On the Validity of the Euler–Lagrange Equation

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Under some regularity assumptions on the boundary datum \(u_0\) (assumptions automatically satisfied in the classical case when the growth of the integrand is bounded by \(a + b |\xi|^p\)), we prove the validity of the Euler–Lagrange equation for the functional

\[
\int_{\Omega} \left[ f(|\nabla u(x)|) + g(x, u(x)) \right] dx
\]

under general growth assumptions on \(f\), for instance for \(f(|\xi|) = P(|\xi|) e^{K|\xi|}\) with \(P\) a nontrivial polynomial and \(K \geq 0\), or for \(f(|\xi|) = |\xi|^{2\nu}\).

Key Words: Euler–Lagrange equation.

1. INTRODUCTION

Although the Euler–Lagrange equation is basic to the investigation of the properties of solutions to minimization problems of the calculus of variations, the proof of its validity is still restricted either to the standard case of growth (in the gradient variable) of the kind \(t \to t^p\) or to a few very special cases of integrands that are extended valued, typically \(+\infty\) outside the unit ball \(B\) ([2], with homogeneous boundary conditions; [3], with general boundary conditions but a very specific functional). The faster the growth of the integrand, the more regular the functions that make the functional finite are; this remark suggests that establishing the validity of the Euler–Lagrange equation should be easier in the cases of fast growth. The fact that this is not so depends on several technical reasons. For a functional of the kind

\[
\int_{\Omega} \left[ F(|\nabla u(x)|) + g(x, u(x)) \right] dx
\]
one has to prove that, for every admissible variation \( \eta \),

\[
\int_\Omega \left[ \langle \nabla F(\nabla u(x)), \nabla \eta(x) \rangle + g_u(x, u(x)) \eta(x) \right] \, dx = 0.
\]

The first difficulty concerns the class of admissible variations. It seems reasonable to require that the variations be in the same integrability class as the solution itself. In the case where the growth of \( F(\xi) \) is bounded both below and above by terms growing like \( \| \xi \|^p \), the solutions will be in \( W^{1, p}(\Omega) \) and every function in this space will make the integral finite: therefore, in this case, it is natural to consider the space of variations to coincide with \( W_0^{1, p}(\Omega) \). However, the choice of the space of admissible variations is more difficult in the case of faster growth of the integrand, since the set of those functions that make the integral finite in general is not a linear space.

The next basic question to be answered is about the integrability of the term

\[
\langle \nabla F(\nabla u(x)), \nabla \eta(x) \rangle
\]

for an admissible variation \( \eta \). Again, when one assumes that the function \( F \) is (convex and) bounded above by a term of the form \( a + b \| \xi \|^p \), as is well known, the gradient \( \nabla F(\xi) \) grows, so to say, one power slower, i.e., as \( \| \xi \|^{p-1} \). Hence not only is \( \| \nabla F(\nabla u(x)) \| \) integrable, but \( \| \nabla F(\nabla u(x)) \|^{p/(p-1)} \) also is integrable. At this point the use of the Hölder inequality shows that \( \langle \nabla F(\nabla u(x)), \nabla \eta(x) \rangle \) is integrable for every \( \xi \) in \( L^p \). It is this gain of integrability, depending on the gradient of \( F \) growing one power slower than the function \( F \) itself, that has, so far, allowed one to prove the validity of the Euler–Lagrange equation.

When the function \( F \) is, say, \( e^{\| \xi \|^p} \), the gradient grows exactly as the function \( F \) itself, and the line of thought followed so far breaks down, even more so when the gradient grows faster than the function \( F \) itself, e.g., for \( F(\xi) = \| \xi \|^q \). The purpose of the present paper is to present some results establishing the validity of the Euler–Lagrange equation for the functional

\[
\int_\Omega \left[ f(\| \nabla u(x) \|) + g(x, u(x)) \right] \, dx
\]

under general growth assumptions on \( f \), for instance, for \( f(\| \xi \|) = P(\| \xi \|) e^{K\| \xi \|} \) with \( P \) a nontrivial polynomial and \( K \geq 0 \) or for \( f(\| \xi \|) = | \xi |^{1+\epsilon} \).

