting of imaginary quaternions constitutes the roots of subalgebra $SP(3)$ with the roots as follows

$$\begin{aligned}
&\text{long roots}: \ V_0' = \{\pm e_1, \pm e_2 \pm e_3\},\\
&\text{short roots}: \ \frac{V_1'}{\sqrt{2}} = \left\{\frac{1}{2}(\pm e_2 \pm e_3)\right\},\\
&\qquad \frac{V_2'}{\sqrt{2}} = \left\{\frac{1}{2}(\pm e_3 \pm e_1)\right\}, \quad \frac{V_3'}{\sqrt{2}} = \left\{\frac{1}{2}(\pm e_1 \pm e_2)\right\}.
\end{aligned} \tag{8}$$

From the magic square one can also write the roots of E_7 in the form $(SP(3), F_4)$ consisting of only imaginary octonions which can further be put in the form

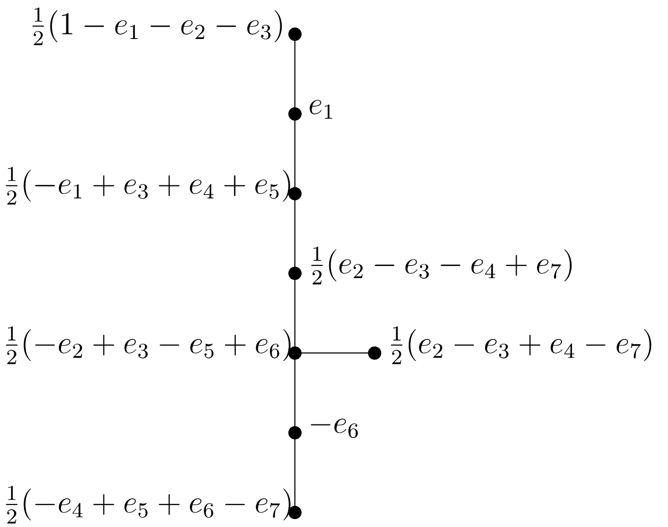

Fig. 1. The Coxeter–Dynkin diagram of E_8 with quaternionic simple roots.

$$\begin{aligned}
&(V_0', 0) = V_0', \quad (0, T) = e_7 T, \\
&\left(\frac{V_1'}{\sqrt{2}}, \frac{V_1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(V_1' + e_7 V_1), \\
&\left(\frac{V_2'}{\sqrt{2}}, \frac{V_3}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(V_2' + e_7 V_3), \\
&\left(\frac{V_3'}{\sqrt{2}}, \frac{V_2}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}(V_3' + e_7 V_2).
\end{aligned} \tag{9}$$

The roots in (9) also follow from a Coxeter–Dynkin diagram of E_8 where the simple roots represented by octonions depicted in Fig. 1. As we stated in the introduction, the automorphism group of octonionic root system of E_7 is the adjoint Chevalley group $G_2(2)$, a maximal subgroup of the Chevalley group $SO_7(2)$. Below we give a proof of this assertion and show how one can construct the explicit elements of $G_2(2)$ without any reference to a computer calculation of the matrix representation.

Theorem 1. *The set of transformations which preserve the octonion algebra of the root system of E_7 in (9) is the adjoint Chevalley group $G_2(2)$.*

Before we proceed further we introduce the well known theorem [22] which states that the set of automorphisms of octonions that take the quaternions H to itself forms a group $[p, q]$, isomorphic to

$$SO(4) \approx \frac{SU(2) \times SU(2)}{Z_2},$$

which is the maximal subgroup of the Lie group G_2. Here p and q are unit quaternions. In a different work [23] we have studied some finite subgroups of O(4) generated by the transformations

$$[p, q] : r \to prq,$$
$$[p, q]^* : r \to p\bar{r}q, \tag{10}$$

where $[p, q]$ represents an $SO(4)$ transformation preserving the norm $r\bar{r} = \bar{r}r$ of the quaternion r. In Ref. [22] it has been proven that the group element $[p, q]$ acts on the octonion represented as a Cayley–Dickson double of quaternion as follows:

$$[p, q] : H + e_7 H \to pH\bar{p} + e_7 pHq. \tag{11}$$

Here H represents an arbitrary quaternion. The theorem in [22] states that an $SO(4)$ transformation in the form $[p, q]$ preserves the Cayley–Dickson double $H + e_7 H$. Now we are in a position to use this theorem to prove that the transformations on the root system of E_7 in (10) preserving the quaternion subalgebra form a finite subgroup of $SO(4)$ of order 192.

Lemma 1. *The set of transformations preserving the quaternion decomposition in the root system of E_7 in (9) is a finite subgroup of $SO(4)$ of order* 192.

In Ref. [10] we have shown that the maximal finite subgroup of $SO(3)$ which preserves the set of quaternions $V_0' = \{\pm e_1, \pm e_2, \pm e_3\}$ representing the long roots of $SP(3)$ as well as the vertices of an octahedron is the octahedral group written in the form $[t, \bar{t}] \oplus [t', \bar{t}']$ where $t \in T$ and $t' \in T'$. This group also preserves the sets of quaternions V_1', V_2', V_3'. On the other hand it can be proven that $e_7 T \to e_7 T$ under the transformations $[p, q] \oplus [p', q']$, $(p, q \in T;\ p', q' \in T')$. A proof is given by Conway and Smith in Ref. [22]. Here the set of elements $[p, q]$ form a group of order 288 since T has 24 quaternions including their negatives. Similarly $[p', q']$ consists of another set of 288 quaternions. Therefore the largest group preserving the structure $(V_0', 0) = V_0'$, $(0, T) = e_7 T$ is a finite subgroup of $SO(4)$ of order $288 + 288 = 576$. We will see that actually we look for a subgroup of this group because it should also preserve the set of roots

$$\frac{1}{\sqrt{2}}(V_1' + e_7 V_1),\ \frac{1}{\sqrt{2}}(V_2' + e_7 V_3),\ \frac{1}{\sqrt{2}}(V_3' + e_7 V_2) \tag{12}$$

as well as keeping the form of (11) invariant.

