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Discrete Applied Mathematics 64 (1996) 207–217

DISCRETE APPLIED MATHEMATICS

Finding Hamiltonian cycles in Delaunay triangulations is NP-complete[☆]

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Received 18 November 1992; revised 27 July 1994

Abstract

It is shown that it is an NP-complete problem to determine whether a Delaunay triangulation or an inscribable polyhedron has a Hamiltonian cycle. It is also shown that there exist nondegenerate Delaunay triangulations and simplicial, inscribable polyhedra without 2-factors.

1. Introduction

The existence of Hamiltonian cycles in Delaunay triangulations and inscribable polyhedra is a question of both practical and theoretical significance. The practical importance stems from the fact that a Hamiltonian cycle in the Delaunay triangulation of a set of points is a natural candidate for a short spanning cycle through the points, and hence might be expected to be a good approximation for the Euclidean Traveling Salesman Cycle (ETSC). Heuristics for approximating the ETSC, using the Delaunay triangulation as a starting point, can be found in [27, 32]. Applications of Hamiltonian cycles in Delaunay triangulations to problems in pattern recognition and solid modeling are discussed in [3, 22, 24, 25].

From a more theoretical viewpoint, there appears to be a close connection between the structure of inscribable polyhedra and Hamiltonian cycles. Hamiltonicity is "almost" sufficient for inscribability. For example, Crapo and Laumond [10, p. 303] have observed that any Hamiltonian polyhedron is inscribable in a certain degenerate sense. More recently, Dillencourt and Smith [15] have shown that any 1-Hamiltonian planar graph is inscribable.[1] Since 4-connected graphs are 1-Hamiltonian [34,

[☆] The support of a UCI Faculty Research Grant is gratefully acknowledged. This research was supported in part by the University of California, Irvine through an allocation of computer resources.

[1] A graph G is k-Hamiltonian if any graph obtained by removing k vertices from G is Hamiltonian.

SSDI 0166-218X(94)00125-1

36] and 2-Hamiltonian [33], it follows that all 4-connected planar graphs and all graphs obtained by deleting a single vertex from a 4-connected planar graph are inscribable. Conversely, empirical evidence suggests that Delaunay triangulations of moderate size are Hamiltonian with high probability [17]. It has also been shown that inscribable polyhedra and Delaunay triangulations have certain Hamiltonian-like properties. For example, Delaunay triangulations are 1-tough [14].[2] This implies, in particular, that all Delaunay triangulations have perfect matchings (1-factors) [14].

The question of whether Delaunay triangulations necessarily have Hamiltonian cycles was posed in [22, 24], and, in a closely related form, in [31]. Counterexamples satisfying progressively more restrictive conditions can be found in [19, 11, 12]. These counterexamples suggest the *computational* question: what is the computational complexity of finding Hamiltonian cycles in Delaunay triangulations? There have been some partial results aimed at addressing this question [7, 8, 10, 13, 20]. In the present paper, we settle the computational question by showing that it is an NP-complete problem to determine whether there is a Hamiltonian cycle in a simplicial inscribable graph (Theorem 3.1) or in a nondegenerate Delaunay triangulation (Theorem 3.4). We also strengthen the non-Hamiltonian counterexamples cited above by showing that there exist inscribable polyhedra (and Delaunay triangulations) that fail to have 2-factors (Section 4).

2. Preliminaries

Except as noted, we use the graph-theoretic terminology of [4]. $V(G)$ and $E(G)$ denote, respectively, the set of vertices and edges of a graph G. A *Hamiltonian cycle* in a graph is a spanning cycle. A *2-factor* in a graph is a spanning collection of disjoint cycles. The *link-distance* between two vertices of a graph is the minimum number of edges in a path connecting them. A graph is *trivalent* if all vertices have degree 3. A *plane graph* is a planar graph together with a combinatorial embedding in the plane and, in particular, identification of the unbounded face. A plane graph is *simplicial*, or *maximal planar*, if all its faces are triangles. A *cutset* in a graph G is a minimal set of edges whose removal increases the number of components of G. A *noncoterminous cutset* is a cutset in which not all edges have a common endpoint. A *dual cycle* (*dual path*) in G is a cycle (path) in the planar dual of G. A cutset in a plane graph G corresponds, in a natural fashion, to a dual cycle.

