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Mellin transform analysis and integration by parts for Hadamard-type fractional integrals

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Abstract

This paper is devoted to the study of four integral operators that are basic generalizations and modifications of fractional integrals of Hadamard, in the space X_c^p of Lebesgue measurable functions f on $\mathbf{R}_+ = (0, \infty)$ such that

$$\int_0^\infty \left|u^c f(u)\right|^p \frac{du}{u} < \infty \quad (1 \leqslant p < \infty), \qquad \operatorname*{ess\,sup}_{u>0}\left[u^c \left|f(u)\right|\right] < \infty \quad (p = \infty),$$

for $c \in \mathbf{R} = (-\infty, \infty)$, in particular in the space $L^p(0, \infty)$ $(1 \leqslant p \leqslant \infty)$. Formulas for the Mellin transforms of the four Hadamard-type fractional integral operators are established as well as relations of fractional integration by parts for them. © 2002 Elsevier Science (USA). All rights reserved.

Keywords: Hadamard-type fractional integration; Spaces of p-summable functions; Mellin transforms; Fractional integration by parts

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1. Introduction

In [1] we have discussed an approach to fractional integration and differentiation in the framework of the Mellin transform $\mathcal{M}$ of $f:\mathbf{R}_+ \to \mathbf{C}$ defined by

$$\mathcal{M}[f](s) := \int_0^\infty u^{s-1} f(u)\,du \quad (s = c + it;\ c, t \in \mathbf{R}). \tag{1.1}$$

In this approach fractional integration $\mathcal{J}^\alpha_{0+,\mu}$ and differentiation $\mathcal{D}^\alpha_{0+,\mu}$ of order $\alpha > 0$ can best be defined by

$$\mathcal{M}\big[\mathcal{J}^\alpha_{0+,\mu} f\big](s) = (\mu - s)^{-\alpha} \mathcal{M}[f](s) \quad (\mu \in \mathbf{R};\ x > 0) \tag{1.2}$$

and

$$\mathcal{M}\big[\mathcal{D}^\alpha_{0+,\mu} f\big](s) = (\mu - s)^{\alpha} \mathcal{M}[f](s) \quad (\mu \in \mathbf{R};\ x > 0), \tag{1.3}$$

respectively.

The explicit form of the fractional integration of (1.2) is given by

$$\big(\mathcal{J}^\alpha_{0+,\mu} f\big)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \left(\frac{u}{x}\right)^\mu \left(\log \frac{x}{u}\right)^{\alpha-1} \frac{f(u)\,du}{u} \quad (x > 0), \tag{1.4}$$

where $\Gamma(\alpha)$ is the Euler gamma-function. When $\alpha = r \in \mathbf{N} = 1, 2, \ldots$ and $\mu = 0$, this integral coincides with the rth integral of the form

$$\begin{aligned} \big(J^r f\big)(x) &= \int_0^x \frac{du_1}{u_1} \int_0^{u_1} \frac{du_2}{u_2} \cdots \int_0^{u_{r-1}} f(u_r) \frac{du_r}{u_r} \\ &= \frac{1}{(r-1)!} \int_0^x \left(\log \frac{x}{u}\right)^{r-1} f(u) \frac{du}{u} \quad (x > 0). \end{aligned} \tag{1.5}$$

The fractional version of (1.5),

$$\big(\mathcal{J}^\alpha_{0+} f\big)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \left(\log \frac{x}{u}\right)^{\alpha-1} \frac{f(u)\,du}{u} \quad (x > 0;\ \alpha > 0), \tag{1.6}$$

was introduced by Hadamard [2]; and it is often referred to as Hadamard fractional integral of order $\alpha > 0$, see [3, Section 18.3 and Section 23.1, notes to Section 18.3]. Thus (1.6) is the particular case of (1.4) for $\mu = 0$.

In our previous papers [1] and [4] we have studied the more general operator (1.4) and its modifications of the form

$$\left(\mathcal{J}^{\alpha}_{-,\mu} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} \left(\frac{x}{u}\right)^{\mu} \left(\log \frac{u}{x}\right)^{\alpha-1} \frac{f(u)\,du}{u} \quad (x > 0), \tag{1.7}$$

$$\left(\mathcal{I}^{\alpha}_{0+,\mu} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{x} \left(\frac{u}{x}\right)^{\mu} \left(\log \frac{x}{u}\right)^{\alpha-1} \frac{f(u)\,du}{x} \quad (x > 0), \tag{1.8}$$

and

$$\left(\mathcal{I}^{\alpha}_{-,\mu} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} \left(\frac{x}{u}\right)^{\mu} \left(\log \frac{u}{x}\right)^{\alpha-1} \frac{f(u)\,du}{x} \quad (x > 0) \tag{1.9}$$

with $\alpha > 0$ and complex $\mu \in \mathbf{C}$. By analogy with the classical Riemann–Liouville and Liouville fractional integrals (see, for example, [3, Sections 2 and 5]) the integrals (1.4) and (1.8) will be called left-hand-sided Hadamard-type integrals, while (1.7) and (1.9) are called right-hand-sided.