The multiplication table shown in Table 2 for the elements of the binary octahedral group [19] will be useful to follow the details of the proof. Now we look for the transformation (11) acting on the roots in (12) and seek the form of $[p, q]$ which preserves (12). More explicitly, we look for the invariance

Table 2
Multiplication table of the binary octahedral group

	V_0	V_+	V_-	V_1	V_2	V_3
V_0	V_0	V_+	V_-	V_1	V_2	V_3
V_+	V_+	V_-	V_0	V_3	V_1	V_2
V_-	V_-	V_0	V_+	V_2	V_3	V_1
V_1	V_1	V_2	V_3	V_0	V_+	V_-
V_2	V_2	V_3	V_1	V_-	V_0	V_+
V_3	V_3	V_1	V_2	V_+	V_-	V_0

$$\begin{aligned}&\frac{1}{\sqrt{2}}(pV_1'\bar{p}+e_7pV_1q)\oplus\frac{1}{\sqrt{2}}(pV_2'\bar{p}+e_7pV_3q)\oplus\frac{1}{\sqrt{2}}(pV'\bar{p}_3+e_7pV_2q)\\&\quad=\frac{1}{\sqrt{2}}(V_1'+e_7V_1)\oplus\frac{1}{\sqrt{2}}(V_2'+e_7V_3)\oplus\frac{1}{\sqrt{2}}(V_3'+e_7V_2).\end{aligned}\tag{13}$$

We should check all pairs in $[V_0\oplus V_+\oplus V_-, V_0\oplus V_+\oplus V_-]$ and see that only the set of elements $[V_0, V_0]$, $[V_+, V_0]$, $[V_-, V_0]$ satisfy the relation (13). This follows from the Table 2. Just to see why $[V_+, V_+]$, for example, does not work let us apply it to the set of roots $\frac{1}{\sqrt{2}}(V_1'+e_7V_1)$:

$$[V_+, V_+]: \frac{1}{\sqrt{2}}(V_1'+e_7V_1)\to\frac{1}{\sqrt{2}}(V_+V_1'V_-+e_7V_+V_1V_+).$$

Using Table 2 we obtain that

$$[V_+, V_+]: \frac{1}{\sqrt{2}}(V_1'+e_7V_1)\to\frac{1}{\sqrt{2}}(V_2'+e_7V_1),$$

which does not belong to the set of roots of E_7. Similar considerations eliminate all the subsets of elements in $[T, T]$ but leave only the elements $[V_0, V_0]$, $[V_+, V_0]$, $[V_-, V_0]$. Note that $[V_+, V_0]^3=[V_0, V_0]$ and it permutes the three sets of roots of E_7 in (12). Now we study the action of $[T', T']$ on the roots in (12). The set of elements $[V_1, V_1]$ does the job:

$$\begin{aligned}[V_1, V_1]:&\frac{1}{\sqrt{2}}(V_1'+e_7V_1)\to\frac{1}{\sqrt{2}}(V_1'+e_7V_1)\\&\frac{1}{\sqrt{2}}(V_2'+e_7V_3)\leftrightarrow\frac{1}{\sqrt{2}}(V_3'+e_7V_2).\end{aligned}\tag{14}$$

We can check easily that the set of elements $[V_2, V_1]$ and $[V_3, V_1]$ also satisfy the requirements. Note that $[V_i, V_1]^2=[V_0, V_0]$, $(i=1,2,3)$; any one of these sets of group elements, while preserving one set of roots in (12), exchanges the other two. This concludes the proof that the set of elements $[T, V_0]\oplus[T', V_1]$ forms a group of order 192 and that it has 17 conjugacy classes.

It is interesting to note that $[V_0, V_0]$ is an invariant subgroup of order 32 of the group $[T, V_0]\oplus[T', V_1]$ where the factor group is isomorphic to the symmetric group S_3 of order 6. The set of elements $[T, V_0]\oplus[T', V_1]$ now can be written as the union of cosets of $[V_0, V_0]$ where the coset representatives can be obtained from, say, $[V_+, V_0]$ and $[V_1, V_1]$. When $[V_0, V_0]$ is taken as a unit element then $[V_+, V_0]$ and $[V_1, V_1]$ generate a group isomorphic to the symmetric group S_3. Symbolically, the group of interest can be written as the semi-direct product of the group $[V_0, V_0]$ with S_3 which is a maximal subgroup of the group of order 576.

It is also interesting to note that the group $[T, T]\oplus[T', T']$ has another maximal subgroup of order 192 with 13 conjugacy classes whose elements can be written as

$$[V_0, V_0]\oplus[V_+, V_-]\oplus[V_-, V_+]\oplus[V_1, V_1]\oplus[V_2, V_2]\oplus[V_3, V_3].\tag{15}$$

This group does not preserve the root system of E_7. However, it preserves the quaternion algebra in the set of imaginary octonions $\pm e_i$ $(i=1,2,\ldots,7)$. This is also an interesting group which turns out to be maximal in an another finite subgroup of $G_2(2)$ of order 1344 [24]. The group in (15) can also be written as the semi-direct product of $[V_0, V_0]$ and S_3. However, two groups are not isomorphic because the symmetric group S_3 here is generated by $[V_+, V_-]$ and $[V_1, V_1]$ instead of $[V_+, V_0]$ and $[V_1, V_1]$ as in the previous case.

