Edge and total coloring of interval graphs

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Abstract

An edge coloring of a graph is a function assigning colors to edges so that incident edges acquire distinct colors. The least number of colors sufficient for an edge coloring of a graph $G$ is called its chromatic index and denoted by $\chi'(G)$. Let $\Delta(G)$ be the maximal degree of $G$; if $\chi'(G) = \Delta(G)$, then $G$ is said to belong to class 1, and otherwise $G$ is said to belong to class 2. A total coloring of a graph is a function assigning colors to its vertices and edges so that adjacent or incident elements acquire distinct colors. The least number of colors sufficient for a total coloring of a graph $G$ is called its total chromatic number and denoted by $\chi_T(G)$. If $\chi_T(G) = \Delta(G) + 1$ then $G$ is said to belong to type 1, and if $\chi(G) = \Delta(G) + 2$ then $G$ is said to belong to type 2. We consider the problem of classifying interval graphs and prove that every interval graph with odd maximal degree belongs to class 1; its edges can be colored in the minimal number of colors in time $O(|V_G| + |E_G| + (\Delta(G))^2)$. Then we show that the conjecture of Behzad and Vizing that $\chi_T(G) \leq \Delta(G) + 2$ holds for interval graphs. We also prove that every interval graph with even maximal degree belongs to type and its elements can be totally colored in time $O(|V_G| + |E_G| + (\Delta(G))^2)$. © 2001 Published by Elsevier Science B.V.

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1. Basic definitions and facts

All the notions which are not defined here can be found in [15]. An edge coloring of a graph is a function assigning colors to edges so that incident edges acquire distinct colors. The least number of colors sufficient for an edge coloring of a graph $G$ is called its chromatic index and denoted by $\chi'(G)$. The maximal degree of $G$ is denoted by $\Delta(G)$. It is known [4] that the problem of finding a minimal edge coloring of a graph is in general case NP-complete.

The following result is classical:

**Theorem 1** (Vizing [12]). *The chromatic index of a graph* $G$ *is equal either to* $\Delta(G)$ *or to* $\Delta(G) + 1$.

A graph $G$ is said to belong to *class 1* if $\chi'(G) = \Delta(G)$ and to *class 2* otherwise, and the problem of classification of graphs according to their classes arises naturally. This problem is solved for some particular families, including complete and bipartite graphs, planar graphs with maximal degree at least 8, and graphs whose maximal degree is large with respect to the number of vertices. Polynomial-time algorithms are known for edge coloring of such graphs in minimal number of colors [3].

A graph $G$ is called *$\rho$-critical* if $\Delta(G) = \rho$, $G$ belongs to class 2, and every graph obtained from $G$ by deleting an edge has a lesser chromatic index. The properties of critical graphs have been studied by many authors [2,3]; the main achievement in this theory is now the following adjacency lemma by Vizing.

**Lemma 1** (Vizing [13]). Let $G$ be a $\rho$-critical graph, $v$ and $w$ be two adjacent vertices in $G$, and $\deg(v) = k$. Then

1. If $k < \rho$, then $w$ is adjacent to at least $\rho - k + 1$ vertices of degree $\rho$;
2. If $k = \rho$, then $w$ is adjacent to at least two vertices of degree $\rho$.

A total coloring of a graph is a function assigning colors to its vertices and edges so that adjacent or incident elements acquire distinct colors. The least number of colors sufficient for a total coloring of a graph $G$ is called its *total chromatic number* and denoted by $\chi_T(G)$. It is known [11] that the problem of finding a minimal total coloring of a graph is in general case NP-complete.

**Behzad’s and Vizing’s Total Coloring Conjecture** (Behzad[1] and Vizing [13]). *For all* $G$ *the inequality* $\chi_T(G) \leq \Delta(G) + 2$ *holds.*

Nowadays the total coloring conjecture is proved for some particular families of graphs, including complete $r$-partite graphs, graphs whose maximal degree is large with respect to the number of vertices, graphs with $\Delta(G) \leq 5$, and planar graphs with maximal degree at least 8 [5,6,14].

A graph $G$ is said to belong to *type 1* if $\chi(G) = \Delta(G) + 1$; it is said to belong to *type 2* if $\chi(G) = \Delta(G) + 2$.

