On a new geometric constant related to the von Neumann–Jordan constant

Changsen Yang\textsuperscript{a}, Fenghui Wang\textsuperscript{b,*}

\textsuperscript{a} Department of Mathematics, Henan Normal University, Xinxiang, Henan 453007, China
\textsuperscript{b} Department of Mathematics, Luoyang Normal University, Luoyang, Henan 471022, China

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Abstract
The von Neumann–Jordan constant \( C_{NJ}(X) \) is computed for \( X \) being \( \ell_2 - \ell_1 \) and \( \ell_\infty - \ell_1 \) space by introducing a new geometric constant \( \gamma_X(t) \). These partly give an answer to an open question posed by Kato et al. Some basic properties of this new coefficient are investigated. Moreover, we obtain a new class of Banach spaces with uniform normal structure.

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1. Introduction

Many recent studies have focused on the von Neumann–Jordan (NJ) constant (cf. [4,6,11–14]). It is proved that the NJ constant is strongly connected with the geometric structure of Banach spaces, such as uniform non-squareness and uniform normal structure. The computation of the NJ constant attracted the interest of several authors, and many papers on this topic have appeared.

However there exist Banach spaces, for example Day–James space, whose NJ constant is not easy to compute. This paper presents the exact value of Day–James space for some particular cases by calculating a new geometric constant \( \gamma_X(t) \). We can establish an equality between this...
new coefficient \( \gamma_X(t) \) and \( C_{NJ}(X) \) easily. According to this equality we can compute the NJ constant for some Day–James \( \ell_p - \ell_q \) space such as \( p = \infty , q = 1 \) and \( p = 2 , q = 1 \). We also investigate some basic properties of \( \gamma_X(t) \). In particular, we prove that \( X \) is a Hilbert space if and only if \( \gamma_X(t) = 1 + t^2 \) for any \( t \in [0,1] \). Finally, we get a new class of Banach spaces with uniform normal structure, which are defined by \( 2\gamma_X(t) < 1 + (1 + t)^2 \) for some \( t \in (0,1] \).

We shall assume throughout this paper that \( X \) and \( X^* \) stand for Banach space and its dual space, respectively. We will use \( B_X \) and \( S_X \) to denote the unit ball and unit sphere of \( X \), respectively. If \( K \) is a bounded closed subset of \( X \), we will use \( e_x(K) \) to denote the set of extreme points of \( K \). \( x_n \xrightarrow{w} x \) stands for the weak convergence of sequence \( \{x_n\} \) in \( X \) to a point \( x \) in \( X \). For \( x \in X \), let \( \nabla_x \) denote the set of norm 1 supporting functionals at \( x \). The von Neumann–Jordan constant of a Banach space \( X \) was introduced by Clarkson [2] as the smallest constant \( C \) for which

\[
\frac{1}{C} \leq \frac{\|x+y\|^2 - \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C
\]

holds for all \( x, y \in X \) with \((x, y) \neq (0, 0)\). An equivalent definition of the NJ constant is found in [13] as the following form:

\[
C_{NJ}(X) = \sup \left\{ \frac{\|x+y\|^2 - \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x \in S_X, \; y \in B_X \right\}.
\]

Now let us collect some properties of this constant:

1. \( C_{NJ}(X) = C_{NJ}(X^*) \) [12].
2. \( 1 \leq C_{NJ}(X) \leq 2 ; X \) is a Hilbert space if and only if \( C_{NJ}(X) = 1 \).
3. \( X \) is uniformly non-square if and only if \( C_{NJ}(X) < 2 \) [13].
4. If \( 1 \leq p \leq \infty \) and \( \dim L_p(\mu) \geq 2 \), then \( C_{NJ}(L_p(\mu)) = 2^{2/r-1} \) with \( r = \min\{p, q\} \) [2].
5. \( X \) has uniform normal structure if \( C_{NJ}(X) < (3 + \sqrt{3})/4 \) [4].