In [3] we gave conditions for the operators (1.4), (1.7), (1.8), and (1.9) to be bounded in the space X^p_c of those complex-valued Lebesgue measurable functions f on $\mathbf{R}_+$ for which $\|f\|_{X^p_c} < \infty$, where

$$\|f\|_{X^p_c} = \left(\int_0^{\infty} \left|u^c f(u)\right|^p \frac{du}{u}\right)^{1/p} \quad (1 \leqslant p < \infty,\ c \in \mathbf{R}) \tag{1.10}$$

and

$$\|f\|_{X^{\infty}_c} = \operatorname*{ess\,sup}_{u>0} \left[u^c \left|f(u)\right|\right] \quad (c \in \mathbf{R}). \tag{1.11}$$

In particular, when $c = 1/p$ $(1 \leqslant p \leqslant \infty)$, these results hold in $L^p(\mathbf{R}_+)$-space:

$$\|f\|_p = \left(\int_0^{\infty} \left|f(u)\right|^p du\right)^{1/p} \quad (1 \leqslant p < \infty),$$

$$\|f\|_{\infty} = \operatorname*{ess\,sup}_{u>0} \left|f(u)\right|. \tag{1.12}$$

The corresponding results were also proved for the operator (1.6) and its modification

$$\left(\mathcal{J}^{\alpha}_{-} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} \left(\log \frac{u}{x}\right)^{\alpha-1} \frac{f(u)\,du}{u} \quad (x > 0;\ \alpha > 0). \tag{1.13}$$

In [1] we also investigated the connections of the operators (1.4) and (1.7) with the Liouville fractional integration operators $I_+^\alpha f$ and $I_-^\alpha f$ defined on the whole real line **R** by

$$\big(I_+^\alpha f\big)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{f(u)\,du}{(x-u)^{1-\alpha}} \quad (x \in \mathbf{R};\ \alpha > 0) \tag{1.14}$$

and

$$\big(I_-^\alpha f\big)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{f(u)\,du}{(u-x)^{1-\alpha}} \quad (x \in \mathbf{R};\ \alpha > 0) \tag{1.15}$$

respectively; see for example [3, Section 5.1].

In [4] we established the semigroup property and its generalizations for the operators (1.4), (1.7), (1.8), and (1.9) and gave conditions for the boundedness in X_c^p of Hadamard-type fractional integration operators, which are more general than (1.4), (1.7), (1.8), and (1.9); they involve Kummer confluent hypergeometric functions in the kernels.

The present paper is concerned with the establishment of the Mellin transform of the Hadamard-type fractional integration operators (1.4), (1.7), (1.8), and (1.9) in the space X_c^p, as well as with relations of fractional integration by parts for these four operators. The corresponding results are also given for the Hadamard fractional integration operators (1.6) and (1.13). Similar assertions for the above operators in the space $L^p(\mathbf{R}_+)$ are also presented.

The paper is organized as follows. Section 2 contains some results from the theory of Mellin tranforms in the space X_c^p. Section 3 is devoted to the formulas for the Mellin transforms in X_c^p of the operators in question. The relations of fractional integration by parts for these operators in X_c^p-spaces are presented in Sections 4 and 5.

2. Mellin transform in X_c^p-space

The Mellin transform $\mathcal{M}^p$ of $f \in X_c^p$ with $1 \leqslant p \leqslant 2$ is defined by

$$\big(\mathcal{M}^p f\big)(s) = \underset{\varrho\to\infty}{\mathrm{l.i.m.}} \int_{1/\varrho}^{\varrho} f(u)u^{s-1}\,du \quad (s = c + it;\ c \in \mathbf{R}) \tag{2.1}$$

in the sense that

$$\lim_{\varrho\to\infty} \left\| \big(\mathcal{M}^p f\big)(c+it) - \int_{1/\varrho}^{\varrho} f(u)u^{c+it-1}\frac{du}{u} \right\|_{L_p(\{c\}\times i\mathbf{R})} = 0, \tag{2.2}$$

where $L^p(\{c\} \times i\mathbf{R})$ for some $c \in \mathbf{R}$ is the set of all functions $g(\{c\} \times i\mathbf{R})$ with $g(c+it) \in L^p(\mathbf{R})$. It is directly checked similarly to the proof of Lemma 2 in [5] that if $f \in X_c^p \cap X_c^1$, then the Mellin transform $\mathcal{M}^p$ coincides with the classical Mellin transform given by (1.1), i.e., $(\mathcal{M}f)(c+it) = (\mathcal{M}^p f)(c+it)$ for almost all $t \in \mathbf{R}$.

For a function $f(x)$ defined almost everywhere on $\mathbf{R}_+ = (0,\infty)$ the elementary operators M_ζ, τ_h^r, $N_{a,r}$, R and Q are defined as

$$(M_\zeta f)(x) = x^\zeta f(x) \qquad (\zeta \in \mathbf{C}), \tag{2.3}$$

$$\left(\tau_h^r f\right)(x) = h^r f(hx) \qquad (h \in \mathbf{R}_+,\ r \in \mathbf{R}), \tag{2.4}$$

$$(N_{a,r} f)(x) = |a|^r f(x^a) \qquad (a \in \mathbf{R},\ a \neq 0;\ r \in \mathbf{R}), \tag{2.5}$$

$$(Rf)(x) = \frac{1}{x} f\left(\frac{1}{x}\right) \tag{2.6}$$

and

$$(Qf)(x) = f\left(\frac{1}{x}\right). \tag{2.7}$$

Mapping properties of these operators in the space X_c^p were presented in [1, Lemma 1].

The following assertions giving the Mellin transform of the operators (2.3)–(2.7) in the space X_c^p are checked directly.

