Remarks on between estimator in the intraclass correlation model with missing data

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Abstract

In this paper, we consider the between estimator under the intraclass correlation model with missing data. We give a necessary and sufficient condition for existing exact simultaneous confidence intervals for all contrasts in the means under the between transformed model, which indicates the F-test statistic and simultaneous confidence intervals, constructed by Seo et al. [T. Seo, J. Kikuchi, K. Koizumi, On simultaneous confidence intervals for all contracts in the means of the intraclass correlation model with missing data, J. Multivariate Anal. 97 (2006) 1976–1983] based on the between estimator, is invalid. Furthermore, using the distribution of the between estimator, we present the exact test statistics and confidence intervals for partial contrasts.

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1. Introduction

Consider the mixed linear model

\[ x_{ij} = \mu_i + \alpha_j + \varepsilon_{ij}, \quad i = 1, \ldots, p, \quad j = 1, \ldots, n, \] (1.1)
where \( \mu_i \) is the mean of the \( i \)th observation, \( \alpha_j \) is the random individual effect, \( \alpha_j \sim N(0, \sigma_\alpha^2) \) and \( \varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2) \) are all mutually independent. Denote \( x_j = (x_{i1}, \ldots, x_{ip_j})' \), \( \sigma_\alpha^2 = \sigma_\alpha^2 + \sigma_\varepsilon^2 \), \( \rho = \sigma_\alpha^2 / \sigma^2 \). It is easy to see that \( x_1, \ldots, x_n \) are independent and

\[
\text{Cov}(x_j) = \Sigma_j = \sigma^2((1 - \rho)I_{p_j} + \rho J_{p_j}), \quad j = 1, \ldots, n, \tag{1.2}
\]

where \( I_{p_j} \) is the \( p_j \times p_j \) identity matrix, \( J_{p_j} = 1_{p_j}1_{p_j}' \), and \( 1_{p_j} = (1, \ldots, 1)' \). When the covariance matrix of observation vector is of the above structure, it is called an intraclass correlation model.

A special case of Model (1.1) is the two-way crossed classification mixed linear model \( x_{ij} = \mu + \beta_i + \alpha_j + \varepsilon_{ij} \), where \( \beta_i \) is a fixed effect. Here \( \mu_i = \mu + \beta_i \). It is easy to see that all contrasts on \( \beta_i \)'s of interest are equivalent to the corresponding contrasts on \( \mu_i \)'s.

When all \( p_j = p \), (balanced data), there are many optimal properties on the hypothesis test and estimation of parameters in model (1.1), we can give optimally powerful unbiased tests and the uniformly minimum variance unbiased estimate of \( (\mu_1, \ldots, \mu_p, \sigma_\alpha^2, \sigma_\varepsilon^2) \), see Searle, et al. [7]. However, the above optimal properties are lost in the unbalanced data case. The problem of missing data occurs frequently in many practical situations, there are a few missing patterns considered in the literature, among which the incomplete data with monotone pattern, not only often occurs, but also is convenient for making inference. Several authors have considered the monotone pattern under normal assumption, and provided asymptotic as well as approximate test procedures about the normal mean vector, such as Anderson [1], Bhargave [2] Kanda and Fujkoshi [4]. Krishnamoorthy and Pannala [5, 6] provided an accurate simple approach to construct test and confidence regions for a normal mean vector. The above work mainly considered the case of the covariance matrix of random variable \( x \) with dimension \( p \) being any arbitrary unknown positive definite matrix. For the intraclass correlation covariance matrix, the above methods usually have low power or efficiency because of ignoring the information on covariance matrix.