The *Delaunay triangulation* is the dual of the Voronoi diagram; see [2, 16, 26] for details. In particular, a Delaunay triangulation is *nondegenerate* if all interior faces are triangles. An *inscribed polyhedron* is a convex polyhedron all of whose vertices lie on a common sphere. A graph is *inscribable* (respectively, *Delaunay realizable*) if it can be

[2] A graph G is *1-tough* if for any nonempty set S of vertices of G, $c(G - S) \leqslant |S|$, where $c(G - S)$ is the number of components of $G - S$ and $|S|$ denotes the cardinality of S.

realized as a combinatorially equivalent inscribed polyhedron (respectively, *Delaunay realizable*) if it can be realized as a combinatorially equivalent inscribed polyhedron (respectively, Delaunay triangulation). If G is a plane graph, and f is a face of G, the operation of *stellating* the face f consists of adding a new vertex in the interior of f and connecting all vertices incident on f to the new vertex. The following lemma, which is closely related to a result in [5], is an easy consequence of standard properties of stereographic projection [9].

Lemma 2.1. *A plane graph G is Delaunay realizable, with face f as its unbounded face, if and only if the graph obtained from G by stellating f is inscribable.*

Our proof also makes use of the following numerical characterization of inscribable polyhedra, due to Rivin [28] (also see [18, 29, 30]).

Theorem 2.2. *A graph G is inscribable if and only if it is planar, 3-connected, and weights w can be assigned to its edges such that:*

(W1) *For each edge e, $0 < w(e) < 1/2$.*

(W2) *For each vertex v, the total weight of all edges incident on v is equal to* 1.

(W3) *For each noncoterminous cutset $C \subseteq E(G)$, the total weight of all edges in C is strictly greater than* 1.

A weighting satisfying conditions (W1)–(W3) will be called a *proper weighting.*

3. Proof of NP-completeness of recognizing Hamiltonian inscribable graphs

Theorem 3.1. *It is an* NP*-complete problem to determine whether a simplicial, inscribable graph is Hamiltonian.*

The problem is clearly in NP, so it is only necessary to show NP-hardness. The reduction is from the recognition problem for Hamiltonian 2-connected bipartite trivalent planar graphs (H2BTP), which was shown to be NP-hard in [1].

Our method extends the construction used by Chvátal [6, p. 427] to show that the recognition problem for Hamiltonian maximal planar graphs is NP-hard. Our construction is more delicate, because the maximal planar graphs produced by our reduction must be inscribable. It turns out that this task is facilitated if the reduction starts from instances of H2BTP in which vertex cuts of size 2 are sparse in a certain sense. So our proof proceeds in two stages. First we show that a restricted version of H2BTP is NP-hard (Lemma 3.3). We then reduce the restricted H2BTP to the Hamiltonian cycle problem for inscribable simplicial graphs.

Let G be any 2-connected plane graph. A *separating pair* of G is a pair of vertices whose removal causes G to become disconnected. Define the *separator set* S of G to be the set of all vertices that are in some separating pair of G. A 2-connnected, bipartite

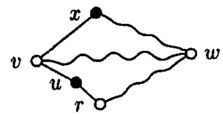

Fig. 1. Proof of Lemma 3.2.

graph has *isolated same-color separators* if the link distance between any two vertices in its separator set that have the same color is at least 4.