Lemma 1. *If $c \in \mathbf{R}$ and $1 \leqslant p \leqslant 2$, then the following relations hold for the Mellin transform of the elementary operators defined by* (2.3)–(2.7) *for $f \in X_c^p$:*

$$\left(\mathcal{M}^p M_\zeta f\right)(s) = \left(\mathcal{M}^p f\right)(s+\zeta) \qquad \left(\mathrm{Re}(s) = c - \mathrm{Re}(\zeta)\right), \tag{2.8}$$

$$\left(\mathcal{M}^p\left(\tau_h^r f\right)\right)(s) = h^{r-s}\left(\mathcal{M}^p f\right)(s) \qquad \left(\mathrm{Re}(s) = c\right), \tag{2.9}$$

$$\left(\mathcal{M}^p N_{a,r} f\right)(s) = |a|^{r-1}\left(\mathcal{M}^p f\right)\left(\frac{s}{a}\right) \qquad \left(\mathrm{Re}(s) = ac\right), \tag{2.10}$$

$$\left(\mathcal{M}^p Rf\right)(s) = \left(\mathcal{M}^p f\right)(1-s) \qquad \left(\mathrm{Re}(s) = 1-c\right), \tag{2.11}$$

$$\left(\mathcal{M}^p Qf\right)(s) = \left(\mathcal{M}^p f\right)(-s) \qquad \left(\mathrm{Re}(s) = -c\right). \tag{2.12}$$

Remark 1. In case of the spaces X_c^p with $p = 1$ and $p = 2$, formulas (2.8), (2.9), (2.10) and some of their generalizations were given in [6] and [5].

Let $Kf = k \star f$ be the Mellin convolution operator defined by

$$(Kf)(x) \equiv (k \star f)(x) = \int_0^\infty k\left(\frac{x}{u}\right) f(u) \frac{du}{u}. \tag{2.13}$$

There holds the following Mellin convolution theorem.

Lemma 2. *Let $c \in \mathbf{R}$ and let $1 \leqslant p \leqslant 2$. If $f \in X_c^p$ and $k \in X_c^1$, then $Kf \in X_c^p$ and the Mellin tranform of* (2.13) *is given by*

$$\big(\mathcal{M}^p Kf\big)(s) = (\mathcal{M}k)(s)\big(\mathcal{M}^p f\big)(s) \quad (s = c + it;\ t \in \mathbf{R}). \tag{2.14}$$

Proof. If $f \in X_c^p$ and $k \in X_c^1$, then by [1, Theorem 1] $Kf \equiv k \star f \in X_c^p$. The relation (2.14) was proved in [6, Theorem 3(b)] and [5, Lemma 2.7] for $p = 1$ and $p = 2$, respectively. For $1 < p < 2$ this formula is proved similarly to the proof of Lemma 2.7 in [5] on the basis of the fact that the characteristic function $f_\varrho = f_{[1/\varrho,\rho]}$ of the interval $[1/\varrho, \rho]$ belongs to the space $X_c^p \cap X_c^1$. □

3. Mellin transform of Hadamard-type fractional integrals

In this section we obtain formulas for the Mellin transform $\mathcal{M}^p$ of the Hadamard-type fractional integrals (1.4), (1.7), (1.8), and (1.9). First we consider $\mathcal{J}^\alpha_{0+,\mu} f$ and $\mathcal{J}^\alpha_{-,\mu} f$ defined by (1.4) and (1.7). These integrals have form (2.13), namely

$$\big(\mathcal{J}^\alpha_{0+,\mu} f\big)(x) \equiv (k_1 \star f)(x) = \int_0^\infty k_1\left(\frac{x}{u}\right) f(u)\frac{du}{u} \tag{3.1}$$

and

$$\big(\mathcal{J}^\alpha_{-,\mu} f\big)(x) \equiv (k_2 \star f)(x) = \int_0^\infty k_2\left(\frac{x}{u}\right) f(u)\frac{du}{u}, \tag{3.2}$$

where the functions $k_1(x)$ and $k_2(x)$ are given by

$$k_1(x) = 0 \quad (0 < x < 1), \qquad k_1(x) = \frac{x^{-\mu}}{\Gamma(\alpha)}(\log x)^{\alpha-1} \quad (x > 1), \tag{3.3}$$

and

$$k_2(x) = \frac{x^{\mu}}{\Gamma(\alpha)}\left(\log\frac{1}{x}\right)^{\alpha-1} \quad (0 < x < 1), \qquad k_2(x) = 0 \quad (x > 1), \tag{3.4}$$

respectively.

The folowing assertions give conditions for the functions $k_1(x)$ and $k_2(x)$ to belong to the space X_c^1.

Lemma 3. *Let $\alpha > 0$, $\mu \in \mathbf{C}$ and $c \in \mathbf{R}$.*

(a) $k_1 \in X_c^1$ *if and only if* $\mathrm{Re}(\mu) > c$.
(b) $k_2 \in X_c^1$ *if and only if* $\mathrm{Re}(\mu) > -c$.

Proof. According to (1.10) and (3.3) we have

$$\|k_1\|_{X_c^1} = \frac{1}{\Gamma(\alpha)} \int_1^\infty \left| x^{c-\mu} \big[\log(x)\big]^{\alpha-1} \right| \frac{dx}{x}$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-[\mathrm{Re}(\mu-c)]u} u^{\alpha-1}\, du,$$

the integral being convergent if and only if $\alpha > 0$ and $\mathrm{Re}(\mu) > c$. Thus assertion (a) follows. Part (b) is proved similarly, which completes the proof of the lemma. □

The next result yields the Mellin transforms of $k_1(x)$ and $k_2(x)$.

Lemma 4. *Let* $\alpha > 0$, $\mu \in \mathbf{C}$ *and* $s \in \mathbf{C}$.