The modulus of smoothness [16] of \( X \) is the function \( \rho(t) \) defined by

\[
\rho(t) = \sup \left\{ \frac{\|x+ty\|^2 - \|x-ty\|^2}{2} - 1 : x, y \in S_X \right\}.
\]

\( X \) is called uniformly smooth if \( \lim_{t \to 0} \rho(t)/t = 0 \). \( X \) is called uniformly non-square [10] if there exists \( \delta > 0 \) such that if \( x, y \in S_X \) then \( \|x+y\|/2 \leq 1 - \delta \) or \( \|x-y\|/2 \leq 1 - \delta \). The number \( r(A) = \inf\{\sup\{\|x-y\| : y \in A\} : x \in A\} \) is called Chebyshev radius of \( A \). The number \( \text{diam} A = \sup\{\|x-y\| : x, y \in A\} \) is called diameter of \( A \).

A Banach space \( X \) has normal structure provided \( r(A) < \text{diam} A \)

for every bounded closed convex subset \( A \) of \( X \) with \( \text{diam} A > 0 \). When the above inequality holds for every weakly compact convex subset \( A \) of \( X \), \( X \) is said to have weak normal structure. \( X \) is said to have uniform normal structure if \( \inf\{\text{diam} A/r(A)\} > 1 \), where the infimum is taken over all bounded closed convex subsets \( A \) of \( X \) with \( \text{diam} A > 0 \).

A Banach space \( Y \) is said to be finitely representable in \( X \) provided for any \( \lambda > 1 \) and each finite-dimensional subspace \( Y_1 \) of \( Y \), there is an isomorphism \( T \) of \( Y_1 \) into \( X \) for which

\[
\lambda^{-1}\|x\| \leq \|Tx\| \leq \lambda\|x\| \quad \text{for all } x \in Y_1.
\]

Let \( \mathcal{P} \) be a Banach space property. We say that a Banach space \( X \) has the property super-\( \mathcal{P} \) if every Banach space finitely representable in \( X \) has property \( \mathcal{P} \). Therefore \( X \) has super-normal structure if every Banach space finitely representable in \( X \) has normal structure.
Theorem A. (See [1,15].) Let $X$ be a Banach space. If $X$ has super-normal structure, then $X$ has uniform normal structure.

2. A new geometric constant

Definition 2.1. Let $X$ be a Banach space. The function $\gamma_X(t) : [0, 1] \to [1, 4]$ is defined by

$$\gamma_X(t) = \sup \left\{ \frac{\|x + ty\|^2 + \|x - ty\|^2}{2} : x \in S_X, y \in tS_X \right\}$$

$$= \sup \left\{ \frac{\|x + ty\|^2 + \|x - ty\|^2}{2} : x \in S_X, y \in S_X \right\}.$$

Remark. Obviously, $1 \leq 1 + t^2 \leq \gamma_X(t) \leq (1 + t)^2 \leq 4$.

According to the definition of $\gamma_X(t)$ we can easily obtain the following:

$$CNJ(X) = \sup \left\{ \frac{\gamma_X(t)}{1 + t^2} : 0 \leq t \leq 1 \right\},$$

thus calculating the NJ constant can be reduced to computing $\gamma_X(t)$. Moreover, to calculate $\gamma_X(t)$ is easier than to compute the NJ constant in general case.

Proposition 2.2. Let $X$ be a Banach space. Then

$$\gamma_X(t) = \sup \left\{ \frac{\|x + ty\|^2 + \|x - ty\|^2}{2} : x \in S_X, y \in B_X \right\}$$

$$= \sup \left\{ \frac{\|x + ty\|^2 + \|x - ty\|^2}{2} : x, y \in B_X \right\}.$$

Proof. Note that $f(t) := \|x + ty\|^2 + \|x - ty\|^2$ is a convex and even function. Let $0 < t_1 \leq t_2 \leq 1, x, y \in S_X$. Then we have

$$\|x + t_1y\|^2 + \|x - t_1y\|^2 = f(t_1) = f \left( \frac{t_2 + t_1}{2t_2}t_2 + \frac{t_2 - t_1}{2t_2}(-t_2) \right)$$

$$\leq f(t_2) = \|x + t_2y\|^2 + \|x - t_2y\|^2$$

$$\leq 2\gamma_X(t_2),$$

which implies that $\gamma_X(t_1) \leq \gamma_X(t_2)$. Therefore, we have

$$\frac{1}{2} \sup_{x \in S_X} \sup_{y \in B_X} \left\{ \frac{\|x + ty\|^2 + \|x - ty\|^2}{\|y\|^2} \right\} \leq \gamma_X \left( \|y\| \right) \leq \gamma_X(t).$$

Since the opposite inequality holds obviously, we get the first equality.