Note that the likelihood method under mixed linear models usually needs iterative numerical algorithms and its inference is based on approximate properties. In practical situations, the transformation is often adopted in order to obtain some simple exact tests and estimator of parameters of interest. For example, the between transformation and the within transformation (see Hsiao [3]) are often considered under an intraclass correlation model. Seo and Srivastava [8], based on the within transformation \( y_j = C_j x_j \), present a simple exact test and the exact simultaneous confidence intervals for linear contrasts of the mean components in model (1.3), where \( C_j \) satisfies that \( C_j C_j' = I_{p_j-1}, C_j' C_j = I_{p_j} - J_{p_j} / p_j \). Notice the transformed data

\[
y_1, \ldots, y_n \text{ are mutually independent and } \text{Cov}(y_j) = \sigma_\varepsilon^2 I_{p_j-1}. \tag{1.3}
\]

Recently, Seo, et al. [9] consider the between transformation \( \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_p)' \) and construct an \( F \)-test statistic and the simultaneous confidence intervals for all contrasts in the mean based on their following conclusion:

**Conclusion 1.1.** For \( n_1 \geq n_2 \geq \cdots \geq n_p \), the quadratic form \( \sum_{j=1}^p ((\tilde{x}_i - \tilde{x}_.) / (\gamma / \sqrt{n_i}))^2 \) has a \( \chi^2 \) distribution with \( p - 1 \) degrees of freedom, where \( \gamma = \sigma \sqrt{(1 - \rho)} = \sigma_\varepsilon \),

\[
\tilde{x}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij}, \quad \tilde{x}. = \frac{1}{p} \sum_{j=1}^{p} \tilde{x}_i, \tag{1.3}
\]

and \( n_i \) is the number of the subjects observed in \( i \)th observation.
Unfortunately, Conclusion 1.1 does not hold because \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_p)' \) has a more complicated covariance matrix than (1.2) in the general case.

This paper is organized as follows. In Section 2, the covariance matrix of \( \bar{x} \) is given, and the quadratic form \( \sum_{i=1}^{p} ((\bar{x}_i, - \bar{x})/(\gamma/\sqrt{n_i}))^2 \) above is proven to be a \( \chi^2 \) variable if and only if \( n_1 = n_2 = \cdots = n_p \). This shows that Conclusion 1.1 is not true and exact simultaneous confidence intervals for all contrasts in the mean, which constructed by Seo, et al. [9], is invalid. Section 3 presents some exact test statistics and exact confidence intervals for partial contrasts based on \( \bar{x} \).

2. Distribution of the between estimator

In this section, we consider the covariance matrix of \( \bar{x} \) and the distribution of the quadratic form \( \sum_{i=1}^{p} ((\bar{x}_i, - \bar{x})/(\gamma/\sqrt{n_i}))^2 \). Without loss of generality, we can rewrite the observations \( \{x_{ij}\} \) with monotone pattern in the following form:

\[
\begin{pmatrix}
  x_{11} & x_{12} & \cdots & \cdots & x_{1n} \\
  x_{21} & x_{22} & \cdots & \cdots & \vdots \\
  \vdots & \vdots & \cdots & \cdots & \vdots \\
  \vdots & \vdots & \cdots & \cdots & \vdots \\
  x_{p1} & x_{p2} & \cdots & \cdots & x_{pn}
\end{pmatrix}
= \begin{pmatrix}
  x_{11} & x_{12} & \cdots & \cdots & x_{1n_1} \\
  x_{21} & x_{22} & \cdots & \cdots & \vdots \\
  \vdots & \vdots & \cdots & \cdots & \vdots \\
  \vdots & \vdots & \cdots & \cdots & \vdots \\
  x_{p1} & x_{p2} & \cdots & \cdots & x_{pn_p}
\end{pmatrix},
\]

where \( p = p_1 \geq p_2 \geq \cdots \geq p_n \) and \( n = n_1 \geq n_2 \geq \cdots \geq n_p \).

Note that the between-transformed model of (1.1) is

\[
\bar{x} = u + e, \quad \text{Cov}(e) = \text{Cov} (\bar{x}), \tag{2.1}
\]

where \( u = (\mu_1, \mu_2, \ldots, \mu_p)' \). So the between estimator of \( u \) is \( \bar{x} \). It is easy to see that \( \bar{x} \) is also the least squares estimator of \( u \) under model (1.1).