Lemma 3.2. *If a 2-connected, bipartite, trivalent graph G has isolated same-color separators, then any vertex in the separator set of G has exactly one neighbor in the separator set.*

Proof. Let S be the separator set of G, $v \in G$. It follows immediately from the definition of isolated same-color separators that v has at most one neighbor in S. Let w be a vertex of G such that $\{v, w\}$ is a separating pair (see Fig. 1). If w is a neighbor of v, we are done, so assume it is not. Since G is trivalent, some component of $G - \{v, w\}$ contains exactly one neighbor of v. Call this neighbor u, and let x be some other neighbor of v. By trivalency, u has at least one neighbor, r, distinct from v and w. Any path from r to x must pass through either v or w, and if it passes through v without passing through w it must first pass through u. So $\{u, w\}$ is a separating pair which implies $u \in S$. □

The problem H2BTPX is the problem of determining whether a 2-connected, bipartite, trivalent, planar graph with isolated same-color separators is Hamiltonian.

Lemma 3.3. *H2BTPX is* NP-*complete.*

Proof. Let G be a 2-connected, bipartite, trivalent plane graph with n vertices. For each vertex v of G, let u, w, and x be its neighbors. Replace v with the 25-vertex configuration inside the circle in the right half of Fig. 2, with connections to the three neighbors of v as illustrated. Let G' be the graph obtained by transforming each vertex of G in this fashion (so G' has $25n$ vertices). It is easy to see that this transformation preserves 2-connectedness, bipartiteness, planarity, and trivalency. G is Ham iltonian if and only if G' is. This is illustrated in Fig. 2, which shows how the path uvx in a Hamiltonian cycle through G is transformed into a path from u to x in a corresponding Hamiltonian cycle through G'. Finally, G' has isolated same-color separators. This last statement follows from the fact that if any vertex inside the circle in Fig. 2 other than u', w', or x' is deleted, there is still a path through the vertices inside the circle connecting any two of u, w, and x. Hence H2BTP can be reduced to H2BTPX in polynomial (actually linear) time. □

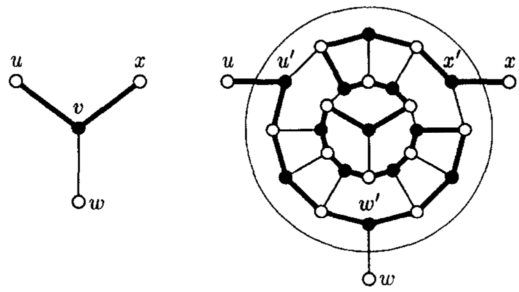

Fig. 2. Construction used to prove H2BTPX is NP-complete. The thick edges illustrated the mapping from a Hamiltonian cycle in G to a Hamiltonian cycle in G'.

To complete the proof of Theorem 3.1 we reduce H2BTPX to the problem of detecting Hamiltonian cycles in simplicial, inscribable graphs. Let G be an n-vertex, 2-connected bipartite trivalent plane graph with isolated same-color separators. Two-color the vertices of G red and blue. Let L be the *medial graph* of G [23]. That is, the vertices of L are the midpoints of the edges of G, and two vertices of L are joined by an edge if and only if the corresponding edges of G are consecutive edges on a common face of G. Since G is planar, L is planar and regular of degree 4. Since G is trivalent, every vertex of G corresponds to a triangular face of L. Call such a triangular face of L a *distinguished triangle*. A distinguished triangle of L is a *red triangle* (respectively, a *blue triangle*) if it corresponds to a red vertex (respectively, a blue vertex) of G. An *s-triangle* is a triangle that corresponds to a vertex in the separator set of G. An *s-vertex* is a vertex of L shared by two s-triangles of L. An s-vertex corresponds to an edge of G connecting two vertices in the separating set of G. By Lemma 3.2, every s-triangle has exactly one s-vertex.

Since G is trivalent, n is even. It follows from Euler's formula and the regularity of G and L that L has $3n/2$ vertices, $3n/2 + 2$ faces (of which n are distinguished triangles), and $3n$ edges.