(a) *If* $\mathrm{Re}(\mu - s) > 0$, *then*

$$(\mathcal{M}k_1)(s) = (\mu - s)^{-\alpha}. \tag{3.5}$$

(b) *If* $\mathrm{Re}(\mu + s) > 0$, *then*

$$(\mathcal{M}k_2)(s) = (\mu + s)^{-\alpha}. \tag{3.6}$$

Proof. Let $\mu \in \mathbf{R}$ and $s \in \mathbf{R}$ be such that $\mu - s > 0$. Using (1.1) and (3.3) and making the changes of variables $x = e^u$ and $u(\mu - s) = \tau$, we have

$$(\mathcal{M}k_1)(s) = \frac{1}{\Gamma(\alpha)} \int_1^\infty x^{s-\mu-1} \big[\log(x)\big]^{\alpha-1}\, dx$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-u(\mu-s)} u^{\alpha-1}\, du = \frac{(\mu - s)^{-\alpha}}{\Gamma(\alpha)} \int_0^\infty e^{-\tau} \tau^{\alpha-1}\, d\tau.$$

This yields (3.5) according to the definition of the gamma-funtion [7, 1.1(1)]. Part (b) with (3.6) is proved similarly for $\mu \in \mathbf{R}$ and $s \in \mathbf{R}$ such that $\mu + s > 0$. Hence the lemma is proved for real μ and s.

For complex $\mu \in \mathbf{C}$ and $s \in \mathbf{C}$ the relations (3.5) and (3.6) stay true by analytic continuation when $\mathrm{Re}(\mu - s) > 0$ and $\mathrm{Re}(\mu + s) > 0$, respectively. This completes the proof of Lemma 4.

Using (3.1), (3.3) and applying Lemmas 2, 3(a), and 4(a) we obtain the following statement involving the Mellin transforms of the Hadamard-type fractional integrals $\mathcal{J}^\alpha_{0+,\mu} f$ and $\mathcal{J}^\alpha_{-,\mu} f$ given by (1.4) and (1.7), respectively. □

Theorem 1. *Let* $\alpha > 0$, $\mu \in \mathbf{C}$, $c \in \mathbf{R}$ *and* $1 \leqslant p \leqslant 2$.

(a) *If* $\mathrm{Re}(\mu) > c$ *and* $f \in X_c^p$, *then the Mellin transform of* $\mathcal{J}_{0+,\mu}^{\alpha} f$ *is given by*

$$\big(\mathcal{M}^p \mathcal{J}_{0+,\mu}^{\alpha} f\big)(s) = (\mu - s)^{-\alpha}\big(\mathcal{M}^p f\big)(s) \quad (s = c + it,\ t \in \mathbf{R}). \tag{3.7}$$

(b) *If* $\mathrm{Re}(\mu) > -c$ *and* $f \in X_c^p$, *then the Mellin transform of* $\mathcal{J}_{-,\mu}^{\alpha} f$ *is given by*

$$\big(\mathcal{M}^p \mathcal{J}_{-,\mu}^{\alpha} f\big)(s) = (\mu + s)^{-\alpha}\big(\mathcal{M}^p f\big)(s) \quad (s = c + it,\ t \in \mathbf{R}). \tag{3.8}$$

Proof. According to (3.1) the Hadamard-type operator $\mathcal{J}_{0+,\mu}^{\alpha} f$ is a Mellin convolution operator (2.13) with the kernel $k(x) = k_1(x)$ of (3.3). By Lemma 3(a) $k_1 \in X_c^1$ if and only if $\alpha > 0$ and $\mathrm{Re}(\mu) > c$. So we may apply Lemma 2 to (3.1). Using (2.14) and Lemma 4(b) we obtain for $s = c + it$, $t \in \mathbf{R}$,

$$\big(\mathcal{M}^p \mathcal{J}_{0+,\mu}^{\alpha} f\big)(s) = (\mathcal{M}k_1)(s)\big(\mathcal{M}^p f\big)(s) = (\mu - s)^{-\alpha}\big(\mathcal{M}^p f\big)(s),$$

which proves assertion (a). Part (b) is proved similarly on the basis of the representation (3.2) for the operator $\mathcal{J}_{-,\mu}^{\alpha} f$ by using Lemmas 2, 3(b), and 4(b):

$$\big(\mathcal{M}^p \mathcal{J}_{-,\mu}^{\alpha} f\big)(s) = (\mathcal{M}k_2)(s)\big(\mathcal{M}^p f\big)(s) = (\mu + s)^{-\alpha}\big(\mathcal{M}^p f\big)(s). \qquad \square$$

Corollary 1. *Let* $\alpha > 0$, $\mu \in \mathbf{C}$ *and* $1 \leqslant p \leqslant 2$.

(a) *If* $\mathrm{Re}(\mu) > 1/p$ *and* $f \in L^p(\mathbf{R}_+)$, *then the Mellin transform of* $\mathcal{J}_{0+,\mu}^{\alpha} f$ *is given by* (3.7).
(b) *If* $\mathrm{Re}(\mu) > -1/p$ *and* $f \in L^p(\mathbf{R}_+)$, *then the Mellin transform of* $\mathcal{J}_{-,\mu}^{\alpha} f$ *is given by* (3.8).

Remark 2. When $\alpha = n \in \mathbf{N} = \{1, 2, \ldots\}$, $\alpha = \mu$ and $p = 1$ formula (3.8) was established in [6, (8.9)].

Now we obtain formulas for the Mellin transform of the Hadamard-type fractional integrals $\mathcal{I}_{0+,\mu}^{\alpha} f$ and $\mathcal{I}_{-,\mu}^{\alpha} f$ defined by (1.8) and (1.9). These integrals can be repesented in the forms (3.1) and (3.2), and their Mellin transfoms could be deduced from Lemma 2 as was carried out above for the Hadamard fractional integrals $\mathcal{J}_{0+,\mu}^{\alpha} f$ and $\mathcal{J}_{-,\mu}^{\alpha} f$. We shall use a more simple procedure based on the connections of (1.8), (1.9) with (1.4), (1.7), given by

$$\big(\mathcal{I}_{0+,\mu}^{\alpha} f\big)(x) = \big(\mathcal{J}_{0+,\mu+1}^{\alpha} f\big)(x) \tag{3.9}$$

and

$$\big(\mathcal{I}_{-,\mu}^{\alpha} f\big)(x) = \big(\mathcal{J}_{-,\mu-1}^{\alpha} f\big)(x), \tag{3.10}$$

respectively, see [1, (4.14)].