The problem of classification of graphs according to their types is solved completely, for example, for such graphs $G$ that $\Delta(G) \geq |V_G| - 2$, for bipartite graphs, and for some families of regular graphs.

A graph is called *interval* if it can be represented by finite intervals on the real line so that two vertices are adjacent if and only if the two corresponding intervals intersect.
A well-known characterization of interval graphs is the following: a graph \( G \) is interval if and only if the set of its maximal cliques \( \{ K_i \}_{i \in I} \) can be linearly ordered so that for each vertex \( v \) of \( G \)

if \( v \in K_i, v \in K_j, \) and \( i < j, \) then \( v \in K_l, \) \( l = i, i + 1, \ldots, j. \)

We shall call such an ordered sequence of maximal cliques a **characteristic maximal cliques sequence**.

Let \( G \) be an interval graph, and let \( \{ K_i \}_{i \in I} \) be its characteristic maximal cliques sequence. The set of maximal cliques containing a vertex \( v \) will be called a **cluster induced by** \( v \) and denoted by \( KL(v): KL(v) = \{ K_i | i \in I, v \in K_i \}. \)

It follows from the definitions that the cluster induced by a vertex is a segment of the characteristic maximal cliques sequence.

2. Main results

The main results of the paper are Theorems 2–4. To prove them, we use the following auxiliary facts:

**Lemma 2** (Fiorini [3]). Let \( K_n \) be the complete graph with \( n \) vertices. Then

\[
\chi'(K_n) = \begin{cases} 
  n & \text{if } n = 2m + 1, \\
  n - 1 & \text{if } n = 2m.
\end{cases}
\]

**Lemma 3** (Yap [14]). Let \( K_n \) be the complete graph with \( n \) vertices. Then

\[
\chi_1(K_n) = \begin{cases} 
  n & \text{if } n = 2m + 1, \\
  n + 1 & \text{if } n = 2m.
\end{cases}
\]

**Theorem 2.** Let \( G \) be an interval graph with \( \Delta(G) = 2m + 1. \) Then \( G \) can be colored with \( 2m + 1 \) colors in time \( O(|V(G)| + |E(G)| + (\Delta(G))^2) \).

**Proof.** Let us consider a characteristic maximal cliques sequence \( K_1, K_2, \ldots, K_N \) of a graph \( G \). Without loss of generality, we may assume that there exists a vertex \( v' \in K_1 \) such that \( \text{deg}(v') = \Delta(G) = 2m + 1 \) (otherwise we may add to \( K_1 \) a needed number of dummy vertices). Since every edge of \( G \) belongs to at least one maximal clique, an edge coloring of \( G \) is equivalent to an edge coloring of all maximal cliques of \( G \) satisfying the following conditions:

1. if an edge belongs to several cliques, then it has the same color in all of them;
2. any two edges incident to the same vertex have distinct colors.

In what follows, we will use the notation \( \text{Trans}(r, r + 1) = \{ v | v \in K_r \cap K_{r+1} \}. \)
An edge coloring of all maximal cliques of $G$ satisfying the conditions above can be obtained by the following algorithm.

**Algorithm.**

*Step 0.*
Take a clique on $2m + 2$ vertices.
Number the vertices of the clique arbitrarily.
Color the edges of the clique in $2m + 1$ colors according to Lemma 2.
Create a $(2m + 2) \times (2m + 2)$ matrix $A = (a_{ij})$.
Fill in $A$ as follows: if an edge $(i, j)$ is colored by the color $c$, then $a_{ij} := c$, $c \in \{1, 2, \ldots, 2m + 1\}$.

*Step 1.*
Number the vertices of $KL(v')$ arbitrarily.
Color the edges of $KL(v')$ as follows: if $a_{ij} = c$, then assume the edge connecting vertices $i$ and $j$ colored in color $c$.

*Step i.*
Let $K_1, K_2, \ldots, K_r$ be the cliques whose edges are already colored. Compute Trans$(r, r + 1)$.
Choose a vertex $v^*$ as the first vertex of Trans$(r, r + 1)$ occurring while viewing the maximal cliques sequence from the left to the right.
Consider the cluster $KL(v^*)$. Some of its vertices are already colored at previous steps. Since $\deg(v^*) \leq 2m + 1$, we have $|KL(v^*)| \leq 2m + 2$. That is why all the other vertices from $KL(v^*)$ can be numbered so that the number of any of them would not exceed $2m + 2$ and no two vertices had the same number.
After the vertices are numbered, assign the color $c$ to each non-colored edge from $KL(v^*)$ connecting vertices numbered $i$ and $j$ if and only if $a_{ij} = c$.