We add edges and vertices to L to turn it into a simplicial graph in two steps.

1. Replace each blue triangle with the graph shown in Fig. 3(a), identifying the three outer vertices of the figure with the three vertices of the triangle. Replace each red triangle with the graph shown in Fig. 3(b), identifying the three outer vertices of the figure with the three vertices of the triangle. Call the resulting graph K.

2. Add new edges to K to obtain a simplicial graph. This requires triangulating each face that is not part of a distinguished triangle. The new edges are added in such a way that no s-vertex is incident on an edge. This is possible because no two s-vertices appear consecutively along any face of L. Call the resulting simplicial graph H. The number of edges added at this stage is the number of edges required to transform L into a simplicial graph which, by Euler's formula is $3n/2 - 6$.

We claim that H is Hamiltonian if and only if G is. Suppose H has a Hamiltonian cycle, Z. A path through a red triangle that visits all interior vertices must visit all

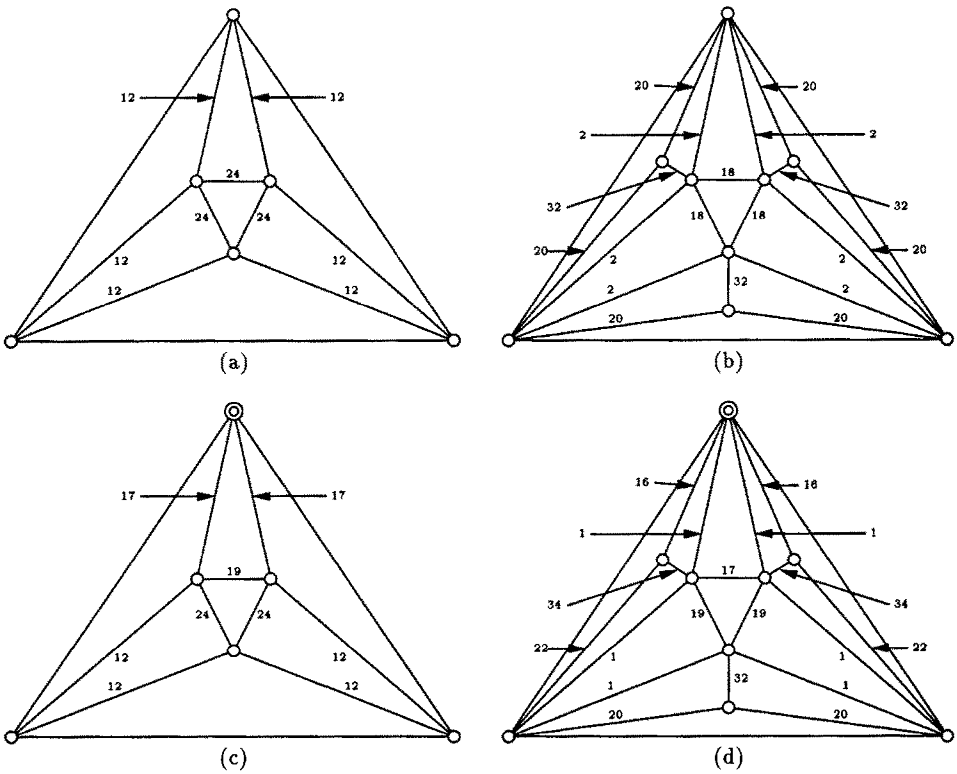

Fig. 3. Construction and weight assignments used to reduce H2BPX to the Hamiltonian cycle problem for inscribable graphs. For clarity, all weights are shown multiplied by 72. (a) A blue triangle that is not an s-triangle. (b) A red triangle that is not an s-triangle. (c) A blue s-triangle. (d) A red s-triangle. In (c) and (d), the double circle represents the s-vertex.