3. Sixty-three embeddings of the quaternion-preserving group in the Chevalley group $G_2(2)$

We go back to Eq. (6) and note that the binary tetrahedral group $T = V_0 \oplus V_+ \oplus V_-$ played an important role in the above analysis for it represents the root system of $SO(8)$. Any one element of the quaternionic elements of the hypercube $V_+ \oplus V_- = \frac{1}{2}\{\pm 1 \pm e_1 \pm e_2 \pm e_3\}$ satisfies the relation $p^3 = \pm 1$. Actually we have 112 octonionic elements of this type in the roots of E_8.

In an earlier paper [25] one of us has proven that the transformation[1]

$$b \to ab\bar{a}, \tag{16}$$

where $a^3 = \pm 1$ is an associative product of octonions which preserve the octonion algebra. More explicitly, when e_i $(i = 1, 2, \ldots, 7)$ represent the imaginary octonions the transformation

$$e_i' = ae_i\bar{a} \quad (a^3 = \pm 1) \tag{17}$$

preserves the octonion algebra

$$e_i'e_j' = (e_ie_j)' = a(e_ie_j)\bar{a}. \tag{18}$$

To work with octonionic root systems makes life difficult because of non-associativity. However, the following theorem [25] proves to be useful.

Theorem 2. *Let p be any root of those 112 roots and q be any root of E_8. Consider the transformations on q:*

$$\pm p : q_1 \equiv q, \quad q_2 \equiv (p)q(\bar{p}), \quad q_3 \equiv (\bar{p})q(p).$$

Then q_1, q_2, q_3 form an associative triad $(q_1q_2)q_3 = q_1(q_2q_3)$ satisfying the relations

$$\begin{aligned} &q_1q_2q_3 = q_1p \quad \text{for } q_i.\bar{p} = 0 \quad (42 \text{ triads}),\\ &q_1q_2q_3 = -1 \quad \text{for } q_i \cdot \bar{p} = -\tfrac{1}{2}\ (18 \text{ triads}),\\ &q_1q_2q_3 = 1 \quad \text{for } q_i \cdot \bar{p} = \tfrac{1}{2}\ (18 \text{ triads}).\\ &i = 1, 2, 3. \end{aligned} \tag{19}$$

Actually this decomposition of E_8 roots is the same as its branching under $SU(2) \times E_7$ where the non-zero roots decompose as $240 = 126 + 2 + (2, 56)$. The first 42 triads are the 126 non-zero roots of E_7 and $\pm\bar{p}$ are those of $SU(2)$. The remaining $36 \times 3 = 108$ roots with $\pm 1, \pm p$ constitute the 112 roots of the coset space. In general one can show that 24 triads out of 42 triads, corresponding to the roots of E_6 are imaginary octonions and the remaining 18 triads are those with non-zero scalar parts. The 9 triads of those octonionic roots which satisfy the relation $q_i \cdot \bar{p} = -\frac{1}{2}$ are imaginary octonions and their negatives satisfy the relation $q_i \cdot \bar{p} = \frac{1}{2}$. When ± 1 represent the roots of $SU(2)$ then all the roots of E_7 are pure imaginary as depicted in Fig. 1. For a given octonion p with non-zero real part one can classify the imaginary roots of E_7 as follows:

(i) 72 imaginary octonions are grouped in 24 triads satisfying the relation $q_i \cdot \bar{p} = 0$.
(ii) 27 imaginary roots are classified in 9 associative triads whose products satisfy the relation $q_1q_2q_3 = -1$ are the quaternionic units. They represent the weights of the 27-dimensional representation of E_6.
(iii) The remaining 9 triads are the conjugates of those in (ii) and represent the weights of the representation $\overline{27}$ of E_6.

We use the result of this theorem to prove the following lemma.

[1] It seems that this proof was given much earlier by Zorn [26].

Lemma 2. *The quaternionic roots of F_4 can be embedded in the octonionic roots of E_8 63 different ways.*

We recall that we have 18 associative triads with non-zero scalar part, each being orthogonal to $\bar{p}$. To distinguish the imaginary octonions we use the notation q_i and we denote the roots with non-zero scalar part by r_i satisfying the relation $r_i \cdot \bar{p} = 0$ where $r_i^3 = \pm 1$, $i = 1, 2, 3$. They are permuted as follows:

$$r_1, r_2 = p r_1 \bar{p}, \quad r_3 = \bar{p} r_1 p.$$

The scalar product $r_i \cdot \bar{p} = 0$ can be written as

$$r_i p + \bar{p}\bar{r}_i = \bar{r}_i p + \bar{p} r_i = 0. \tag{20}$$

We can use (19) to show that $r_1 r_2 = r_2 r_3 = r_3 r_1 = p$ with conjugates $\bar{r}_2\bar{r}_1 = \bar{r}_3\bar{r}_2 = \bar{r}_1\bar{r}_2 = \bar{p}$. Now the octonions r_1, r_2 and r_3 are mutually orthogonal to each other:

$$\begin{aligned} r_1 \cdot r_2 = r_2 \cdot r_3 = r_3 \cdot r_1 = 0 &\to r_1\bar{r}_2 + r_2\bar{r}_1 \\ &= r_2\bar{r}_3 + r_3\bar{r}_2 = r_3\bar{r}_1 + r_1\bar{r}_3 = 0, \end{aligned} \tag{21}$$

which also implies that $r_1\bar{r}_2, r_2\bar{r}_3, r_3\bar{r}_1$ are imaginary octonions.