**Theorem 2.1.** Let \( \bar{x} \) be defined as in (2.1), then \( \bar{x} \sim N(u, \sigma^2 \Sigma) \), where

\[
\Sigma = (\sigma_{ij}) = \begin{pmatrix}
  1 & \rho & \cdots & \rho & \\
  \rho & 1 & \cdots & \rho & \\
  \rho & \rho & \cdots & 1 & \\
  n_1 & n_2 & \cdots & n_{p-1} & n_p \\
  n_1 & n_2 & \cdots & n_{p-1} & n_p
\end{pmatrix}.
\]

**Proof.** It is clear from the assumptions on distributions of \( \alpha_j \) and \( \varepsilon_{ij} \) in model (1.1) that \( \bar{x} = (\bar{x}_1, \ldots, \bar{x}_p)' \) has a normal distribution with mean vector \( u \), and

\[
\text{Cov}(x_{ij}, x_{kl}) = \begin{cases} 
  \sigma^2, & \text{if } i = k \text{ and } j = l, \\
  \sigma^2 \rho, & \text{if } i \neq k \text{ and } j = l, \\
  0, & \text{otherwise}.
\end{cases}
\]
Thus
\[
\sigma_{ii} = \text{Cov}(\bar{x}_i, \bar{x}_i) = \frac{1}{n_i} \text{Cov}(x_{i1}, x_{i1}) = \frac{1}{n_i} \sigma^2, \quad i = 1, \ldots, p, \tag{2.2}
\]
\[
\sigma_{il} = \text{Cov}(\bar{x}_i, \bar{x}_l) = \frac{1}{n_i n_l} \sum_j \text{Cov}(x_{ij}, x_{lj}) = \frac{\min(n_{ij}, n_{jl})}{n_i n_l} \rho \sigma^2
\]
\[
= \frac{\rho \sigma^2}{\max(n_i, n_l)}, \quad (i \neq l). \tag{2.3}
\]
Combined with \(n_1 \geq n_2 \geq \cdots \geq n_p\), (2.3) can be simplified as
\[
\sigma_{il} = \sigma_{li} = \text{Cov}(\bar{x}_i, \bar{x}_l) = \frac{\rho \sigma^2}{\max(n_i, n_l)} = \frac{\rho \sigma^2}{n_i}, \quad (i < l). \tag{2.4}
\]
That is, \(\text{Cov}(\bar{x}_i, \bar{x}_l) = \sigma^2 \Sigma\). The proof of Theorem 2.1 is completed. □

Denote
\[
A = (I_p - \bar{J}_p)V^{-1}(I_p - \bar{J}_p)/\gamma^2,
\]
where \(\bar{J}_p = J_p/p\), \(V = \text{diag}(1/n_i)\). The quadratic form in Conclusion 1.1 can be rewritten as
\[
\sum_{i=1}^{p} \left( \frac{\bar{x}_i - \bar{x}_.}{\gamma/\sqrt{n_i}} \right)^2 = (\bar{x}_. - \bar{x}_..1_p)'V^{-1}(\bar{x}_. - \bar{x}_..1_p)/\gamma^2 = \bar{x}_.'A\bar{x}_., \tag{2.5}
\]

Theorem 2.2. The quadratic form \(\bar{x}_.'A\bar{x}_\). has a \(\chi^2\) distribution if and only if
\[
n_1 = n_2 = \cdots = n_p.
\]

Proof. According to Corollary 3.4.3 in Wang and Chow [10], \(\bar{x}_.'A\bar{x}_\). is a \(\chi^2\) variable if and only if
\[
A(\sigma^2 \Sigma)A = A. \tag{2.6}
\]
Denote \(D = (d_{ij})_{p \times p}\) and \(d_{ij} = d_{ji} = 1/n_i, (i \leq j)\). Then
\[
\Sigma = (1 - \rho)V + \rho D.
\]
Combining with the facts \(\gamma^2 = \sigma^2(1 - \rho) = \sigma_\varepsilon^2\) and
\[
(I_p - \bar{J}_p)V^{-1}(I_p - \bar{J}_p)V\bar{J}_p = -(I_p - \bar{J}_p)V^{-1}\bar{J}_p V\bar{J}_p,
\]
(2.6) can be simplified as
\[
-\rho BVB' = (1 - \rho)A_0DA_0', \tag{2.7}
\]
where \(A_0 = \gamma^2 A\) and \(B = (I_p - \bar{J}_p)V^{-1}\bar{J}_p\).