three boundary vertices of the triangle. A path through a blue triangle may visit all interior vertices while visting only two boundary vertices of the triangle (the entry and exit vertices). Hence any portion of a path that visits the interior of a red triangle followed by the interior of a blue triangle must visit four boundary vertices of interior triangles (including the entry vertex to the red triangle and the exit vertex from the blue triangle). Since Z can enter the interior of a distinguished triangle only once, visiting the interiors of all n distinguished triangles of H requires visiting all $3n/2$ vertices of distinguished triangles. So Z necessarily alternates between red and blue triangles, never using any of the edges of $H - K$. It follows that the sequence of points in G corresponding to the distinguished triangles visited by Z represents a Hamiltonian cycle of G. Conversely, a Hamiltonian cycle through G corresponds to a sequence of triangles in H that give rise to a Hamiltonian cycle in H.

To complete the proof, we must show that H has a proper weighting, and hence is inscribable. To do this, we construct a weighting of K, extend it to H, and verify that the weighting has the required properties. We begin by assigning each edge of the medial graph L a weight of 1/72. Since L is 4-valent, these edges contribute a total weight of 1/18 to each vertex. We then assign weights to edges inside a distinguished triangle according to Fig. 3. There are four cases: (a) a blue triangle that is not an s-triangle; (b) a red triangle that is not an s-triangle; (c) a blue s-triangle; and (d) a red s-triangle. Note that the labels in the figures are the weights multiplied by 72. By Lemma 3.2 an s-triangle has exactly one s-vertex. It may be verified by inspection that this weighting satisfies all (W1) and (W2) constraints. In addition, all (W3) constraints corresponding to cutsets that dualize to cycles remaining within a single distinguished triangle are satisfied.

To extend the weighting to H, we let $\tau = 1/(288n)$, and we process each edge $e = uv$ in $H - K$ as follows. We assign e a weight of τ. By our construction, neither endpoint of e is an s-vertex. Hence u is incident on at least one distinguished triangle T that is not an s-triangle. Decrease the weights of the two edges of T incident on u by $\tau/2$, and increase the weight on the third edge of T by $\tau/2$. Do the same thing with the other endpoint v. When we process an edge of $H - K$ in this manner, the sum of the weights of edges incident on each vertex is preserved, so all (W2) constraints continue to hold. As each edge of $H - K$ is added, no edge of K has its weight changed by more than τ. Since the number of edges of $H - K$ is $3n/2 - 6 < 2n$, each edge of K has its weight decreased by less than $(2n)\cdot\tau = 1/144$, so all edges of K continue to have positive weights and hence all (W1) constraints remain satisfied. All (W3) constraints for dual cycles that remain within a single distinguished triangle also remain satisfied, as the weights of edges that are inside distinguished triangles are unchanged.

To verify the (W3) constraints for arbitrary dual cycles, we first make the following observations, which may be verified by inspecting Fig. 3:

(C1) Any dual path that enters and leaves a distinguished triangle picks up a total weight $> 1/3$ from that triangle.

(C2) Any dual path that enters and leaves a distinguished triangle in such a way that not all the edges it crosses in the triangle have a common endpoint picks up a total weight $> 2/3$ from that triangle.

(C3) Any dual path that enters and leaves an s-triangle in such a way that all the edges it crosses are incident on the s-vertex picks up a total weight of 1/2 from that triangle.

Now suppose we are given a dual cycle, Z, that does not remain within a single distinguished triangle. Z must pass through at least two distinguished triangles. There are three cases.

Case 1: Z passes through at least three distinguished triangles. In this case, the total weight of Z is > 1 by (C1).

Case 2: Z passes through exactly two distinguished triangles, T and U, and these two triangles share a common vertex, v. If the edges crossed within at least one of the triangles (say T) do not all have a common endpoint, then the edges crossed in T have total weight $> 2/3$ and the edges crossed in U have total weight $> 1/3$ (by (C2) and (C1), respectively). If the edges crossed within either triangle have a common endpoint, the common endpoint must be v. Thus if the edges crossed within both triangles have a common endpoint, the cycle is the face ring about v, so the appropriate constraint is a vertex constraint, rather than a cycle constriant, and we have already observed that all vertex constraints are satisfied.