This will require some additional regularity assumptions on the boundary datum \( u^0 \) (these assumptions are automatically satisfied in the classical case of an integrand bounded by \( a + b \| \xi \|^p \)) and, more particularly, a different proof.
The core of the proof of the validity of the Euler–Lagrange equation is, clearly, passing to the limit under the integral sign in the difference quotient. It is remarkable that the proofs presented so far nowhere exploit, in this passing to the limit, the fact that \( u \) is a solution to the problem, i.e., \( u \) could very well be any function making the integral finite. In the proof we present here, instead, we first gain some further regularity (higher integrability) of the solution \( u \) and then, through these properties, establish the validity of the equation.

2. MAIN RESULTS

Let \( \Omega \) be a bounded and open subset of \( \mathbb{R}^d \) with Lipschitzian boundary, and consider the problem (P),

\[
\minimize \int_\Omega \left[ f(|\nabla u(x)|) + g(x, u(x)) \right] \, dx : u - u^0 \in W_0^{1,1}(\Omega) \quad (P)
\]

We will assume that \( u^0 \) is such that the above functional is finite. In general, the subset of \( W_0^{1,1} \) of those \( \eta \) that make the integral \( \int f(|\nabla \eta(x)|) \, dx \) finite is not a linear subspace. On the other hand, proving the validity of the Euler–Lagrange equation amounts to proving that a certain functional is zero on a linear subspace \( W_0 \), the space of admissible variations. It is natural to assume that \( W_0 \) should be in the span of those functions \( \eta \) that make the integral \( \int f(|\nabla \eta(x)|) \, dx \) finite. The idea of a variation being zero at the boundary of \( \Omega \) demands a little additional care, since \( \int f(|\nabla \eta(x)|) \, dx \) does not necessarily behave like a norm at zero (\( f \) could be constant on a non-trivial interval \([0, d])

**Definition.** \( W_0 \) is the linear span of \( \{ w \in W^{1,1} : \int f(|\nabla w(x)|) \, dx < \infty, \) and there exists a sequence \( (\eta_n) \) of \( C^1 \) maps with support in \( \Omega \) such that \( \eta_n \) converges to \( w \) in \( W^{1,1} \) and the sequence \( (f(|\nabla \eta_n|)) \) is equi-integrable.\}

Note that, in the case when \( f(\xi) \) is bounded below by \( \beta \xi^p \), \( W_0 \) contains \( W_0^p \); when \( f(\xi) \) is bounded above by \( a + b \xi^p \), \( W_0^p \) contains \( W_0 \).

A class of integrands we shall consider is described by the following definition.

**Definition.** We say that the differentiable map \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) is of exponential class if there exist non-negative reals \( a \) and \( \zeta_0 \) such that for \( \zeta \geq \zeta_0 \) we have

\[
f'(\zeta) \leq a f(\zeta).
\]
Maps as different as $\xi \rightarrow \sqrt{1 + \xi^2}$, $\xi \rightarrow a + b\xi^p$, $\xi \rightarrow \xi \log(\xi + 1)$, and $\xi \rightarrow P(\xi)e^{\kappa \xi}$, $P$ a polynomial, are all in the above class.

The following assumption on the boundary datum $u^0$ will play a major role.

**Assumption A.** Let $f: [0, b) \rightarrow \mathbb{R}^+$ be convex and differentiable. Let $u$ be any solution to the minimization problem (P). We assume that either

(i) $f'(\|Vu^0(\cdot)\|)\|Vu^0(\cdot)\| \in L^1(\Omega)$

and

$$\int_\Omega f'(\|Vu(x)\|)\left(\frac{Vu(x)}{\|Vu(x)\|}, Vu^0\right)dx < +\infty$$

or

(ii) $\text{ess sup}\ |\|Vu^0(x)\|| < b$.

The above assumption depends in part on an unknown function, $u$, and therefore it is not easily checked. The next proposition provides some useful special cases. We recall that $f$ satisfies the $A_2$ condition [1] if there exists a $k$ such that $f(2\xi) \leq kf(\xi)$.