Suppose the parameter $t$ is fixed beforehand. Let $g(\lambda) = \|\lambda x + ty\|^2 + \|\lambda x - ty\|^2$. Then $g(\lambda)$ is a convex and even function and therefore $g(\lambda) \geq g(1)$ for all $\lambda \geq 1$. For $x, y \in B_X$ we have

$$\left\| \frac{x}{\|x\|} + ty \right\|^2 + \left\| \frac{x}{\|x\|} - ty \right\|^2 \geq \|x + ty\|^2 + \|x - ty\|^2.$$
Therefore
\[
\sup_{x \in S_X} \sup_{y \in B_X} (\|x + ty\|^2 + \|x - ty\|^2) \geq \sup_{x \in B_X} \sup_{y \in B_X} (\|x + ty\|^2 + \|x - ty\|^2),
\]
then we obtain the second equality. □

**Proposition 2.3.** Let \( X \) be a Banach space. Then

1. \( \gamma_X(t) \) is a non-decreasing function.
2. \( \gamma_X(t) \) is a convex function.
3. \( \gamma_X(t) \) is continuous on \([0, 1]\).
4. \( \frac{\gamma_X(t) - 1}{t} \) is non-decreasing on \((0, 1]\).

**Proof.** (1) Obvious.

(2) Let \( x, y \in S_X, t_1, t_2 \in [0, 1], \lambda \in (0, 1) \). Then we have
\[
\|x + (\lambda t_1 + (1 - \lambda)t_2)y\|^2 + \|x - (\lambda t_1 + (1 - \lambda)t_2)y\|^2 \\
\leq \left[\lambda \|x + t_1 y\| + (1 - \lambda)\|x + t_2 y\|\right]^2 + \left[\lambda \|x - t_1 y\| + (1 - \lambda)\|x - t_2 y\|\right]^2 \\
\leq \lambda \left[\|x + t_1 y\|^2 + \|x - t_1 y\|^2\right] + (1 - \lambda)\left[\|x + t_2 y\|^2 + \|x - t_2 y\|^2\right] \\
\leq \lambda \gamma(t_1) + (1 - \lambda) \gamma(t_2).
\]

Since \( x, y \) are arbitrary, we have
\[
\gamma(\lambda t_1 + (1 - \lambda)t_2) \leq \lambda \gamma(t_1) + (1 - \lambda) \gamma(t_2).
\]

(3) Since (2) implies that \( \gamma_X(t) \) is continuous on \((0, 1]\), it suffices to show that \( \gamma_X(t) \) is continuous at \( t = 1 \). Let \( x, y \in S_X, u = tx, v = ty \). Then \( \|u\| \leq 1, \|v\| \leq t, \) and
\[
\left(\|x + y\|^2 + \|x - y\|^2\right) - 2\gamma_X(t) \leq \left(\|x + y\|^2 + \|x - y\|^2\right) - \left(\|u + v\|^2 + \|u - v\|^2\right) \\
= (1 - t^2)(\|x + y\|^2 + \|x - y\|^2) \\
\leq 8(1 - t^2).
\]

Since \( x, y \) are arbitrary, we have \( \gamma_X(1) - \gamma_X(t) \leq 4(1 - t^2) \). Thus our proof is completed.

(4) Let \( 0 < t_1 < t_2 < 1 \), then \( t_1 = \lambda t_2 \) \((0 < \lambda < 1)\). Thus
\[
\frac{\gamma(t_1) - 1}{t_1} \leq \frac{\gamma((1 - \lambda)0 + \lambda t_2) - 1}{\lambda t_2} \leq \frac{\gamma(t_2) - 1}{t_2}.
\]

The following facts are readily seen according to Krein–Milman theorem.