Case 3: Z passes through exactly two distinguished triangles, T and U, which do not share a vertex. In this case, T and U correspond to a separating pair cutset of G, so they are both s-triangles. If the edges crossed in either triangle are not all incident on a common vertex, then the total weight of the edges crossed is > 1, by (C1) and (C2). Hence we may assume that the edges crossed within T are all incident on a common vertex t, and the edges crossed within U are all incident on a common vertex u. If we were to "slide" Z across t from T into the adjacent distinguished triangle, we could preserve the property that Z is a dual cycle and only passes through two distinguished triangles. Consequently, t is an s-vertex. A similar argument shows that u is an s-vertex. By (C3), it follows that the edges crossed on or inside T and the edges crossed on or inside U have total edge weight 1. Since Z also crosses edges of $H - K$, and these edges have positive weight, the total sum of the crossed edges is > 1.

Since all cycle constraints are satisfied, the weighting is a proper weighting, so H is inscribable. Since H is Hamiltonian if and only if G is, the NP-completeness follows from Lemma 3.3. □

Theorem 3.4. *It is* NP-*complete to determine whether a nondegenerate Delaunay triangulation is Hamiltonian.*

Proof. It suffices to prove that the graph H constructed in the course of the proof of Theorem 3.1 is Delaunay realizable. Choose one blue triangle of H that is not an s-triangle. The edges inside this triangle will be as shown in Fig. 3(a). Let T be the triangle at the center of this figure (the one with all three edges labeled 24). Stellate T, and alter the edge weights so that the three edges of T are labeled 12 and the three edges incident on the stellating vertex are labeled 24, bearing in mind that the labels are the edge weights multiplied by 72. Treat all other distinguished triangles as in the proof of Theorem 3.1. The argument of Theorem 3.1 shows that the resulting graph is inscribable. Hence H is Delaunay realizable, by Lemma 2.1. □

4. An inscribable graph with no 2-factor

Consider the 25-vertex graph shown in Fig. 4. We claim this graph is inscribable and has no 2-factor. To show inscribability, we describe a proper weighting. For clarity, each edge weight is multiplied by 132. Each edge incident on the degree-3 vertex in the center of the graph has weight 44, and the three edges connecting two vertices denoted by squares have weight 2. The remainder of the graph consists of three copies of the graph shown in Fig. 5, weighted as indicated.

The absence of a 2-factor follows from the following special case of Tutte's factor theorem [35].

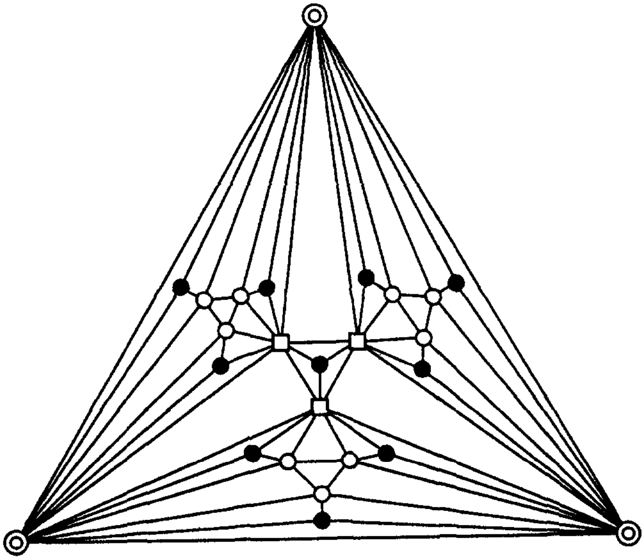

Fig. 4. An inscribable graph with no 2-factor.