**Proposition 1.** Let $f: [0, b) \rightarrow \mathbb{R}^+$ be differentiable. Each of the following conditions implies that Assumption A is satisfied:

(i) $b = \infty$, $f$ is a convex function satisfying the $A_2$ condition (such as $f(\xi) = a + b\xi^p$ or $f(\xi) = \xi \log(\xi + 1)$), and $u^0$ is such that $\int_\Omega f(\|Vu^0(x)\|)dx < \infty$.

(ii) $b = \infty$ and $u^0 \in W^{1, \infty}(\Omega)$.

(iii) $u^0 = 0$.

**Proof.** Only (i) has to be proved. From the convexity of $f$ we have $f(2t) \geq f(t) + tf'(t)$, i.e., $tf'(t) \leq (k - 1)f(t)$. Consider $f^*$, the polar of $f$ [5], defined for $p$ in the image of $f'$ and given by $f^*(p) = pt_0 - f(t_0)$, where $f'(t_0) = p$. In particular, for $p = f'(\|Vu(x)\|)$, we have

$$f^*(f'(\|Vu(x)\|)) = f'(\|Vu(x)\|)\|Vu(x)\| - f(\|Vu(x)\|).$$

For every $p$ where $f^*$ is defined and every $s$ in the domain of $f$, Fenchel’s inequality holds: $ps \leq f(s) + f^*(p)$. Hence we obtain
\[ \int_{\Omega} f'(\|\nabla u(x)\|) \|\nabla u^0(x)\| \, dx \]
\[ \leq \int_{\Omega} f'(\|\nabla u(x)\|) \|\nabla u(x)\| - f(\|\nabla u(x)\|) \, dx + \int_{\Omega} f(\|\nabla u^0(x)\|) \, dx \]
\[ \leq (k-2) \int_{\Omega} f(\|\nabla u(x)\|) \, dx + \int_{\Omega} f(\|\nabla u^0(x)\|) \, dx, \]
proving (i).

Although the emphasis of this paper is on the term \( f(\|\nabla u(.\|)), \) we will need a result about the convergence of the term
\[ \int_{\Omega} \frac{g(x, u(x) + \lambda \eta(x)) - g(x, u(x))}{\lambda} \, dx \]
to
\[ \int_{\Omega} g_d(x, u(x)) \eta(x) \, dx. \]

Conditions ensuring this fact are classical: several possible assumptions on \( g \) exist that guarantee this convergence, depending on the growth of \( f[4]. \)

Namely, we assume:

**Assumption B.** The function \( g \) is a Carathéodory function. Let \( p \) be such that \( f'(\xi) \geq a + b \xi^p \). We assume that

(i) for \( p > d \), for every \( R \) there exists \( \alpha_d(.\) \in \( L^1(\Omega) : |g_d(x, u)| \leq \alpha_d(x) \) for \( |u| \leq R; \)

(ii) for \( p = d \), there exist \( \beta(\.) \in L^1(\Omega) \) and \( q \geq 1 : |g_d(x, u)| \leq \beta(x) + u^q; \)

(iii) for \( p < d \), there exist \( \gamma \in L^1(\Omega) : |g_d(x, u)| \leq \gamma + u^{\alpha_d(\xi - d) - 1}. \)

The next theorem provides the additional integrability on \( \|\nabla u(.\|) \) required to establish the validity of the Euler–Lagrange equation.

**Theorem 1.** Let \( f : [0, b) \rightarrow \mathbb{R}^+ \) be convex, increasing, differentiable, and such that \( \lim_{t \to 0^+} f'(t) = 0. \) Let \( u \) be a solution to Problem \((P)\). Let Assumption A and Assumption B hold. Then:

(i) \( f''(f'(\|\nabla u(.)\|)) \) \in \( L^1(\Omega), \)

(ii) for every \( w \) such that \( \int_{\Omega} f(\|\nabla w(x)\|) \, dx < \infty, \) we have \( f'(\|\nabla u(.)\|) \|\nabla w(.)\| \) \in \( L^1(\Omega). \)
Proof. Set $\eta \in W_0$ to be $u^0 - u$. Since both $u$ and $u^0$ make the integral finite and $f$ is convex, for every $\lambda$, $0 \leq \lambda \leq 1$, the integral computed on $u + \lambda \eta$ is finite.