Using (3.9) and (3.10) and applying Theorem 1 and Corollary 1 with μ being replaced by $\mu + 1$ and $\mu - 1$, respectively, we obtain the corresponding statements for the Hadamard-type fractional integrals (1.8) and (1.9).

Theorem 2. *Let* $\alpha > 0$, $\mu \in \mathbf{C}$, $c \in \mathbf{R}$, *and* $1 \leqslant p \leqslant 2$.

(a) *If* $\mathrm{Re}(\mu) > c-1$ *and* $f \in X_c^p$, *then the Mellin transform of* $\mathcal{I}_{0+,\mu}^{\alpha} f$ *is given by*

$$\big(\mathcal{M}^p \mathcal{I}_{0+,\mu}^{\alpha} f\big)(s) = (\mu+1-s)^{-\alpha}\big(\mathcal{M}^p f\big)(s) \quad (s = c+it,\ t \in \mathbf{R}). \tag{3.11}$$

(b) *If* $\mathrm{Re}(\mu) > 1-c$ *and* $f \in X_c^p$, *then the Mellin transform of* $\mathcal{J}_{-,\mu}^{\alpha} f$ *is given by*

$$\big(\mathcal{M}^p \mathcal{I}_{-,\mu}^{\alpha} f\big)(s) = (\mu-1+s)^{-\alpha}\big(\mathcal{M}^p f\big)(s) \quad (s = c+it,\ t \in \mathbf{R}). \tag{3.12}$$

Corollary 2. *Let* $\alpha > 0$, $\mu \in \mathbf{C}$ *and* $1 \leqslant p \leqslant 2$.

(a) *If* $\mathrm{Re}(\mu) > -1/p'$ *and* $f \in L^p(\mathbf{R}_+)$, *then the Mellin transform of* $\mathcal{I}_{0+,\mu}^{\alpha} f$ *is given by* (3.11).
(b) *If* $\mathrm{Re}(\mu) > 1/p'$ *and* $f \in L^p(\mathbf{R}_+)$, *then the Mellin transform of* $\mathcal{I}_{-,\mu}^{\alpha} f$ *is given by* (3.12).

Here p' is conjugate to p $(1 \leqslant p \leqslant \infty)$

$$\frac{1}{p} + \frac{1}{p'} = 1, \tag{3.13}$$

with $p' = \infty$ for $p = 1$, while $p' = 1$ for $p = \infty$.

When $\mu = 0$, the corrresponding assertions for the Hadamard fractional integrals (1.6) and (1.13) also follow from Theorem 1.

Theorem 3. *Let* $\alpha > 0$, $c \in \mathbf{R}$, *and* $1 \leqslant p \leqslant 2$.

(a) *If* $c < 0$ *and* $f \in X_c^p$, *then the Mellin tranform of* $\mathcal{J}_{0+}^{\alpha} f$ *is given by*

$$\big(\mathcal{M}^p \mathcal{J}_{0+}^{\alpha} f\big)(s) = (-s)^{-\alpha}\big(\mathcal{M}^p f\big)(s) \quad (s = c+it,\ t \in \mathbf{R}). \tag{3.14}$$

(b) *If* $c > 0$ *and* $f \in X_c^p$, *then the Mellin tranform of* $\mathcal{J}_{-}^{\alpha} f$ *is given by*

$$\big(\mathcal{M}^p \mathcal{J}_{-}^{\alpha} f\big)(s) = s^{-\alpha}\big(\mathcal{M}^p f\big)(s) \quad (s = c+it,\ t \in \mathbf{R}). \tag{3.15}$$

Corollary 3. *Let* $\alpha > 0$, $1 \leqslant p \leqslant 2$ *and* $f \in L^p(\mathbf{R}_+)$, *then the Mellin transform of* $\mathcal{J}_{-}^{\alpha} f$ *is given by* (3.15).

Remark 3. A relation of the form (3.15) was indicated in [8] as an example of the theory of multipliers based on the Mellin transform, which is different from that in (2.1)–(2.2).

4. Relations of fractional integration by parts in X_c^p

The following formula for fractional integration by parts is known [3, (5.16)] for the Liouville fractional integration operators (1.14) and (1.15):

$$\int_{-\infty}^{\infty} f(x)\big(I_+^\alpha g\big)(x)\,dx = \int_{-\infty}^{\infty} g(x)\big(I_-^\alpha f\big)(x)\,dx; \tag{4.1}$$

it is valid for $f \in L^p(\mathbf{R})$ and $g \in L^r(\mathbf{R})$ with $p > 1$, $r > 1$ and $1/p + 1/r = 1 + \alpha$.