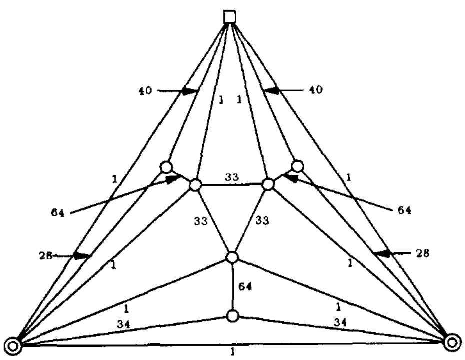

Fig. 5. Edge weightings for the graph of Fig. 4.

Theorem 4.1. *A graph G fails to have a 2-factor if and only if the vertices of G can be partitioned into three sets R, S, and T such that*

$$2|T| < c_e(R) + 2|S| - \sum_{s \in S} d_{S \cup R}(s), \tag{1}$$

where $|\cdot|$ denotes the number of vertices, $c_e(R)$ is the number of components of R that are joined to S by an odd number of edges, and $d_{S \cup R}(s)$ is the degree of s in the subgraph of G induced by $S \cup R$.

In Fig. 4, let R consist of the nine light vertices denoted by single circles, let S consist of the ten dark vertices, and let T consist of the remaining six vertices (denoted by squares and double circles). R has three components, each of which is joined to S by exactly three edges, so $c_e(R) = 3$. All but one vertex of S has one neighbor in R, the tenth has none, and S is an independent set, so the sum in (1) is 9. Hence the left-hand side of (1) is 12 while the right-hand side is $3 + 20 - 9 = 14$, so (1) holds and the graph has no 2-factor.

If any face containing at least one degree-three vertex is stellated, then the graph of Fig. 4 remains inscribable, so it follows from Lemma 2.1 that there exists a nondegenerate Delaunay triangulation with no 2-factor.

Acknowledgements

It is a pleasure to acknowledge stimulating conversations with Warren Smith and Igor Rivin. Drew Bliss implemented the first release of **GraphTool** [21], which played a critical role in the discovery of the example of Section 4.

References

[1] T. Akiyama, T. Nishizeki and N. Saito, NP-completeness of the Hamiltonian cycle problem for bipartite graphs, J. Inform. Process. 3 (1980) 73–76.

[2] F. Aurenhammer, Voronoi diagrams – a survey of a fundamental geometric data structure, ACM Comput. Surveys 23 (1991) 345–405.

[3] J.-D. Boissonnat, Geometric structures for three-dimensional storage representation. ACM Trans. Graph. 3 (1984) 256–286.

[4] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (North-Holland, New York, 1976).

[5] K.Q. Brown, Voronoi diagrams from convex hulls, Inform. Process. Lett. 9 (1979) 223–228.

[6] V. Chvátal, Hamiltonian cycles, in: E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan and D.B. Shmoys, eds., The Traveling Salesman Problem, Ch. 11 (Wiley, New York, 1985) 403–429.

[7] R.J. Cimikowski, Finding Hamiltonian cycles in certain planar graphs, Inform. Process. Lett. 35 (1990) 249–254.

[8] R.J. Cimikowski, On certain Hamiltonian inner triangulations, Discrete Appl. Math. 43 (1993) 185–195.

[9] H.S.M. Coxeter, Introduction to Geometry (Wiley, New York, 2nd ed., 1969).

[10] H. Crapo and J.-P. Laumond, Hamiltonian cycles in Delaunay complexes, in: J.-D. Boissonnat and J.-P. Laumond, eds., Geometry and Robotics Workshop, Toulouse, France (1989), Lecture Notes in Computer Science 391 (Springer, Berlin, 1989).

[11] M.B. Dillencourt, A non-Hamiltonian, nondegenerate Delaunay triangulation, Inform. Process. Lett. 25 (1987) 149–151.