(a) The convergence of the term $\frac{1}{\lambda} \int_{\Omega} \left[ g(x, u(x) + \lambda \eta(x)) - g(x, u(x)) \right] dx$ to the integral $\int_{\Omega} g(x, u(x)) \eta(x) dx$ is classical [4]. By assumption there are $C^1$ maps $(\eta_n)$ with support in $\Omega$ such that: $\eta_n$ converges to $\eta$ in $W^{1,1}(\Omega)$ and $(f(\|\nabla \eta_n\|))$ is equi-integrable. Consider Assumption B.

Case (i). $u$ is bounded on $\Omega$. Moreover, the sequence $(\frac{1}{\lambda} \int_{\Omega} g(x, u(x) + \lambda \eta(x)) dx)$ is equi-bounded and, by Poincaré's inequality, so is $(\|\eta_n\|_{H^1})_n$, i.e., the sequence $(\|\eta_n\|_{1, p})_n$ is equi-bounded. By the Imbedding Theorem, the sequence $(\|\eta_n\|_{\infty})_n$ is equi-bounded. So $\eta$ is bounded. Hence for some $R$, $\|g(x, u(x) + \epsilon \eta(x))\| \leq \pi g(x)$ and the convergence to $\int_{\Omega} g(x, u(x)) \eta(x) dx$ follows by dominated convergence.

Similar treatments are used for the cases (ii) and (iii).

(b) Consider $(1/\lambda)(f(\|\nabla u(x) + \lambda \nabla \eta(x)\|) - f(\|\nabla u(x)\|))$. As $\lambda \to 0^+$, it converges pointwise to $f'(\|\nabla u(x)\|)(\|\nabla u(x)\|, \nabla \eta(x))$. Set $E^+$ to be $\{x \in \Omega : \|\nabla u(x)\| > \|\nabla u^0(x)\| \}$ and set $E^-$ to be $\{x \in \Omega : \|\nabla u(x)\| \leq \|\nabla u^0(x)\| \}$.

Let $x$ be in $E^-$. Then, for some $0 \leq \lambda_0(x) \leq \lambda$,

$$
\left(1/\lambda\right)(f(\|\nabla u(x) + \lambda \nabla \eta(x)\|) - f(\|\nabla u(x)\|))
$$

$$
= f'(\|\nabla u(x) + \lambda_0(x) \nabla \eta(x)\|) \left(\frac{\nabla u(x) + \lambda_0(x) \nabla \eta(x)}{\|\nabla u(x) + \lambda_0(x) \nabla \eta(x)\|}, \nabla \eta(x)\right).
$$

Since $\|\nabla u + \lambda_0 \nabla \eta\| = \|\nabla u + (1 - \lambda_0)(\nabla u)\| \leq \|\nabla u^0\|$, we have

$$
\left(1/\lambda\right)(f(\|\nabla u(x) + \lambda \nabla \eta(x)\|) - f(\|\nabla u(x)\|))
$$

$$
\leq f'(\|\nabla u^0(x)\|) \|\nabla u^0(x) - \nabla u(x)\| \leq 2f'(\|\nabla u^0(x)\|) \|\nabla u^0(x)\|.
$$

By Assumption A and dominated convergence we have then

$$
\lim_{\lambda \to 0^+} \int_{E^-} \left\{ f(\|\nabla u(x) + \lambda \nabla \eta(x)\|) - f(\|\nabla u(x)\|) \right\} dx
$$

$$
= \int_{E^-} f'(\|\nabla u(x)\|) \left(\frac{\nabla u(x)}{\|\nabla u(x)\|}, \nabla u^0(x) - \nabla u(x)\right) dx = 0.
$$
(c) Since $u$ is a minimum, for $\lambda > 0$,

$$
\int_{\partial} \frac{f(\|\nabla u(x) + \lambda \nabla \eta(x)\|) - f(\|\nabla u(x)\|) + g(x, u(x) + \lambda \eta(x)) - g(x, u(x))}{\lambda} \, dx
$$