Such relations for the Hadamard-type fractional integration operators (1.4), (1.7), (1.8), and (1.9) have the following forms in succession:

$$\int_0^{\infty} f(x)\big(\mathcal{J}_{0+,\mu}^\alpha g\big)(x)\frac{dx}{x} = \int_0^{\infty} g(x)\big(\mathcal{J}_{-,\mu}^\alpha f\big)(x)\frac{dx}{x}, \tag{4.2}$$

$$\int_0^{\infty} f(x)\big(\mathcal{J}_{0+,\mu}^\alpha g\big)(x)\,dx = \int_0^{\infty} g(x)\big(\mathcal{I}_{-,\mu}^\alpha f\big)(x)\,dx, \tag{4.3}$$

$$\int_0^{\infty} f(x)\big(\mathcal{I}_{0+,\mu}^\alpha g\big)(x)\,dx = \int_0^{\infty} g(x)\big(\mathcal{J}_{-,\mu}^\alpha f\big)(x)\,dx, \tag{4.4}$$

and

$$\int_0^{\infty} f(x)\big(\mathcal{I}_{0+,\mu}^\alpha g\big)(x)x\,dx = \int_0^{\infty} g(x)\big(\mathcal{I}_{-,\mu}^\alpha f\big)(x)x\,dx. \tag{4.5}$$

First we prove the result in (4.2).

Theorem 4. *Let* $\alpha > 0$ *and let* $\mu \in \mathbf{C}$, $c \in \mathbf{R}$, $1 \leqslant p \leqslant \infty$, *and* $1 \leqslant r \leqslant \infty$ *be such that* $\mathrm{Re}(\mu) > -c$ *and* $1/p + 1/r \geqslant 1$.

If $f \in X_c^p$ *and* $g \in X_{-c}^r$, *then there holds the relation* (4.2) *of fractional integration by parts.*

Proof. For "sufficiently good" functions f and g, (4.2) is verified directly by using (1.4) and (1.7), changing the order of integration and applying the Dirichlet formula (for example, see [3, (1.32)]):

$$\int_0^{\infty} f(x)\big(\mathcal{J}_{-,\mu}^\alpha g\big)(x)\frac{dx}{x}$$

$$= \int_0^{\infty} f(x)\left[\frac{1}{\Gamma(\alpha)}\int_x^{\infty}\left(\frac{x}{u}\right)^{\mu}\left(\log\frac{u}{x}\right)^{\alpha-1}\frac{g(u)\,du}{u}\right]\frac{dx}{x},$$

$$\int_0^\infty g(u)\left[\frac{1}{\Gamma(\alpha)}\int_0^u \left(\frac{x}{u}\right)^\mu \left(\log\frac{u}{x}\right)^{\alpha-1}\frac{f(x)\,dx}{x}\right]\frac{du}{u}$$
$$=\int_0^\infty g(u)\big(\mathcal{J}^\alpha_{0+,\mu}f\big)(u)\frac{du}{u}.$$

To prove the theorem in general, it is suffices to show that both sides of (4.2) represent bounded bilinear functionals on $X^p_c \times X^r_{-c}$. For the left side of (4.2) we apply the Hoelder inequality to deduce

$$\left|\int_0^\infty f(x)\big(\mathcal{J}^\alpha_{0+,\mu}g\big)(x)\frac{dx}{x}\right| \leqslant \int_0^\infty \big[x^{c-1/p}\big|f(x)\big|\big]\big[x^{-c-1/p'}\big|\big(\mathcal{J}^\alpha_{0+,\mu}g\big)(x)\big|\big]\,dx$$
$$\leqslant \|f\|_{X^p_c}\big\|\mathcal{J}^\alpha_{0+,\mu}g\big\|_{X^{p'}_{-c}}. \tag{4.6}$$

Since $1/p + 1/r \geqslant 1$, then $p' \geqslant r$ in accordance with (3.13), and hence if $\mathrm{Re}(\mu) > -c$, then $\mathcal{J}^\alpha_{0+,\mu} \in [X^r_{-c}, X^{p'}_{-c}]$ by Theorem 7(a) in [1], and

$$\big\|\mathcal{J}^\alpha_{0+,\mu}g\big\|_{X^{p'}_{-c}} \leqslant K_1\|g\|_{X^r_{-c}}, \tag{4.7}$$

where $K_1 > 0$ is a bound for $\mathcal{J}^\alpha_{0+,\mu}$ as a member of the set $[X^r_{-c}, X^{p'}_{-c}]$ of all linear mappings from X^r_{-c} into $X^{p'}_{-c}$. Substituting this estimate into (4.6) we obtain

$$\left|\int_0^\infty f(x)\big(\mathcal{J}^\alpha_{0+,\mu}g\big)(x)\frac{dx}{x}\right| \leqslant K_1\|f\|_{X^p_c}\|g\|_{X^r_{-c}}. \tag{4.8}$$

Hence the left side of (4.2) represents a bounded bilinear functional on $X^p_c \times X^r_{-c}$.

Using the Hoelder inequality and Theorem 7(b) in [1] similarly as for (4.6)–(4.8) we have

$$\left|\int_0^\infty g(x)\big(\mathcal{J}^\alpha_{-,\mu}f\big)(x)\frac{dx}{x}\right| \leqslant \int_0^\infty \big[x^{-c-1/p'}\big|g(x)\big|\big]\big[x^{c-1/p}\big|\big(\mathcal{J}^\alpha_{-,\mu}g\big)(x)\big|\big]\,dx$$
$$\leqslant \|g\|_{X^r_{-c}}\big\|\mathcal{J}^\alpha_{-,\mu}f\big\|_{X^{r'}_c} \leqslant K_2\|g\|_{X^r_{-c}}\|f\|_{X^p_c},$$

where $K_2 > 0$ is a bound for $\mathcal{J}^\alpha_{-,\mu}$ as a member of $[X^p_c, X^{r'}_c]$. Hence the right side of (4.2) also represents a bounded bilinear functional on $X^p_c \times X^r_{-c}$; this completes the proof of the theorem. □

Corollary 4. *Let $\alpha > 0$, $1 \leqslant p \leqslant \infty$, and let $\mu \in \mathbf{C}$ and $c \in \mathbf{R}$ be such that* $\mathrm{Re}(\mu) > -c$.