[12] M.B. Dillencourt, An upper bound on the shortness exponent of inscribable graphs, J. Combin. Theory Ser. B 46 (1989) 66–83.

[13] M.B. Dillencourt, Hamiltonian cycles in planar triangulations with no separating triangles, J. Graph Theory 14 (1990) 31–40.

[14] M.B. Dillencourt, Toughness and Delaunay triangulation, Discrete Comput. Geom. 5 (1990) 575–601.

[15] M.B. Dillencourt and W.D. Smith, Graph-theoretical conditions for inscribability and Delaunay realizability, in: Proceedings of the Sixth Canadian Conference on Computational Geometry, Saskatoon, Sask. (1994).
[16] H. Edelsbrunner, Algorithms in Combinatorial Geometry, EATCS Monographs on Theoretical Computer Science 10 (Springer, Berlin, 1987).
[17] T. Genoud, Etude du caractère hamiltonien de Delaunays aléatoires, Working Paper, Ecole Polytechnique Federale de Lausanne (1989).
[18] C.D. Hodgson, I. Rivin and W.D. Smith, A characterization of convex hyperbolic polyhedra and of convex polyhedra inscribed in the sphere, Bull. Amer. Math. Soc. 27 (1992) 246–251.; see also Erratum, Bull. Amer. Math. Soc. 28 (1993) 213.
[19] V. Kantabutra, Traveling salesman cycles are not always subgraphs of Voronoi duals, Inform. Process. Lett. 16 (1983) 11–12.
[20] J.-P. Laumond, Connectivity of planar triangulations, Inform. Process. Lett. 34 (1990) 87–96.
[21] V.J. Leung, M.B. Dillencourt and A.L. Bliss, **GraphTool**: A tool for interactive design and manipulation of graphs and graph algorithms, in: N. Dean and G.E. Shannon, eds., Computational Support for Discrete Mathematics: DIMACS Workshop, New Brunswick, NJ (1992), DIMACS Series in Discrete Mathematics and Theoretical Computer Science 15 (American Mathematical Society, Providence, RI, 1994).
[22] C. Mathieu, Some problems in computational geometry, Algorithmica 2 (1987) 131–133.
[23] O. Ore, The Four-Color Problem (Academic Press, New York, 1967).
[24] J. O'Rourke, The computational geometry column #2, Comput. Graph. 21 (1987) 155–157.
[25] J. O'Rourke, H. Booth and R. Washington, Connect-the-dots: A new heuristic, Comput. Graph. Vision Image Process. 39 (1987) 258–266.
[26] F.P. Preparata and M.I. Shamos, Computational Geometry: An Introduction (Springer, New York, 1985).
[27] G. Reinelt, Fast heuristics for large geometric traveling salesman problems, ORSA J. Comput. 4 (1992) 206–217.
[28] I. Rivin, A characterization of ideal polyhedra in hyperbolic 3-space, preprint (1992).
[29] I. Rivin, On geometry of convex ideal polyhedra in hyperbolic 3-space, Topology 32 (1993) 87–92.
[30] I. Rivin and W.D. Smith, Inscribable graphs, Manuscript, NEC Research Institute, Princeton, NJ (1991).
[31] M.I. Shamos, Computational geometry, Ph.D. Thesis, Yale University, New Haven, CT (1978).
[32] W.R. Stewart Jr, Euclidean traveling salesman problems and Voronoi diagrams, Manuscript, College of William and Mary (1992).
[33] R. Thomas and X. Yu, 4-connected projective-planar graphs are Hamiltonian, J. Combin. Theory Ser. B 62 (1994) 114–132.
[34] C. Thomassen, A theorem on paths in planar graphs, J. Graph Theory 7 (1983) 169–176.
[35] W.T. Tutte, The factors of graphs, Canad. J. Math. 4 (1952) 314–328.
[36] W.T. Tutte, A theorem on planar graphs, Trans. Amer. Math. Soc. 82 (1956) 99–116.