**Corollary 2.4.** Let \( X \) be a finite dimensional Banach space. Then
\[
\gamma_X(t) = \sup \left\{ \frac{\|x + ty\|^2 + \|x - ty\|^2}{2} : x, y \in \text{ex}(B_X) \right\}.
\]

**Proposition 2.5.** \( X \) is a Hilbert space if and only if \( \gamma_X(t) = 1 + t^2 \) for any \( t \in [0, 1] \).

**Proof.** \((\Rightarrow)\) Obvious.

\((\Leftarrow)\) Taking \( x, y \in X \), if necessary, we can assume \( \|x\| \geq \|y\| > 0 \). On one hand we have
Proof. Recall Clarkson inequality [3,9], that when

\[ \frac{x + y}{\|x + y\|} + \frac{x - y}{\|x - y\|} \leq 2 \left( \frac{\|x\|^2 + \|y\|^2}{\|x\|^2} \right) \leq 2 \|x\|^2 YX \left( \frac{\|y\|}{\|x\|} \right) \]

\[ = 2 \|x\|^2 \left( 1 + \frac{\|y\|^2}{\|x\|^2} \right) = 2 (\|x\|^2 + \|y\|^2). \]

On the other hand, if we put \( u = (x + y)/2, v = (x - y)/2 \) then

\[ \|u + v\|^2 + \|u - v\|^2 \leq 2 (\|u\|^2 + \|v\|^2), \]

which implies \( 2 (\|x\|^2 + \|y\|^2) \leq \|x + y\|^2 + \|x - y\|^2 \). Thus

\[ \|x + y\|^2 + \|x - y\|^2 = 2 (\|x\|^2 + \|y\|^2) \quad (\forall x, y \in X) \]

and we complete the proof. \( \square \)

**Corollary 2.6.** \( X \) is a Hilbert space if and only if \( C_{NJ}(X) = 1 \).

**Theorem 2.7.** Let \( X \) be the \( \ell_p \) space, then

\[ \gamma_p(t) = \begin{cases} \frac{(1+t)^p + (1-t)^p}{2}^{2/p}, & 2 \leq p < \infty, \\ (1 + t)^2, & p = \infty. \end{cases} \]

**Proof.** Recall Clarkson inequality [3,9], that when \( p \geq 2 \),

\[ \|x + y\|^p + \|x - y\|^p \leq (\|x\| + \|y\|)^p + (\|x\| - \|y\|)^p, \quad x, y \in \ell_p. \] (2)

Then for any \( x \in S_X, y \in t S_X \) we have

\[ \|x + y\|^2 + \|x - y\|^2 \leq 2^{1-2/p} (\|x + y\|^p + \|x - y\|^p)^{2/p} \]

\[ \leq 2^{1-2/p} ((1 + t)^p + (1 - t)^p)^{2/p} \]

\[ = 2 \left( \frac{(1 + t)^p + (1 - t)^p}{2} \right)^{2/p}. \]

Put \( x = (1/2^{1/p}, 1/2^{1/p}, 0, \ldots), y = (t/2^{1/p}, t/2^{1/p}, 0, \ldots), \) then

\[ \|x + y\|^2_p + \|x - y\|^2_p = 2 \left( \frac{(1 + t)^p + (1 - t)^p}{2} \right)^{2/p}. \]

Therefore we obtain the equality as desired. For \( \ell_\infty \) if we take \( x = (1, 1, 0, \ldots), y = (t, -t, 0, \ldots) \) then we obtain the second equality.

**Remark.** The above assertion also holds for \( \ell_p^n \) (\( n \geq 2 \)), since \( \gamma_X(t) \) is a two-dimensional coefficient.

**Corollary 2.8.** If \( 1 \leq p \leq \infty \) and \( 1/p + 1/q = 1 \), then \( C_{NJ}(\ell_p) = 2^{2/r-1} \) with \( r = \min\{p, q\} \).

**Proof.** For \( p \geq 2 \) we have

\[ f(t) := \frac{(1 + t)^p + (1 - t)^p}{2^{2/p}(1 + t^2)} \leq \frac{(1 + t)^2 + (1 - t)^2}{2^{2/p}(1 + t^2)} = 2^{1-2/p} = 2^{2/q-1}. \]

Note that \( f(1) = 2^{2/q-1} \), which implies \( C_{NJ}(\ell_p) = 2^{2/q-1} \).