Notice both \(BV B\) and \(A_0DA_0\) being nonnegative definite matrices. Thus for any \(\rho \in (0, 1)\), (2.7) holds if and only if
\[
\begin{cases}
BV B' = 0, \\
A_0DA_0' = 0. \tag{2.8}
\end{cases}
\]
It is clear from the fact $V > 0$ and $D \geq 0$ that (2.8) is equivalent to $B = 0$, $A_0 D = 0$, that is
\[
\begin{cases}
(I_p - \bar{J}_p) V^{-1} I_p = 0, \\
(I_p - \bar{J}_p) V^{-1} (I_p - \bar{J}_p) D = 0,
\end{cases}
\] (2.9)
and the last equality of (2.9) is equivalent to
\[V^{-1/2}(I_p - \bar{J}_p) D = 0 .\]
Using the fact $V > 0$, (2.9) is simplified as
\[
\begin{align*}
& n_i - \sum_{j=1}^{p} n_j / p = 0, \quad i = 1, \ldots, p, \\
& (I_p - \bar{J}_p) D = 0 .
\end{align*}
\] (2.10)
It is easy to see that both equalities of (2.10) are equivalent to $n_1 = n_2 = \cdots = n_p$.

Notice that (2.6) is equivalent to the second (or first) equality of (2.10) if $\rho = 0$ (or 1). Thus (2.6) is equivalent to $n_1 = n_2 = \cdots = n_p$ for any $\rho \in [0, 1]$. The proof of Theorem 2.2 is completed. \hfill \Box

For balanced data, it has $A = n_1 (I_p - \bar{J}_p) / \sigma_e^2$. Combined with Theorem 2.2, we obtain the following result.

**Corollary 2.1.** If $n_1 = n_2 = \cdots = n_p$, then $\tilde{x}' A \tilde{x} = n_1 \sum_{i=1}^{p} (\bar{x}_i - \bar{x})^2 / \sigma_e^2$ has a noncentral $\chi^2$ distribution with $p - 1$ degrees of freedom and noncentral parameter $\lambda = u' A u$, where $\lambda = 0$ if and only if $\mu_1 = \mu_2 = \cdots = \mu_p$.

Theorem 2.2 and Corollary 2.1 indicate Conclusion 1.1 given by Seo, et al. [9] is wrong for unequal $n_i$. Thus both an $F$-test statistic and the simultaneous confidence intervals for all contrasts in the means constructed based on Conclusion 1.1 usually do not perform well in the unbalanced case.

It is the key mistake in their proof to take $\bar{z}_i - \bar{z}_{..} = \bar{x}_i - \bar{x}_c$, where $z_j = C_j x_j$, $C_j = \Sigma_j^1 / (\sigma \sqrt{1 - \rho}) = I_{p_1} - (j_1 / p_1) I_{1'}$, here $v_j = 1 \pm (1 - \rho)(1 + (p_1 - 1) \rho)$, $\bar{z}_i$, and $\bar{z}_c$ are defined similarly as (1.3). In fact, we can notice that
\[
\bar{z}_i - \bar{z}_c = \bar{x}_i - \bar{x}_c - \left( \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij} \bar{x}_c - \frac{1}{p} \sum_{i=1}^{p} \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij} \bar{x}_c \right) \neq \bar{x}_i - \bar{x}_c ,
\]
for $n_1 \geq n_2 \geq \cdots \geq n_p$ and $p_1 \geq p_2 \geq \cdots \geq p_n$, where $x_{ij} = \sum_{i=1}^{p_j} x_{ij} / p_j$.