$$
= \int_{E^+} \frac{f(\|\nabla u(x) + \lambda \nabla \eta(x)\|) - f(\|\nabla u(x)\|)}{\lambda} \, dx
$$

$$
+ \int_{E^-} \frac{f(\|\nabla u(x) + \lambda \nabla \eta(x)\|) - f(\|\nabla u(x)\|)}{\lambda} \, dx
$$

$$
+ \int_{\Omega} \frac{g(x, u(x) + \lambda \eta(x)) - g(x, u(x))}{\lambda} \, dx \geq 0,
$$

so that

$$
- \int_{E^+} \frac{f(\|\nabla u(x) + \lambda \nabla \eta(x)\|) - f(\|\nabla u(x)\|)}{\lambda} \, dx
$$

$$
\leq \int_{E^-} \frac{f(\|\nabla u(x) + \lambda \nabla \eta(x)\|) - f(\|\nabla u(x)\|)}{\lambda} \, dx
$$

$$
+ \int_{\Omega} \frac{g(x, u(x) + \lambda \eta(x)) - g(x, u(x))}{\lambda} \, dx.
$$

Hence, from the convergence results obtained in (a) and (b), the following inequality holds true: for all $\lambda > 0$ sufficiently small,

$$
\int_{E^+} \frac{f(\|\nabla u(x) + \lambda \nabla \eta(x)\|) - f(\|\nabla u(x)\|)}{\lambda} \, dx
$$

$$
\leq \varepsilon + \int_{\Omega} g_d(x, u(x)) \eta(x) \, dx + \varepsilon.
$$

(d) Consider $x$ in $E^+$: we have that $\|\nabla u + \lambda (\nabla u_0 - \nabla u)\| \leq \|\nabla u\|$, hence, for $\lambda > 0$, the restriction to $E^+$ of each map

$$
x \rightarrow - \frac{f(\|\nabla u(x) + \lambda \nabla \eta(x)\|) - f(\|\nabla u(x)\|)}{\lambda}
$$

is non-negative. Moreover, setting $v$ to be $(\nabla u_0 - \nabla u)$, we have that

$$
\|\nabla u + v\|^2 = \|\nabla u_0\|^2 \leq \|\nabla u\|^2, \text{ hence } \|v\|^2 + 2 \langle v, \nabla u \rangle \leq 0, \text{ i.e.,}
$$

$$
- \frac{1}{\|v\|^2} \langle v, \nabla u \rangle \geq \frac{1}{2}.
$$
Consider the map \( \lambda \mapsto \|\nabla u + \lambda \nabla u\|^2 \); a computation shows that it is decreasing for \( \lambda \leq -(1/\|v\|^2) \langle v, \nabla u \rangle \). Hence, from the previous remarks, we have obtained that for all (fixed) \( x \) in \( E^+ \) and for all \( \lambda \) satisfying \( 0 < \lambda \leq \lambda_1 \), the map \( \lambda \mapsto \|\nabla u(x) + \lambda \nabla u(x)\| \) is decreasing.

(e) Let \((\lambda_n)_n\) decrease to zero. Then \( \|\nabla u(x) + \lambda_n \nabla u(x)\| \) increases to \( \|\nabla u(x)\| \). By the convexity of the map \( \xi \mapsto f(\xi) \), the sequence of maps

\[
\lambda_n \rightarrow -f(\|\nabla u(x) + \lambda_n \nabla u(x)\|) - f(\|\nabla u(x)\|)
\]

is monotonically increasing, on \( E^+ \), to the map \( x \mapsto -f'(\|\nabla u(x)\|) \langle \nabla u(x)/\|\nabla u(x)\|, \nabla u(x) \rangle \). Since each map in the sequence is non-negative and, from the results in (c), its integral

\[
(1/\lambda_n) \int_{E^+} - \left[ f(\|\nabla u(x) + \lambda_n \nabla u(x)\|) - f(\|\nabla u(x)\|) \right] \, dx \]

is bounded above by a constant, independent of \( n \), we can apply the Monotone Convergence Theorem to infer that

\[
\int_{E^+} -f(\|\nabla u(x) + \lambda_n \nabla u(x)\|) - f(\|\nabla u(x)\|) \, dx
\]

\[
\rightarrow \int_{E^+} -f'(\|\nabla u(x)\|) \left( \frac{\nabla u(x)}{\|\nabla u(x)\|}, \nabla u(x) \right) \, dx.
\]