For \( 1 \leq p < 2 \), \( C_{NJ}(\ell_p) = C_{NJ}(\ell_q) = 2^{2/p-1} \). \( \square \)
3. Some examples

Example 3.1 (Day–James $\ell_p - \ell_q$ space). For $1 \leq p, q \leq \infty$ denote by $\ell_p - \ell_q$ the Day–James spaces, i.e., $\mathbb{R}^2$ with the norm defined by

$$\|x\| = \begin{cases} \|x\|_p, & x_1x_2 \geq 0, \\ \|x\|_q, & x_1x_2 \leq 0. \end{cases}$$

In [13, Examples 2, 4 and 8] the authors studied the $\ell_p - \ell_q$ spaces, and gave the following estimates:

$$C_{NJ}(\ell_{\infty} - \ell_1) \geq \frac{5}{4}, \quad C_{NJ}(\ell_2 - \ell_1) \geq \frac{3}{2}.$$ 

The exact value of these spaces is obtained in the following, which answers a question posed by Kato et al. partly [13, Problem 2].

Example 3.2 ($\ell_{\infty} - \ell_1$ space). Let $X = \mathbb{R}^2$ with the norm defined by

$$\|x\| = \begin{cases} \|x\|_{\infty}, & x_1x_2 \geq 0, \\ \|x\|_1, & x_1x_2 \leq 0, \end{cases}$$

then $C_{NJ}(\ell_{\infty} - \ell_1) = (3 + \sqrt{5})/4$.

Proof. Since $\rho(t) = \max\{t/2, t - 1/2\}$ (cf. [13, Example 4]), we have

$$\|x + y\|^2 + \|x - y\|^2 \leq 1 + (1 + t)^2, \quad \forall x \in S_X, \quad y \in tS_X. \quad \text{(3)}$$

In fact, if $\|x + y\| \leq 1$, then (3) holds obviously; if $\|x + y\| = a$ ($1 \leq a \leq 1 + t$), then

$$\|x + y\|^2 + \|x - y\|^2 \leq a^2 + [2(\rho(t) + 1) - a]^2 = a^2 + (2 + t - a)^2 = 2a^2 - 2a(2 + t) + (2 + t)^2 =: f(a).$$

Note that the function $f(a)$ attains its maximum at $a = 1$, thus we obtain the above inequality (3).

Put $x = (1, 1), \quad y = (0, t)$, then

$$\|x + y\|^2 + \|x - y\|^2 = 1 + (1 + t)^2.$$ 

Thus we have $2\gamma_X(t) = 1 + (1 + t)^2$, which implies that

$$C_{NJ}(\ell_{\infty} - \ell_1) = \sup_{t \in [0, 1]} \left\{ \frac{1 + (1 + t)^2}{2(1 + t^2)} \right\} = \frac{3 + \sqrt{5}}{4}. \quad \Box$$

Example 3.3 ($\ell_2 - \ell_1$ space). Let $X = \mathbb{R}^2$ with the norm defined by

$$\|x\| = \begin{cases} \|x\|_2, & x_1x_2 \geq 0, \\ \|x\|_1, & x_1x_2 \leq 0, \end{cases}$$

then $C_{NJ}(\ell_2 - \ell_1) = 3/2$.

Proof. Note that $\text{ex}(B_X) = \{(x_1, x_2) \mid x_1^2 + x_2^2 = 1, \quad x_1x_2 \geq 0\}$. Then for any $x, y \in \text{ex}(B_X)$, we have

$$\|x + ty\|^2 + \|x - ty\|^2 \leq 2(1 + t^2) + 2t. \quad \text{(4)}$$
We may assume that $x = (a, b), y = (c, d) \in \text{ex}(B_X)$ where $a, b, c, d \geq 0$, then $x + ty = (a + ct, b + dt)$, $x - ty = (a - ct, b - dt)$. If $(a - ct)(b - dt) \geq 0$, then
\[
\|x + ty\|^2 + \|x - ty\|^2 = \|(a + ct, b + dt)\|^2_2 + \|(a - ct, b - dt)\|^2_2 \\
\leq 2\gamma_2(t) = 2(1 + t^2) \\
\leq 2(1 + t^2) + 2t.
\]
If $a - ct \geq 0$, $b - dt \leq 0$, then
\[
\|x + ty\|^2 + \|x - ty\|^2 = \|(a + ct, b + dt)\|^2_2 + \|(a - ct, b - dt)\|^2_1 \\
= 2(1 + t^2) - 2ab - 2c dt^2 + 2adt + 2bct \\
\leq 2(1 + t^2) + 2t.
\]
A similar discussion shows that inequality (4) also holds in the remaining cases. Put $x = (1, 0)$, $y = (0, t)$ then $\|x + y\|^2 + \|x - y\|^2 = 2(1 + t + t^2)$. Therefore $\gamma_X(t) = 1 + t + t^2$, which implies
\[
C_{NJ}(\ell_2 - \ell_1) = \sup_{t \in [0, 1]} \left\{ \frac{1 + t + t^2}{1 + t^2} \right\} = 3/2. \quad \square
\]

Example 3.4. For $\lambda > 0$, let $Z_\lambda$ be $R^2$ with the norm
\[
|x|_\lambda = (\|x\|^2_2 + \lambda \|x\|^2_\infty)^{1/2},
\]
then $C_{NJ}(Z_\lambda) = 2(\lambda + 1)/(\lambda + 2)$.