If $f \in X^p_c$ and $g \in X^{p'}_{-c}$, then (4.2) *holds.*

Corollary 5. *Let $\alpha > 0$ and let $\mu \in \mathbf{C}$, $1 \leqslant p \leqslant \infty$ and $1 \leqslant r \leqslant \infty$ be such that $1/p + 1/r \geqslant 1$ and $\mathrm{Re}(\mu) > -1/p$.*

If $f \in L^p(\mathbf{R}_+)$ and $g \in X^r_{-1/p}$, then (4.2) *holds.*

Corollary 6. *Let $\alpha > 0$ and let $\mu \in \mathbf{C}$ and $1 \leqslant p \leqslant \infty$ be such that $\mathrm{Re}(\mu) > -1/p$.*

If $f \in L^p(\mathbf{R}_+)$ and $g \in X^{p'}_{-1/p}$, then (4.2) *holds.*

Corollary 4 is a particular case of Theorem 4 when $r = p'$. Corollary 5 follows from Theorem 4 when $c = 1/p$, while Corollary 6 when $c = 1/p$ and $r = p'$.

The assertions in (4.3), (4.4), and (4.5) can be established similarly to the proof of Theorem 4 by using Hoelder's inequality and the coresponding results in [1]: Theorems 7(a) and 8(b), Theorems 8(a) and 7(b), and Theorems 8(a) and 8(b), respectively. In this respect we have:

Theorem 5. *Let $\alpha > 0$ and let $\mu \in \mathbf{C}$, $c \in \mathbf{R}$, $1 \leqslant p \leqslant \infty$ and $1 \leqslant r \leqslant \infty$ be such that $1/p + 1/r \geqslant 1$ and $\mathrm{Re}(\mu) > 1 - c$.*

If $f \in X^p_c$ and $g \in X^r_{1-c}$, then there holds the relation (4.3) *of fractional integration by parts.*

Theorem 6. *Let $\alpha > 0$ and let $\mu \in \mathbf{C}$, $c \in \mathbf{R}$, $1 \leqslant p \leqslant \infty$ and $1 \leqslant r \leqslant \infty$ be such that $1/p + 1/r \geqslant 1$ and $\mathrm{Re}(\mu) > -c$.*

If $f \in X^p_c$ and $g \in X^r_{1-c}$, then there holds the relation (4.4) *of fractional integration by parts.*

Theorem 7. *Let $\alpha > 0$ and let $\mu \in \mathbf{C}$, $c \in \mathbf{R}$, $1 \leqslant p \leqslant \infty$ and $1 \leqslant r \leqslant \infty$ be such that $1/p + 1/r \geqslant 1$ and $\mathrm{Re}(\mu) > 1 - c$.*

If $f \in X^p_c$ and $g \in X^r_{2-c}$, then there holds the relation (4.5) *of fractional integration by parts.*

The counterparts of Corollaries 4 to 6 are also valid for the relations (4.3)–(4.5).

Taking $\mu = 0$ in (4.2), we obtain the relation of fractional integration by parts for the Hadamard fractional integration oparators (1.6) and (1.13):

$$\int_0^\infty f(x)\big(\mathcal{J}^\alpha_{0+} g\big)(x)\frac{dx}{x} = \int_0^\infty g(x)\big(\mathcal{J}^\alpha_{-} f\big)(x)\frac{dx}{x}. \tag{4.9}$$

Indeed, Theorem 4 yields the result for the validity of (4.9).

Theorem 8. *Let $\alpha > 0$ and let $c \in \mathbf{R}$, $1 \leqslant p \leqslant \infty$ and $1 \leqslant r \leqslant \infty$ be such that $1/p + 1/r \geqslant 1$ and $c > 0$.*

If $f \in X_c^p$ *and* $g \in X_{-c}^r$*, then there holds the relation* (4.9) *of fractional integration by parts.*

Corollary 7. *Let* $\alpha > 0$, $1 \leqslant p \leqslant \infty$ *and* $c > 0$.

If $f \in X_c^p$ *and* $g \in X_{-c}^{p'}$*, then* (4.9) *holds.*

Corollary 8. *Let* $\alpha > 0$ *and let* $1 \leqslant p \leqslant \infty$ *and* $1 \leqslant r \leqslant \infty$ *be such that* $1/p + 1/r \geqslant 1$.

If $f \in L^p(\mathbf{R}_+)$ *and* $g \in X_{-1/p}^r$*, then* (4.9) *holds.*

Corollary 9. *Let* $\alpha > 0$ *and* $1 \leqslant p \leqslant \infty$.

If $f \in L^p(\mathbf{R}_+)$ *and* $g \in X_{-1/p}^{p'}$*, then* (4.9) *holds.*

5. Relations of fractional integration by parts in X_c^p. Continuation

There hold the formulas, similar to (4.2)–(4.5) and (4.9), of the form

$$\int_0^\infty f(x)\big(\mathcal{J}_{-,\mu}^\alpha g\big)(x)\frac{dx}{x} = \int_0^\infty g(x)\big(\mathcal{J}_{0+,\mu}^\alpha f\big)(x)\frac{dx}{x}, \tag{5.1}$$

$$\int_0^\infty f(x)\big(\mathcal{J}_{-,\mu}^\alpha g\big)(x)\,dx = \int_0^\infty g(x)\big(\mathcal{I}_{0+,\mu}^\alpha f\big)(x)\,dx, \tag{5.2}$$

$$\int_0^\infty f(x)\big(\mathcal{I}_{-,\mu}^\alpha g\big)(x)\,dx = \int_0^\infty g(x)\big(\mathcal{J}_{0+,\mu}^\alpha f\big)(x)\,dx, \tag{5.3}$$

$$\int_0^\infty f(x)\big(\mathcal{I}_{-,\mu}^\alpha g\big)(x)x\,dx = \int_0^\infty g(x)\big(\mathcal{I}_{0+,\mu}^\alpha f\big)(x)x\,dx, \tag{5.4}$$

and

$$\int_0^\infty f(x)\big(\mathcal{J}_{-}^\alpha g\big)(x)\frac{dx}{x} = \int_0^\infty g(x)\big(\mathcal{J}_{0+}^\alpha f\big)(x)\frac{dx}{x}. \tag{5.5}$$

Conditions for the validity of (5.1) are given by the following result.