This fact and the corresponding convergence result on \( E^- \) established in (b) prove that

\[
\int_Q f'(\|\nabla u(x)\|) \left( \frac{\nabla u(x)}{\|\nabla u(x)\|}, \nabla u^0(x) - \nabla u(x) \right) \, dx
\]

exists finite. In case (i) of Assumption A,

\[
\int_Q f'(\|\nabla u(x)\|) \left( \frac{\nabla u(x)}{\|\nabla u(x)\|}, \nabla u^0(x) \right) \, dx \leq +\infty
\]

while

\[
\int_Q f'(\|\nabla u(x)\|) \left( \frac{\nabla u(x)}{\|\nabla u(x)\|}, \nabla u \right) \, dx \geq 0,
\]
so we obtain that

\[ f'(\|\nabla u(. )\|) \|\nabla u(. )\| \in L^1(\Omega). \]

In the case (ii), for a suitable \( \varepsilon > 0 \) consider separately the sets 
\( \{ x \in \Omega : \|\nabla u(x)\| \geq \|\nabla u_0(x)\| + \varepsilon \} \) and 
\( \{ x \in \Omega : \|\nabla u(x)\| < \|\nabla u_0(x)\| + \varepsilon \} \) to obtain the same result.

This proves the theorem in the special case \( w = u. \)

(\textit{f}) For \( p \) in the image of \( f' \), the polar of \( f \), \( f^* \), is given by

\[ f^*(p) = p(t_0) - f(t_0), \] where \( f'(t_0) \) = \( p \). Hence

\[ f^*(p) = p(f')^{-1}(p) - f((f')^{-1}(p)). \]

In particular, for \( p = f'(\|\nabla u(x)\|) \), we have

\[ f^*(f'(\|\nabla u(x)\|)) = f'(\|\nabla u(x)\|) \|\nabla u(x)\| - f(\|\nabla u(x)\|) \]

Apply Fenchel's inequality to obtain

\[
\int_{\Omega} f'(\|\nabla u(x)\|) \|\nabla w(x)\| \, dx \\
\leq \int_{\Omega} f^*(f'(\|\nabla u(x)\|)) \, dx + \int_{\Omega} f(\|\nabla w(x)\|) \, dx \\
= \int_{\Omega} f'(\|\nabla u(x)\|) \|\nabla u(x)\| - f(\|\nabla u(x)\|) \, dx + \int_{\Omega} f(\|\nabla w(x)\|) \, dx,
\]

thus proving the theorem.

Conditions for the validity of the Euler–Lagrange equation are expressed in Theorem 3. The following Lemma 1 and Theorem 2 provide the tools needed for its proof.

**Lemma 1.** Let \( \Phi^* : \mathbb{R}^d \to \mathbb{R} \) be the polar to the function \( \Phi : \mathbb{R}^d \to \mathbb{R} \). Let \( g(.) \), \( h(.) \), \( \Phi(g(.) \) \), \( \Phi^*(h(.) \) be in \( L^1(\Omega) \); let \( g_n \to g \) in \( L^1(\Omega) \) and let \( (\Phi(g_n)) \) be equi-integrable. Then \( \int_{\Omega} \langle h, g - g_n \rangle \to 0. \)

**Proof.** Fix \( \varepsilon > 0 \). There exists \( \delta > 0 \) (we can assume \( \delta \leq \varepsilon \)) such that

\[ |E| \leq \delta \implies \int_E \Phi(g_n) \leq \varepsilon/5; \quad \int_E \Phi(g) \leq \varepsilon/5; \quad \int_E \Phi^*(h) \leq \varepsilon/5. \]
Set \( E_n = \{ x : \| g(x) - g_n(x) \| > \delta / \left( 5 \int |h| \right) \} \). Since \( g_n \to g \) in \( L^1 \), there exists \( n^0 \) such that \( n > n^0 \) implies \( |E_n| < \delta \). Then,

\[
\int_\Omega |\langle h, g - g_n \rangle| = \int_{\Omega \setminus E_n} |\langle h, g - g_n \rangle| + \int_{E_n} |\langle h, g - g_n \rangle|
\]

\[
\leq \frac{\delta}{5} \int_\Omega |h| + \int_{E_n} |\langle h, g \rangle| + \int_{E_n} |\langle h, g_n \rangle|
\]

\[
\leq \delta / 5 + \int_{E_n} \Phi^*(h) + \int_{E_n} \Phi(g) + \int_{E_n} \Phi^*(h) + \int_{E_n} \Phi(g_n)
\]

\[
\leq \varepsilon.
\]