Proof. Since
\[
\sqrt{(\lambda + 2)/2} \|x\|_2 \leq |x|_\lambda \leq \sqrt{\lambda + 1} \|x\|_2 \quad \text{for all} \ x \in Z_\lambda,
\]
then for any $|x|_\lambda = 1$, $|y|_\lambda = t$ we have
\[
|x + y|_\lambda^2 + |x - y|_\lambda^2 \leq (1 + \lambda)(\|x + y\|^2_2 + \|x - y\|^2_2) \\
= 2(1 + \lambda)(\|x\|^2_2 + \|y\|^2_2) \\
\leq 4(\lambda + 1)/\lambda + 2))(|x|^2_2 + |y|^2_2) \\
= 4(\lambda + 1)(1 + t^2)/(\lambda + 2).
\]
Thus $\gamma_{Z_\lambda}(t) \leq 2(\lambda + 1)(1 + t^2)/(\lambda + 2)$, which implies that
\[
C_{NJ}(Z_\lambda) \leq \sup_{t \in [0, 1]} \left\{ \frac{2(\lambda + 1)(1 + t^2)}{(\lambda + 2)(1 + t^2)} \right\} = \frac{2(\lambda + 1)}{\lambda + 2}. \quad (5)
\]
Put $x = (a, a)$, $y = (a, -a)$ where $a = 1/\sqrt{\lambda + 2}$. Then $|x|_\lambda = |y|_\lambda = 1$, and
\[
\frac{|x + y|_\lambda^2 + |x - y|_\lambda^2}{2(|x|^2_2 + |y|^2_2)} = \frac{2(\lambda + 1)}{\lambda + 2}.
\]
Therefore $C_{NJ}(Z_\lambda) = 2(\lambda + 1)/(\lambda + 2). \quad \square
4. Geometric properties

**Theorem 4.1.** A Banach space $X$ is uniformly smooth if $\lim_{t \to 0^+} \frac{\gamma_X(t) - 1}{t} = 0$.

**Proof.** For any $x \in S_X$, $y \in tS_X$ we obtain

$$\left(\frac{\|x + y\| + \|x - y\|}{2}\right)^2 \leq \frac{\|x + y\|^2 + \|x - y\|^2}{2} \leq \gamma_X(t),$$

which implies

$$\frac{(\rho(t) + 1)^2 - 1}{t} \leq \frac{\gamma_X(t) - 1}{t} \to 0 \quad (t \to 0).$$

Thus

$$\lim_{t \to 0} \frac{\rho(t)}{t} = \frac{1}{2} \lim_{t \to 0} \frac{(\rho(t) + 1)^2 - 1}{t} = 0$$

and then $X$ is uniformly smooth. $\square$

**Theorem 4.2.** Let $X$ be a Banach space. Then the following conditions are equivalent:

1. $X$ is not uniformly non-square.
2. $\gamma_X(t) = (1 + t)^2$ for all $t \in (0, 1]$.
3. $\gamma_X(t) = (1 + t_0)^2$ for some $t_0 \in (0, 1]$.

**Proof.** (1) $\Rightarrow$ (2). If $X$ is not uniformly non-square, then there exist $x_n, y_n \in S_X$ such that $\|x_n + y_n\| \to 2, \|x_n - y_n\| \to 2$ ($n \to \infty$) which implies that

$$\|x_n + ty_n\| \to 1 + t, \quad \|x_n - ty_n\| \to 1 + t, \quad \forall t \in (0, 1].$$