The orthogonality of r_1, r_2 and r_3 can be proven as follows. Consider the scalar product

$$r_1 \cdot r_2 = \frac{1}{2}[\bar{r}_1(p r_1 \bar{p}) + (p\bar{r}_1\bar{p}) r_1]. \tag{22}$$

Let us assume without loss of generality that $\bar{p} = 1 - p$, $\bar{r}_1 = 1 - r_1$. Substituting $\bar{p} = 1 - p$ and $\bar{r}_1 = 1 - r_1$ in (22) and using (20) as well as the Moufang identities [22]

$$(pq)(rp) = p(qr)p, \tag{23a}$$

$$p(qrq) = [(pq)r]q, \tag{23b}$$

$$(qrq)p = q[r(qp)], \tag{23c}$$

it follows that $r_1 \cdot r_2 = 0$. Similar considerations for the other pairs of octonions imply that the four octonions r_1, r_2, r_3 and $\bar{p}$ are mutually orthogonal to each other so that $\pm r_1, \pm r_2, \pm r_3$ and $\pm\bar{p}$ form the vertices of a hyperoctahedron. Similarly their conjugates constitute the vertices of another hyperoctahedron, together with which they constitute a cube in 4-dimensional Euclidean space. The imaginary octonions $r_1\bar{r}_2, r_2\bar{r}_3, r_3\bar{r}_1$ are cyclically rotated to each other in the manner $p(r_1\bar{r}_2)\bar{p} = r_2\bar{r}_3$ (cyclic permutations of 1, 2, 3) which satisfies the relation $(r_1\bar{r}_2).\bar{p} = \frac{1}{2}$, (cyclic permutations of 1,2,3 is understood), where the conjugate $r_2\bar{r}_1$ satisfies the relation $(r_2\bar{r}_1) \cdot \bar{p} = -\frac{1}{2}$. If we denote by the imaginary octonions $E_1 = r_3\bar{r}_2$, $E_2 = r_1\bar{r}_3$ and $E_3 = r_2\bar{r}_1$, it is easy to prove the following identities:

$$\begin{aligned} \bar{p} &= \frac{1}{2}(1 - E_1 - E_2 - E_3), \\ r_1 &= \frac{1}{2}(1 + E_1 + E_2 - E_3), \\ r_2 &= \frac{1}{2}(1 - E_1 + E_2 + E_3), \\ r_3 &= \frac{1}{2}(1 + E_1 - E_2 + E_3). \end{aligned} \tag{24}$$

Therefore the set of 24 octonions

$$\left\{\pm 1, \pm E_1, \pm E_2, \pm E_3, \frac{1}{2}(\pm 1 \pm E_1 \pm E_2 \pm E_3)\right\} \tag{25}$$

are the quaternions forming the binary tetrahedral group and representing the roots of $SO(8)$. Once this set of octonions is given we can construct the root system of F_4 and form the roots of E_8 similar to Eq. (5).

It is obvious that for a given $p(\bar{p})$ one can construct the elements of the binary tetrahedral group, in other words, $SO(8)$'s root system, 9 different ways, as we have argued in the previous section. Since we have 112 roots of this type and a choice of p includes always $\pm p$ and $\pm\bar{p}$, we reduce to 28 choices. This number further reduces to $\frac{28}{4} = 7$ because $\bar{p}, r_1, r_2, r_3$ come always in quartets. It is not only $p(\bar{p})$ that rotates r_1, r_2, r_3 in the cyclic order but any one of them rotates the other three cyclically. For example, that

$$r_1\bar{p}\bar{r}_1 = r_2, \quad r_1r_2\bar{r}_1 = r_3, \quad r_1r_3\bar{r}_1 = \bar{p}. \tag{26}$$

The others satisfy similar relations. Therefore the choice of elements of a binary tetrahedral group or equivalently F_4 root system out of octonions is $9 \times 7 = 63$.

Since the group preserving the quaternion structure is of order 192 the overall group which preserves the octonionic root system of E_7 is a group of order $192 \times 63 = 12096$. It has to be a subgroup of G_2 and there is only one finite subgroup as such which is the adjoint Chevalley group $G_2(2)$ [2] of order 12096.

4. Maximal subgroups of $G_2(2)$ and the maximal Lie algebras of E_7

There are four regular maximal Lie algebras of E_7:

$E_6 \times U(1), SU(2) \times SO(12), SU(8), SU(3) \times SU(6)$; and there are four maximal subgroups of the Chevalley group $G_2(2)$. It is interesting to see whether any relations between these groups and the octonionic root systems of these Lie algebras exist (see [2]). There is a one-to-one correspondence between them but with one exception. When one imposes the invariance of the octonion algebra on the root system of $SU(3) \times SU(6)$ one obtains a group which is not maximal in the Chevalley group $G_2(2)$. Yet the maximal subgroup $[T, V_0] \oplus [T', V_1]$ of order 192(17) preserves the quaternion algebra of the magic square structure (SP_3, F_4). The number in the bracket is the number of conjugacy classes and is used to distinguish the groups having the same order. The other maximal subgroups of $G_2(2)$ which are of orders 432(14), 192(14) and 336(9) have one-to-one correspondences with the groups which preserve the octonionic root systems of $E_6 \times U(1)$, $SU(2) \times SO(12)$ and $SU(8)$ respectively. In this section we will discuss the constructions of these three maximal subgroups of $G_2(2)$ as the automorphism groups of the corresponding octonionic root systems. Their character tables and the subgroup structures can be found in Ref. [27].