This proves the lemma.

**Theorem 2.** Under the same assumptions on \( f, g, u^0, \) and \( u \) as in Theorem 1, assume that the Euler–Lagrange equation \((E-L)\),

\[
\int_\Omega \left[ f\left( \frac{\nabla u(x)}{\| \nabla u(x) \|} \right) \cdot \nabla w(x) \right] + g_n(x, u(x)) w(x) \, dx = 0,
\]

holds for \( w \) in \( W_0 \) and \( \nabla w \) in \( L^\infty(\Omega) \). Then it holds for \( w \) in \( W_0 \).

**Proof.** Fix \( w^* \) in \( W_0 \), i.e., \( w^* = \lambda w \), where \( \int f(\| \nabla w(x) \|) \, dx < \infty \) and there are \( C^1 \) maps \( \eta_n \) with support in \( \Omega \) such that \( \eta_n \) converges to \( w \) in \( W^{1,1}(\Omega) \) and \( f(\| \nabla \eta_n \|) \) is equi-integrable. Proving the result for \( w \) amounts to proving it for \( w^* \).

By assumption we have that

\[
\int_\Omega f\left( \frac{\nabla u(x)}{\| \nabla u(x) \|} \right) \cdot \nabla \eta_n(x) \, dx = 0.
\]

The argument for the convergence of \( g_n(x, u(x)) \eta_n(x) \, dx \) is the same as in point (a) of the Proof of Theorem 1. Hence to prove the theorem it is enough to show that

\[
\int_\Omega f\left( \frac{\nabla u(x)}{\| \nabla u(x) \|} \right) \cdot \left( \nabla \eta_n(x) - \nabla w(x) \right) \, dx \to 0.
\]

By assumption, \( f(\| \nabla w \|) \in L^1(\Omega) \) and, by Theorem 1, \( f^*(f(\| \nabla u \|)) \in L^1(\Omega) \). Hence we can apply Lemma 1, setting \( \Phi(.) = f(\| . \|) \), \( g = \nabla w \), \( g_n = \nabla \eta_n \), and \( h = f(\| \nabla u \|) \nabla u/\| \nabla u \| \). This ends the proof.

The next theorem provides conditions for the validity of \((E-L)\). In it, a class of functions \( f \) growing faster than functions in the exponential class,
e.g., \( f(\xi) = \xi^\lambda \), is allowed. For this case, a bound from below on the growth of \( f \) is needed. No such bound from below is needed for functions of exponential class.

**Theorem 3.** Under the same assumptions on \( f, g, u^0, \) and \( u \) as in Theorem 1, let \( f \) satisfy either

(i) \( f \) is of exponential class, or 
(ii) there exists \( \xi_0, a \geq 0, b \geq 0, K \geq 0, \) and \( \lambda \) positive such that, for \( \xi \geq \xi_0 \), we have \( f'(\xi) \leq (a + b \log(\xi)) f(\xi) \) and \( f(\xi) \leq K f'(\xi) \xi^{1-\lambda} \).

Then the Euler–Lagrange equation holds for a solution \( u \) to Problem (P).

**Proof.** By Theorem 2 it is enough to show the validity of the Euler–Lagrange equation for a variation \( \eta \) in \( W \) and such that \( \nabla \eta \in L^\infty(\Omega) \). We have to show that 

\[
\int_\Omega \frac{f(\|\nabla u(x) + \lambda \nabla \eta(x)\|)}{\lambda} \left( f(\|\nabla u(x)\|) - f(\|\nabla u(x)\|) - g(x, u(x)) \eta(x) \right) dx.
\]

As in (a) of Theorem 1, Assumption B guarantees the convergence of the difference quotient concerning the function \( g \).