From the definition of $\gamma_X(t)$, we obtain $\gamma_X(t) = (1 + t)^2$.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (1). If $X$ is uniformly non-square, then there exists $\delta > 0$, such that for any $x, y \in S_X$ either $\|x + y\|/2 \leq 1 - \delta$ or $\|x - y\|/2 \leq 1 - \delta$. Without loss of generality, we can assume $\|x - y\|/2 \leq 1 - \delta$. Then for some $t_0 \in (0, 1]$ we have

$$\|x + t_0y\|^2 + \|x - t_0y\|^2 \leq \|x + t_0y\|^2 + (t_0\|x - y\| + (1 - t_0)\|x\|)^2$$

$$\leq (1 + t_0)^2 + (1 + t_0 - 2\delta t_0)^2$$

$$\leq 2(1 + t_0)^2 - 4\delta t_0,$$

which implies that $\gamma_X(t) < (1 + t_0)^2$. This is a contradiction and thus we complete the proof. $\square$

**Remark.** If $X$ is one of the spaces $L_1[0, 1], C[0, 1], C_0[0, 1], c_0, \ell_1$, then we have $\gamma_X(t) = (1 + t)^2$ since they are not uniformly non-square.

**Corollary 4.3.** $X$ is uniformly non-square if and only if $C_{NJ}(X) < 2$.

Normal structure and uniform normal structure have proved to be useful to obtain fixed points of non-expansive mappings and uniformly Lipschitz mappings (see [7,8]). In 1991 Gao and Lau
Lemma 4.4. [4, Lemma 3.5] Let $X$ be a Banach space without weak normal structure, then for any $0 < \eta < 1$ and each $1/2 < t \leq 1$ there exist $x_1 \in S_X$, $x_2, x_3 \in TS_X$, such that

1. $x_2 - x_3 = ax_1$, with $|a - t| < \eta$,
2. $\|x_1 - x_2\| > 1 - \eta$,
3. $\|x_1 + x_2\| > (1 + t) - \eta$, $\|x_1 + (-x_3)\| > (3t - 1) - \eta$.

The following lemma is an improvement of the above lemma.

Lemma 4.5. Let $X$ be a Banach space without weak normal structure, then for any $0 < \eta < 1$ and each $0 \leq t \leq 1$ there exist $x_1 \in S_X$, $x_2, x_3 \in tS_X$, such that

1. $x_2 - x_3 = ax_1$, with $|a - t| < \eta$,
2. $\|x_1 - x_2\| > 1 - 3\eta$,
3. $\|x_1 + x_2\| > (1 + t) - 3\eta$, $\|x_1 + (-x_3)\| > (1 + t) - 3\eta$.

Proof. If $X$ does not have weak normal structure, by [5, Lemma 2.2], for any $\eta, 0 < \eta < 1$, there exists $z_n \in S_X$ with $z_n \not\rightarrow 0$ and

$$1 - \eta < \|z_{n+1} - z\| < 1 + \eta$$

for sufficiently large $n$ and for any $z \in co\{z_k\}_{k=1}^n$.

Since $0$ belongs to the weakly closed convex hull of $\{z_n\}$, which equals to the norm closed convex hull, we can take $n_0 \in \mathbb{N}$, $y \in co\{z_k\}_{k=1}^{n_0}$ and $z^* \in \partial z_1$, such that

$$\|y\| < \eta, \quad |z^*(z_{n_0})| < \eta, \quad 1 - \eta < \|z_{n_0} - z_1\| < 1 + \eta.$$

Then for any $t \in [0, 1]$,

$$\|z_{n_0} - (1-t)z_1\| \geq \|z_{n_0} - ((1-t)z_1 + ty)\| - t\|y\| > 1 - 2\eta.$$

Put $x_1 = (z_1 - z_{n_0})/\|z_1 - z_{n_0}\|$, $x_2 = tz_1$, $x_3 = tz_{n_0}$, hence (1) holds.