4.1. Octonionic root system of $E_6 \times U(1)$ and the group of order 432(14)

Since the $U(1)$ factor is represented by the zero root we are essentially looking at the roots of E_6 in E_7. Using either the simple roots of E_8 in Fig. 1 or those roots of E_7 already given in Eq. (7b) we may decompose the roots of E_7 to those roots orthogonal to the vector $\frac{1}{2}(1 - e_1 - e_2 - e_3)$ which constitute the 72 roots of E_6 and the ones having a scalar product $\pm\frac{1}{2}$ with it will be the weights of the representations $\underline{27} + \underline{27^*}$. In explicit form they read:

Non-zero roots of E_6:

$$\begin{aligned}&\pm e_4, \pm e_5, \pm e_6, \frac{1}{2}(\pm e_4 \pm e_5 \pm e_6 \pm e_7), \pm\frac{1}{2}(e_2 - e_3 \pm e_4 \pm e_7),\\&\pm\frac{1}{2}(e_2 - e_3 \pm e_5 \pm e_6), \pm\frac{1}{2}(e_3 - e_1 \pm e_6 \pm e_7), \pm\frac{1}{2}(e_3 - e_1 \pm e_4 \pm e_5),\\&\pm\frac{1}{2}(e_1 - e_2 \pm e_5 \pm e_7), \pm\frac{1}{2}(e_1 - e_2 \pm e_4 \pm e_6).\end{aligned} \tag{27}$$

Weights of $\underline{27} + \underline{27^*}$ of E_6:

$$\begin{aligned}&\pm e_1, \pm e_2, \pm e_3,\\&\pm\frac{1}{2}(e_2 + e_3 \pm e_4 \pm e_7), \pm\frac{1}{2}(e_2 + e_3 \pm e_5 \pm e_6),\\&\pm\frac{1}{2}(e_3 + e_1 \pm e_6 \pm e_7), \pm\frac{1}{2}(e_3 + e_1 \pm e_4 \pm e_5),\\&\pm\frac{1}{2}(e_1 + e_2 \pm e_5 \pm e_7), \pm\frac{1}{2}(e_1 + e_2 \pm e_4 \pm e_6).\end{aligned} \tag{28}$$

Now we are in a position to determine the subgroup of the group of order 192(17) which preserves this decomposition. The magic square indicates that the root system of E_6 can be obtained by the Cayley–Dickson procedure as applied to the pair $(SU(3), F_4)$ given in (27). Here the roots of $SU(3)$ are represented by the short roots $\pm\frac{1}{2}(e_2 - e_3), \pm\frac{1}{2}(e_3 - e_1), \pm\frac{1}{2}(e_1 - e_2)$.

The subgroup of the group of order 192(17) preserving this system of roots where the imaginary unit e_7 is left invariant is the group generated by the elements,

$$[t, V_0], \left[\frac{1}{\sqrt{2}}(e_2 - e_3), V_1\right]. \tag{29}$$

Here t is given by $t = \frac{1}{2}(1 + e_1 + e_2 + e_3)$. More explicitly we can write the elements of the group of interest as follows

$$[t, V_0] \subset [V_+, V_0], \quad [\bar{t}, V_0] \subset [V_-, V_0], \quad [1, V_0] \subset [V_0, V_0]; \tag{30a}$$

$$\begin{aligned}&\left[\frac{1}{\sqrt{2}}(e_2 - e_3), V_1\right] \subset [V_1, V_1], \quad \left[\frac{1}{\sqrt{2}}(e_3 - e_1), V_1\right] \subset [V_2, V_1],\\&\left[\frac{1}{\sqrt{2}}(e_1 - e_2), V_1\right] \subset [V_3, V_1].\end{aligned} \tag{30b}$$

Each set contains 8 elements hence the group is of order 48. We recall that in the decomposition of the root system of E_7 in (27) and (28) under E_6 the quaternions $\pm t(\pm\bar{t})$ and thereby the quaternionic imaginary units e_1, e_2, e_3 are used. This implies that the sum $\frac{1}{\sqrt{3}}(e_1 + e_2 + e_3)$ is left invariant under the transformations $tq\bar{t}$ where q is any octonion. This proves that the group of concern is a finite subgroup of $SU(3)$ acting in the 6-dimensional Euclidean subspace. The discussions through the relations (16)–(19) show that one can construct the root system of E_6 in (27), consequently those weights in (28), 9 different ways implying that the group preserving the root system of E_6 in (27) is a finite subgroup of $SU(3)$ of order $48 \times 9 = 432$ with 14 conjugacy classes. The 6×6 irreducible matrix representation of this group as well as its character table are found in Ref. [27].

4.2. The octonionic root system of $SU(2) \times SO(12)$ and the group of order 192(14)

Existence of an automorphism group of order 192 is obvious since the $SU(2)$ roots are any imaginary octonion $\pm q$ which must be left invariant under all transformation. Since we have $126/2 = 63$ choice for the $SU(2)$ roots the group of invariance is $12096/63 = 192$. The structure of this group is totally different than the previous group of order 192(17) as we will discuss below.