Theorem 9. *Let* $\alpha > 0$ *and let* $\mu \in \mathbf{C}$, $c \in \mathbf{R}$, $1 \leqslant p \leqslant \infty$ *and* $1 \leqslant r \leqslant \infty$ *be such that* $1/p + 1/r \geqslant 1$ *and* $\operatorname{Re}(\mu) > c$.

If $f \in X_c^p$ *and* $g \in X_{-c}^r$*, then there holds the relation* (5.1) *of fractional integration by parts.*

Proof. Theorem 9 is proved similarly to Theorem 4 by applying the Hoelder inequality and Theorem 7 in [1]. □

Corollary 10. *Let $\alpha > 0$, $1 \leqslant p \leqslant \infty$ and let $\mu \in \mathbf{C}$ and $c \in \mathbf{R}$ be such that* $\mathrm{Re}(\mu) > c$.

If $f \in X_c^p$ and $g \in X_{-c}^{p'}$, then (5.1) *holds.*

Corollary 11. *Let $\alpha > 0$ and let $\mu \in \mathbf{C}$, $1 \leqslant p \leqslant \infty$ and $1 \leqslant r \leqslant \infty$ be such that* $1/p + 1/r \geqslant 1$ *and* $\mathrm{Re}(\mu) > 1/p$.

If $f \in L^p(\mathbf{R}_+)$ and $g \in X_{-1/p}^r$, then (5.1) *holds.*

Corollary 12. *Let $\alpha > 0$ and let $\mu \in \mathbf{C}$ and $1 \leqslant p \leqslant \infty$ be such that* $\mathrm{Re}(\mu) > 1/p$.

If $f \in L^p(\mathbf{R}_+)$ and $g \in X_{-1/p}^{p'}$, then (5.1) *holds.*

The results in (5.2), (5.3), and (5.4) are proved similarly to the proof of Theorem 9 by using Hoelder inequality and the following results in [1]: Theorems 7(b) and 8(a), Theorems 8(b) and 7(a), and Theorem 8, respectively.

Theorem 10. *Let $\alpha > 0$ and let $\mu \in \mathbf{C}$, $c \in \mathbf{R}$, $1 \leqslant p \leqslant \infty$ and $1 \leqslant r \leqslant \infty$ be such that* $1/p + 1/r \geqslant 1$ *and* $\mathrm{Re}(\mu) > c - 1$.

If $f \in X_c^p$ and $g \in X_{1-c}^r$, then there holds the relation (5.2) *of fractional integration by parts.*

Theorem 11. *Let $\alpha > 0$ and let $\mu \in \mathbf{C}$, $c \in \mathbf{R}$, $1 \leqslant p \leqslant \infty$ and $1 \leqslant r \leqslant \infty$ be such that* $1/p + 1/r \geqslant 1$ *and* $\mathrm{Re}(\mu) > c$.

If $f \in X_c^p$ and $g \in X_{1-c}^r$, then there holds the relation (5.3).

Theorem 12. *Let $\alpha > 0$ and let $\mu \in \mathbf{C}$, $c \in \mathbf{R}$, $1 \leqslant p \leqslant \infty$ and $1 \leqslant r \leqslant \infty$ be such that* $1/p + 1/r \geqslant 1$ *and* $\mathrm{Re}(\mu) > c - 1$.

If $f \in X_c^p$ and $g \in X_{2-c}^r$, then there holds the relation (5.4).

The counterparts of Corollaries 10 to 12 are also valid for the relations (5.2)–(5.4).

When $c = 0$, Theorem 10 yields the result for the validity of (5.5).

Theorem 13. *Let $\alpha > 0$ and let $c \in \mathbf{R}$, $1 \leqslant p \leqslant \infty$ and $1 \leqslant r \leqslant \infty$ be such that* $1/p + 1/r \geqslant 1$ *and* $c < 0$.

If $f \in X_c^p$ and $g \in X_{-c}^r$, then there holds (5.5).

Corollary 13. *Let $\alpha > 0$, $1 \leqslant p \leqslant \infty$, and $c < 0$.*

If $f \in X_c^p$ and $g \in X_{-c}^{p'}$, then (5.5) *holds.*

Remark 4. The relations (4.2)–(4.3), (5.1)–(5.2), (4.4)–(4.5), (5.3)–(5.4), (4.9), and (5.5) could be applied to define the Hadamard-type fractional integration operators $\mathcal{J}^{\alpha}_{0+,\mu}$, $\mathcal{J}^{\alpha}_{-,\mu}$, $\mathcal{I}^{\alpha}_{0+,\mu}$, $\mathcal{I}^{\alpha}_{-,\mu}$, $\mathcal{J}^{\alpha}_{0+}$, and $\mathcal{J}^{\alpha}_{-}$ in the space of generalized functions $(X^p_c)'$ consisting of continuous linear functionals on X^p_c equipped with the norms (1.10) and (1.11).

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