On one hand we have

$$\|x_1 - x_2\| = \|x_1 - tz_1\| = \|x_1 - (z_1 - z_{n_0}) + (1-t)z_1 - z_{n_0}\|
\geq \|z_{n_0} - (1-t)z_1\| - \|x_1 - (z_1 - z_{n_0})\|
> 1 - 2\eta - \|z_1 - z_{n_0}\|
> 1 - 3\eta.$$

Thus (2) holds. On the other hand,

$$\|x_1 + x_2\| = \left\| \frac{z_1 - z_{n_0}}{\|z_1 - z_{n_0}\|} + tz_1 \right\|
\geq \frac{z^*(z_1) - z^*(z_{n_0})}{\|z_1 - z_{n_0}\|} + tz^*(z_1)
> \frac{1 - \eta}{1 + \eta} + t > (1 + t) - 3\eta.$$
Also,
\[
\|x_1 - x_3\| = \|x_1 - tx_{n_0}\| = \|x_1 - (z_1 - z_{n_0}) + (1 + t)z_{n_0}\| \\
\geq \|(1 + t)z_{n_0} - z_1\| - \|x_1 - (z_1 - z_{n_0})\| \\
= (1 + t)\left\|z_{n_0} - \frac{1}{1 + t}z_1\right\| - |1 - \|z_1 - z_{n_0}\|| \\
> (1 + t)(1 - 2\eta) - \eta \\
> (1 + t) - 3\eta.
\]

Thus (3) holds and we complete the proof. □

**Theorem 4.6.** Let \(Y\) be a Banach space which is finitely representable in \(X\). Then \(\gamma_X(t) \geq \gamma_Y(t)\).

**Proof.** Let \(x \in S_Y, y \in tS_Y, Y_1 = \text{span}\{x, y\}\). For any \(\lambda > 1\), there exists an isomorphism \(T\) of \(Y_1\) into \(X\) for which
\[
\lambda^{-1}\|x\| \leq \|Tx\| \leq \lambda\|x\| \quad \text{for all} \quad x \in Y_1.
\]

Put \(x' = Tx/\lambda, y' = Ty/\lambda\), then \(\|x'\| \leq 1, \|y'\| \leq t\) and
\[
2\gamma_X(t) \geq \|x' + y'\|^2 + \|x' - y'\|^2 \geq 1/\lambda^4(\|x + y\|^2 + \|x - y\|^2).
\]

Since \(x, y\) are arbitrary, we have \(\lambda^4\gamma_X(t) \geq \gamma_Y(t)\). Letting \(\lambda \rightarrow 1\), we get the inequality as desired. □

**Theorem 4.7.** If there exists \(t\), \(0 < t \leq 1\), such that \(2\gamma_X(t) < 1 + (1 + t)^2\), then \(X\) has super-normal structure, and therefore uniform normal structure.

**Proof.** Suppose \(X\) does not have weak normal structure, then by Lemma 4.4 for any \(0 < \eta < 1\) and each \(0 < t \leq 1\), there exist \(x_1 \in S_X, x_2 \in tS_X\) such that \(\|x_1 + x_2\| > (1 + t) - 3\eta, \|x_1 - x_2\| > 1 - 3\eta\). From the definition of \(\gamma_X(t)\) we have
\[
2\gamma_X(t) \geq \|x_1 + x_2\|^2 + \|x_1 - x_2\|^2 \geq (1 + t - 3\eta)^2 + (1 - 3\eta)^2,
\]
which implies that \(2\gamma_X(t) \geq 1 + (1 + t)^2\) for any \(t \in (0, 1]\). This is a contradiction and \(X\) must have weak normal structure.

Since \(2\gamma_X(t) < 1 + (1 + t)^2\) for some \(t \in (0, 1]\) implies that \(X\) is uniformly non-square, and consequently, reflexive. Thus normal structure and weak normal structure coincide. Let \(Y\) be a Banach space which is finitely representable in \(X\), then by Theorem 4.5, \(2\gamma_Y(t) \leq 2\gamma_X(t) < 1 + (1 + t)^2\) for some \(t \in (0, 1]\). From the above discussion \(Y\) has normal structure and therefore \(X\) has super normal structure. The last assertion follows from Theorem A. □

**Corollary 4.8.** [4, Theorem 3.6] If \(C_{NJ}(X) < \frac{3 + \sqrt{3}}{4}\), then \(X\) has uniform normal structure.

**Proof.** Suppose \(X\) does not have uniform normal structure, then by Theorem 4.6, \(2\gamma_X(t) \geq 1 + (1 + t)^2\) for any \(t \in (0, 1]\). Therefore
\[
C_{NJ}(X) \geq \sup_{t \in (0, 1]} \left\{ \frac{1 + (1 + t)^2}{2(1 + t^2)} \right\} = \frac{3 + \sqrt{3}}{4}.
\]
This is a contradiction and \(X\) must have uniform normal structure. □
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References