The magic square tells us that the root system of $SO(12)$ can be obtained by pairing two sets of quaternionic roots of $SP(3)$ *ála* Cayley–Dickson procedure $(SP(3), SP(3))$. When we take the quaternionic roots of $SP(3)$ given in (8) we obtain the root system of $SO(12)$ and $SU(2)$ as follows:

$SO(12)$ roots:

$$\begin{aligned}
&\pm e_1, \pm e_2, \pm e_3, e_7(\pm e_1, \pm e_2, \pm e_3) = \pm e_4, \pm e_5, \pm e_6,\\
&\frac{1}{2}(\pm e_2 \pm e_3) + e_7\frac{1}{2}(\pm e_2 \pm e_3) = \frac{1}{2}(\pm e_2 \pm e_3 \pm e_5 \pm e_6),\\
&\frac{1}{2}(\pm e_3 \pm e_1) + e_7\frac{1}{2}(\pm e_1 \pm e_2) = \frac{1}{2}(\pm e_1 \pm e_3 \pm e_4 \pm e_5),\\
&\frac{1}{2}(\pm e_1 \pm e_2) + e_7\frac{1}{2}(\pm e_3 \pm e_1) = \frac{1}{2}(\pm e_1 \pm e_2 \pm e_4 \pm e_6);
\end{aligned} \tag{31}$$

$SU(2)$ roots: $\pm e_7$.

The remaining roots transform as the weights of the representation $(\underline{2}, \underline{32}')$ under $SU(2) \times SO(12)$. Since the root $\pm e_7$ remains invariant under any transformation which preserves the decomposition of E_7 under $SU(2) \times SO(12)$ the group which we seek is a finite subgroup of $SU(3)$. We recall from the previous discussions that the quaternionic root system of $SP(3)$ is preserved by the octahedral group $[T, \bar{T}] \oplus [T', \overline{T'}]$. However, we seek a subgroup of $[T, V_0] \oplus [T', V_1]$ which is also a subgroup of the octahedral group. Since we have V_0 and V_1 on the right of the pairs it should be $[\overline{V}_0, V_0] \oplus [\overline{V}_1, V_1]$. Actually we can write all the group elements explicitly,

$$[1, 1], [e_1, -e_1], \left[\frac{1}{\sqrt{2}}(1 + e_1), \frac{1}{\sqrt{2}}(1 - e_1)\right], \left[\frac{1}{\sqrt{2}}(1 - e_1), \frac{1}{\sqrt{2}}(1 + e_1)\right], \tag{32a}$$

$$\begin{aligned}
&[e_2, -e_2], [e_3, -e_3], \left[\frac{1}{\sqrt{2}}(e_2 + e_3), -\frac{1}{\sqrt{2}}(e_2 + e_3)\right],\\
&\left[\frac{1}{\sqrt{2}}(e_2 - e_3), \frac{1}{\sqrt{2}}(-e_2 + e_3)\right].
\end{aligned} \tag{32b}$$

The elements in (32a) form a cyclic group Z_4 and those in (32b) are the right or left cosets of (32a) where, say, $[e_2, -e_2]$ is a coset representative. Indeed the elements $[1, 1]$ and $[e_2, -e_2]$ form the group Z_2 which leaves the group Z_4 invariant under conjugation. Hence the group of order 8 in (32a)–(32b) has the structure $Z_4 : Z_2$ where Z_4 is an invariant subgroup. We may also allow $e_7 \to -e_7$, extending the group $Z_4 : Z_2$ by the element $[-1, 1]$. Since the element $[-1, 1]$ commutes with the elements of $Z_4 : Z_2$ then we have a group of order 16 with the structure $Z_2 \times (Z_4 : Z_2)$. This is the group of the group automorphisms of the root system in (31) when the quaternionic units are taken to be e_1, e_2 and e_3.

Now the question is how many different ways can we decompose (31) allowing $e_7 \to \pm e_7$ only. In other words, what is the number of quaternionic units one can choose allowing $e_7 \to \pm e_7$? These quaternionic units can be chosen from the set of 112 roots orthogonal to e_7. They are

$$\frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3),\ \frac{1}{2}(\pm 1 \pm e_1 \pm e_5 \pm e_6),$$
$$\frac{1}{2}(\pm 1 \pm e_2 \pm e_4 \pm e_5),\ \frac{1}{2}(\pm 1 \pm e_3 \pm e_4 \pm e_6). \tag{33}$$

Each set of 16 octonions in (33) will yield 3 sets of quaternionic imaginary units not involving e_7. Therefore there are 12 different quaternionic units to build the group structure $Z_2 \times (Z_4 : Z_2)$ and the number of overall elements of the group preserving the root system in (31) is $12 \times 16 = 192$. To give a nontrivial example let us choose $p = \frac{1}{2}(1 + e_2 + e_4 + e_5)$ with $\bar{p} = \frac{1}{2}(1 - e_2 - e_4 - e_5)$. The following set of octonions chosen from (31)

$$\frac{1}{2}(\pm e_1 + e_2 + e_4 \pm e_6),\ \frac{1}{2}(\pm e_3 + e_2 + e_5 \pm e_6),\ \frac{1}{2}(\pm e_1 + e_4 + e_5 \pm e_3) \tag{34}$$

have scalar products $q_i \cdot \bar{p} = 0$ where q_i is one of those in (34). Under the rotation $pq_i\bar{p}$, for example, the quaternionic units

$$E_1 = \frac{1}{2}(e_2 + e_5 + e_5 - e_6), \quad E_2 = \frac{1}{2}(e_1 - e_3 + e_4 + e_5), \tag{35}$$
$$E_3 = \frac{1}{2}(-e_1 + e_2 + e_4 + e_6)$$

are permuted and one can construct (31) with the set of octonions:

$SO(12)$ roots:

$$\begin{aligned}
&\pm E_1, \pm E_2, \pm E_3, e_7(\pm E_1, \pm E_2, \pm E_3) = \pm E_4, \pm E_5, \pm E_6,\\
&\frac{1}{2}(\pm E_2 \pm E_3) + e_7\frac{1}{2}(\pm E_2 \pm E_3) = \frac{1}{2}(\pm E_2 \pm E_3 \pm E_5 \pm E_6),\\
&\frac{1}{2}(\pm E_3 \pm E_1) + e_7\frac{1}{2}(\pm E_1 \pm E_2) = \frac{1}{2}(\pm E_1 \pm E_3 \pm E_4 \pm E_5),\\
&\frac{1}{2}(\pm eE_1 \pm E_2) + e_7\frac{1}{2}(\pm E_3 \pm E_1) = \frac{1}{2}(\pm E_1 \pm E_2 \pm E_4 \pm E_6),
\end{aligned} \tag{36}$$

$SU(2)$ roots: $\pm e_7$.

This is certainly invariant under the quaternion-preserving automorphism group of order 16 as discussed above where the imaginary quaternionic units e_1, e_2, e_3 in (32a)–(32b) are replaced by E_1, E_2, E_3 in (35). One can proceed in the same manner and construct 12 different sets of quaternionic units by which one constructs the group $Z_2 \times (Z_4 : Z_2)$.

4.3. Octonionic root system of $SU(8)$ and the automorphism group of order 336(9)

Using the Coxeter–Dynkin diagram of Fig. 1 we can write the octonionic roots of $SU(8)$ as follows:

$$\begin{aligned}
&\pm e_1, \pm e_2, \pm e_4, \pm e_6,\\
&\frac{1}{2}(\pm e_1 \pm e_2 + e_5 + e_7),\ \frac{1}{2}(\pm e_1 \pm e_4 + e_3 + e_5),\ \frac{1}{2}(\pm e_1 \pm e_6 + e_3 + e_7),\\
&\frac{1}{2}(\pm e_2 \pm e_4 + e_3 - e_7),\ \frac{1}{2}(\pm e_2 \pm e_6 - e_3 + e_5),\ \frac{1}{2}(\pm e_4 \pm e_6 - e_5 + e_7).
\end{aligned} \tag{37}$$

First of all, we note that the roots of E_7 decompose under its maximal Lie algebra $SU(8)$ as $126 = 56 + 70$. Therefore those roots of E_7 in (7b) not displayed in (37) are the weights of the 70-dimensional representation of $SU(8)$.

To determine the automorphism group of the set in (37) we may follow the same method discussed above but here we follow a different way since $SU(8)$ is not in the magic square.

In an earlier paper [16] we have constructed the 7-dimensional irreducible representation of the group $PSL_2(7) : Z_2$ of order 336 and proved that this group preserves the octonionic root system of $SU(8)$. Below we give three matrix generators of Klein's group $PSL_2(7)$, a simple group with 6 conjugacy classes:

$$A = \frac{1}{2}\begin{bmatrix} -1 & -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 1 \\ 1 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$C = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 & 0 \end{bmatrix}. \tag{38}$$

These matrices satisfy relation

$$A^4 = B^2 = C^7 = I. \tag{39}$$

The matrices A and B generate the octahedral subgroup of order 24 of the Klein's group.

The 56 octonionic roots can be decomposed into 7 sets of hyperocthedra in 4 dimensions. The matrix C permutes the seven sets of octahedra to each other. The octahedral group generated by A and B preserves one of the octahedra while transforming the other sets to each other. We display the 7 octahedra as follows:

$$\underline{1}: \quad \begin{matrix} \pm e_2 \\ \pm\frac{1}{2}(e_4 - e_5 + e_6 + e_7) \\ \pm\frac{1}{2}(e_1 - e_3 + e_6 - e_7) \\ \mp\frac{1}{2}(e_1 + e_3 + e_4 + e_5) \end{matrix} \qquad \underline{2}: \quad \begin{matrix} \pm e_1 \\ \pm\frac{1}{2}(e_2 + e_3 - e_5 + e_6) \\ \pm\frac{1}{2}(-e_2 + e_3 - e_4 - e_7) \\ \pm\frac{1}{2}(e_4 + e_5 + e_6 - e_7) \end{matrix}$$

$$\begin{array}{rc}
 & \mp e_4 \\
 & \pm\frac{1}{2}(e_1+e_3+e_6+e_7) \\
\underline{3}: & \pm\frac{1}{2}(-e_1-e_2+e_5+e_7) \\
 & \pm\frac{1}{2}(e_2-e_3+e_5+e_6) \\
 & \mp\frac{1}{2}(e_1+e_2+e_5+e_7) \\
 & \pm e_6 \\
\underline{5}: & \pm\frac{1}{2}(e_2+e_3+e_4-e_7) \\
 & \mp\frac{1}{2}(e_1-e_3+e_4-e_6) \\
 & \pm\frac{1}{2}(e_4-e_5-e_6+e_7) \\
 & \pm\frac{1}{2}(-e_1+e_3+e_6+e_7) \\
\underline{7}: & \pm\frac{1}{2}(e_2-e_3-e_4+e_7) \\
 & \mp\frac{1}{2}(e_1-e_2+e_5+e_7)
\end{array}
\qquad
\begin{array}{rc}
 & \pm\frac{1}{2}(-e_2-e_3+e_5+e_6) \\
 & \pm\frac{1}{2}(-e_4-e_5+e_6+e_7) \\
\underline{4}: & \pm\frac{1}{2}(-e_1-e_3+e_4-e_5) \\
 & \mp\frac{1}{2}(-e_1+e_2+e_5+e_7) \\
 & \pm\frac{1}{2}(-e_1+e_3+e_4+e_5) \\
 & \pm\frac{1}{2}(-e_2+e_3-e_5+e_6) \\
\underline{6}: & \pm\frac{1}{2}(e_1+e_3-e_6+e_7) \\
 & \mp\frac{1}{2}(e_2+e_3-e_4-e_7)
\end{array}$$