**End.**

**Correctness of the Algorithm.** The proof is by induction.

At Step 1, the edges of the first cluster are colored properly because in fact we built an isomorphic embedding of the cluster to a colored clique on $2m + 2$ vertices.

Now suppose that after the first $i$ steps a correct partial edge coloring is built. Let $K_1, K_2, \ldots, K_r$ be the cliques whose edges were colored during the first $i$ steps, and let $K_i, K_{i+1}, \ldots, K_r, K_{r+1}, \ldots, K_{r+c}$ be the cluster induced by the vertex $v^* \in \text{Trans}(r, r + 1)$ chosen at the $(i + 1)$th step.

Suppose that the partial coloring obtained after coloring edges of $KL(v^*)$ is not correct, that is, there exist two edges $e_1 = (a, b)$, $e_2 = (b, c)$ colored by the same color. By the induction hypothesis, the edges of $K_1, K_2, \ldots, K_r$ were colored properly, so that at least one of $\{e_1, e_2\}$ does not belong to $K_1, K_2, \ldots, K_r$. Without loss of generality, we assume that $e_2$ was treated at the $(i + 1)$th step for the first time. Since the vertex $b$
belongs to at least one of the cliques $K_r, \ldots, K_{r+c}$, it follows from the choice of $v^*$ that $a$ and, therefore, $e_1$ lie in $KL(v^*)$. At Step $(i+1)$, all the vertices of $KL(v^*)$ received different numbers. Assume that $N(a) = i$, $N(b) = j$, and $N(c) = k$. It follows from the performance of the algorithm that the edges $(i,j)$ and $(j,k)$ received the same number in the properly colored clique, a contradiction.

**Time Complexity.** All the sets $\text{Trans}(i,i+1)$ and vertices $v^*$ can be found already while constructing the characteristic maximal cliques sequence. Constructing this sequence takes $O(|V_G| + |E_G|)$ steps [7,8]. Coloring a clique with $\Delta(G) + 1$ vertices and filling the matrix $A$ by the information on the coloring will take $O((\Delta(G))^2)$ steps.

Since every vertex receives a unique number, and every edge is colored only once, it follows that the time complexity of the algorithm does not exceed $O(|V_G| + |E_G| + (\Delta(G))^2)$, and the theorem is proved.

The similar theorem does not hold for an interval graph with even maximal degree since a complete graph with odd number of vertices is a counterexample.

The following theorems can be proved analogously to Theorem 2 using Lemma 3.

**Theorem 3.** Let $G$ be an interval graph with $\Delta(G) = 2m$. Then $G$ can be totally colored in $2m + 1$ colors in time $O(|V(G)| + |E(G)| + (\Delta(G))^2)$.

The similar theorem does not hold for an interval graph with odd maximal degree. A counterexample belonging to type 2 is a complete graph with even number of vertices. Thus, if $\Delta(G) = 2m + 1$, then even a single cluster may belong to type 2.

**Theorem 4.** Let $G$ be an interval graph. Then $\chi_T(G) \leq \Delta(G) + 2$.

It follows from Theorem 4 that the total coloring conjecture holds for all interval graphs.

3. Conclusion

As it has been mentioned above, if the maximal degree of the graph $G$ is even, then even a single cluster can belong to class 2. However, the following criterion for clusters is known.

**Theorem 5** (Plantholt [9,10]). Let $G$ be a graph of an odd order $2s + 1$ containing a vertex adjacent to all the other vertices. Then $G$ belongs to class 1 if and only if $G$ has at most $2s^2$ edges.

Clearly, a necessary condition for a graph to belong to class 1 is for all its clusters induced by vertices of degree $\Delta(G)$ to do so. It seems plausible that this condition is also sufficient.
Conjecture. Let $G$ be an interval graph with maximal degree $2m$. Then $G$ belongs to class 2 if and only if it has a vertex $v^*$ such that $\text{deg}(v^*) = 2m$ and $KL(v^*)$ belongs to class 2.

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References