About the first term we have

\[
\left| \frac{f(\|\nabla u(x) + \lambda \nabla \eta(x)\|)}{\lambda} - f(\|\nabla u(x)\|) \right| = \frac{\|\nabla \eta(x)\|}{\lambda} f'(\|\nabla u(x)\| + \theta_\lambda(x))
\]

where \( |\theta_\lambda(x)| \leq \lambda \|\nabla \eta(x)\| \).

**Case (i).** Since for \( \xi \geq \xi_0 \) we have that \( f(\lambda \xi + \theta_\lambda) \leq f(\xi) e^{\alpha |\theta_\lambda|} \), we obtain 

\[
(1/\lambda) \left( \frac{f(\xi)}{f(\xi + \lambda \nabla \eta\|) - f(\xi)} \right) \leq Hf(\xi).
\]

From the differential inequality for \( f \) we obtain, for \( \xi \geq \xi_0 \), that 

\[
f(\xi + \theta_\lambda) \leq K f(\xi) (\xi + \theta_\lambda)^{\lambda |\nabla \eta(x)|/\lambda}.
\]

Let \( \lambda \) be so small that, a.e., 

\[
\lambda \|\nabla \eta(x)\| < \alpha \lambda \text{ and let } \xi_1 \geq \xi_0 \text{ be so large that, for } \xi \geq \xi_1,
\]

\[
(\xi + \theta_\lambda)^{\lambda |\nabla \eta(x)|/\lambda} \leq (\xi)^{\alpha}.
\]

Then 

\[
f'(\xi + \theta_\lambda) \leq (a + b \log(\xi + \theta_\lambda)) f(\xi) (\xi)^{\alpha}.
\]

Hence there exists \( \xi_2 \geq \xi_1 \) such that, for \( \xi \geq \xi_2 \),

\[
f'(\xi + \theta_\lambda) \leq K f(\xi) \xi^\lambda.
\]
Hence, for all $\xi \geq \xi_2$, by the assumption on the growth of $f$, we have

$$f'(\xi + \theta \xi) \leq K_2 f'(\xi) \xi.$$ 

Write

$$\int_Q \frac{f(\|\nabla u(x) + \lambda \nabla \eta(x)\| - f(\|\nabla u(x)\|)}{\lambda} d\lambda$$

as

$$\int_{\{x : \|\nabla u(x)\| \leq \xi_2\}} \frac{f(\|\nabla u(x) + \lambda \nabla \eta(x)\| - f(\|\nabla u(x)\|)}{\lambda} d\lambda + \int_{\{x : \|\nabla u(x)\| > \xi_2\}} \frac{f(\|\nabla u(x) + \lambda \nabla \eta(x)\| - f(\|\nabla u(x)\|)}{\lambda} d\lambda.$$ 

Convergence of the first integral to $\int f'(|\nabla u(x)|)\langle \nabla u(x)/|\nabla u(x)|, \nabla \eta \rangle d\lambda$ is assured by the uniform boundedness of the derivative of $f$ on bounded sets. The integrand of the second term, in both cases (i) and (ii), is bounded by a constant times (the restriction to the set $\{x : \|\nabla u(x)\| > \xi_2\}$ of $f'(|\nabla u(x)|)\|\nabla u(x)\|$, an integrable function from Theorem 1, so that convergence follows again by dominated convergence.

Since $u$ is a solution, we have

$$\int_Q \frac{f(\|\nabla u(x) + \lambda \nabla \eta(x)\| - f(\|\nabla u(x)\|) + g(x, u(x) + \lambda \eta(x)) - g(x, u(x))}{\lambda} d\lambda \geq 0.$$ 

By passing to the limit we obtain

$$\int_Q f'(|\nabla u(x)|) \left( \frac{\nabla u(x)}{|\nabla u(x)|}, \nabla \eta \right) + g(x, u(x)) \eta(x) d\lambda \geq 0.$$ 

$W_0$ being a vector space, we have proved the validity of the Euler–Lagrange equation.

REFERENCES