Note that each set of vectors represents an octahedron in 4 dimensions. The matrix C permutes the set of octahedra as $\underline{1} \to \underline{2} \to \underline{3} \to \underline{4} \to \underline{5} \to \underline{6} \to \underline{7} \to \underline{1}$. The matrices A and B leave the set of vectors in $\underline{1}$ invariant and transform the other sets as follows:

$A : \underline{2} \to \underline{5} \to \underline{6} \to \underline{7} \to \underline{2}$; $\underline{3} \leftrightarrow \underline{4}$ and leaves $\underline{1}$ invariant.

$B : \underline{3} \leftrightarrow \underline{5}$; $\underline{4} \leftrightarrow \underline{7}$ and leaves each of the set $\underline{1}, \underline{2}, \underline{6}$ invariant.

When we decompose the weights of the 70 dimensional representation of $SU(8)$ under the octahedral group the vectors are partitioned into sets of sizes 2, 6, 6, 8, 12, 12, 24. The vector $\pm\frac{1}{2}(e_1+e_2+e_5-e_7)$ is left invariant under the action of the octahedral group which corresponds to its 2 dimensional irreducible representation. The group $PSL_2(7)$ can be further extended to the group $PSL_2 : Z_2$ of order 336 by adding a generator which can be obtained from the transformation $e_1 \to -e_1, e_2 \to e_2, e_4 \to e_4$. One can readily check that the this transformation leaves the root system of $SU(8)$ invariant.

References

[1] C. Chevalley, Tohoku, Math. J. 7 (1955) 14; Am. J. Math. 77 (1955) 778.
[2] A.M. Cohen, D.B. Wales, Commun. Algebra 11 (1983) 441; F. Karsch, M. Koca, J. Phys. A 23 (1990) 4739.
[3] H. Georgi, S.L. Glashow, Phys. Rev. Lett. 32 (1974) 438.
[4] H. Fritzsch, P. Minkowski, Annals Phys. 93 (1975) 193.
[5] F. Gursey, P. Ramond, P. Sikivie, Phys. Lett. 60B (1976) 177.
[6] E. Cremmer, B. Julia, J. Scherk, Phys. Lett. 70B (1978) 409.
[7] I. Bars, M. Günaydin, Phys. Rev. Lett. 45 (1980) 859; M. Koca, Phys. Lett. 107B (1981) 73.
[8] David J. Gross, Jeffrey A. Harvey, Emil J. Martinec, Ryan Rohm, Phys. Rev. Lett. 54 (1985) 502.
[9] T. Pengpan, P. Ramond, Phys. Rep. C 315 (1999) 137.
[10] M. Koca, R. Koç, M. Al-Barwani, J. Math. Phys. 44 (2003) 3123.
[11] For a review see for instance Bobby S. Acharya, Sergei Gukov, Phys. Rep. C 392 (2004) 121.
[12] R. Dijkraaf, S. Gukov, A. Neitzke, C. Vafa, <arXiv:hep-th/0411073>.
[13] M. Koca, N. Özdeş, J. Phys. A 22 (1989) 1469.
[14] J.E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge University Press, Cambridge, 1990; R.W. Carter, Simple Groups of Lie Type, John Wiley & Sons, 1989.
[15] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, Atlas of Finite Groups, Oxford University Press, Oxford, 1985.

[16] M. Koca, R. Koç, J. Math. Phys. A 27 (1994) 2429.
[17] H.S.M. Coxeter, Duke Math. J. 13 (1946) 561; L.E. Dickson, Ann. Math. 20 (1919) 155; F. Gürsey, Mod. Phys. Lett. A 2 (1987) 967.
[18] H. Freudenthal, Advances in Math. I, 145 (1965); R.D. Schafer, An Introduction to Non-Associative Algebras, Academic Press, New York, 1966.
[19] P. du Val, Homographies, Quaternions and Rotations, Cambridge University Press, Cambridge, 1964.
[20] H.S.M. Coxeter, Regular Complex Polytopes, Cambridge University Press, Cambridge, 1973.
[21] M. Günaydin, F. Gürsey, J. Math. Phys. 14 (1973) 1651.
[22] T.A. Springer, F.D. Veldkamp, Octonions, Jordan Algebras and Exceptional Groups, Springer-Verlag, Berlin-Heidelberg, 2000; J.H. Conway, D.A. Smith, On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry, A.K. Peters Ltd., Natick, MA, 2003.
[23] M. Koca, R. Koç, M. Al-Barwani, J. Phys. A 34 (2001) 11201.
[24] M. Koca, R. Koç, Tr. J. Phys. 19 (1995) 304; M. Koca, R. Koç, N.Ö. Koca, in press.
[25] M. Koca, J. Math. Phys. 33 (1992) 497.
[26] M. Zorn, Proc. Natl. Acad. Sci. 21 (1935) 355.
[27] R. Koç, Ph.D. thesis, Çukurova University, Institute of Natural and Applied Sciences, 1995 (unpublished).