Theorems 2.1 and 2.2 indicate that the exact simultaneous confidence intervals for all contrasts in means do not exist under the between-transformed model (2.1) with unbalanced data. In the following section, we will consider exact tests and confidence intervals for partial contrasts in means based on $\tilde{x}$.

**3. Exact test and confidence intervals for partial contrasts**

Note that observed data $\{x_{ij}\}$ can be grouped into $s$ subsets of complete data, respectively, where the $c$th group is a $p^{(c)} \times n^{(c)}$ matrix, $1 \leq c \leq s \leq p$. For convenience, we firstly consider the simple case: $s = 2$ and each $p^{(c)} \geq 2$. Let $p^{(1)} = k$, $p^{(2)} = p - k$,
\[
n^{(1)} = n_1 = n_2 = \cdots = n_k \geq n^{(2)} = n_{k+1} = \cdots = n_p.
\]
Then the two subsets of complete data can be denoted as
\[ X^{(1)}_{k \times n^{(1)}} = (x_{ij}), \quad (1 \leq i \leq k), \quad X^{(2)}_{(p-k) \times n^{(2)}} = (x_{ij}), \quad (k+1 \leq i \leq p). \]  

(3.1)

We consider the between transformation under each subset of complete data:
\[ \tilde{x}^{(1)} = \frac{1}{n^{(1)}} X_1 1_{n^{(1)}}, \quad \tilde{x}^{(2)} = \frac{1}{n^{(2)}} X_2 1_{n^{(2)}}. \]  

(3.2)

Clearly, it has
\[ \tilde{x}^{(1)} = (\bar{x}_1, \ldots, \bar{x}_k), \quad \tilde{x}^{(2)} = (\bar{x}_{(k+1)}, \ldots, \bar{x}_p), \]  

and the between transformed data above \( \tilde{x} \) is \((\tilde{x}^{(1)}', \tilde{x}^{(2)}')'\). Denote
\[ u^{(1)} = (\mu_1, \ldots, \mu_k)', \quad u^{(2)} = (\mu_{k+1}, \ldots, \mu_p)'. \]

By Corollary 2.1, we have
\[ \frac{n^{(1)}}{\sigma^2} \sum_{i=1}^{k} (\bar{x}_i - \bar{x}^{(1)})^2 = n^{(1)} \frac{1}{\sigma^2} \| \tilde{x}^{(1)} - \bar{x}^{(1)} \|_2^2 \sim \chi^2_{(k-1), \delta_1}, \]  

(3.3)

\[ \frac{n^{(2)}}{\sigma^2} \sum_{i=k+1}^{p} (\bar{x}_i - \bar{x}^{(2)})^2 = n^{(2)} \frac{1}{\sigma^2} \| \tilde{x}^{(2)} - \bar{x}^{(2)} \|_{p-k-1}^2 \sim \chi^2_{(p-k-1), \delta_2}, \]  

(3.4)

where \( \|a\| = \sqrt{a^T a} \), the noncentral parameters
\[ \delta_1 = \frac{n^{(1)}}{\sigma^2} u^{(1)}' (I_k - J_k) u^{(1)}, \quad \delta_2 = \frac{n^{(2)}}{\sigma^2} u^{(2)}' (I_{p-k} - J_{p-k}) u^{(2)}. \]

Under the two subsets of complete data, we can obtain the two unbiased estimates of \( \sigma^2 \)
\[ \hat{\sigma}^2_{c}(1) = s_1/f_1 = \sum_{i=1}^{k} \sum_{j=1}^{n^{(1)}} (x_{ij} - \bar{x}_i - \bar{x}^{(1)} - \bar{x}^{(1)})^2 / f_1, \]  

\[ \hat{\sigma}^2_{c}(2) = s_2/f_2 = \sum_{i=k+1}^{p} \sum_{j=1}^{n^{(2)}} (x_{ij} - \bar{x}_i - \bar{x}^{(2)} - \bar{x}^{(2)})^2 / f_2, \]  

where \( f_1 = (k-1)(n^{(1)} - 1), \quad f_2 = (p-k-1)(n^{(2)} - 1), \) and
\[ \tilde{x}^{(1)} = \sum_{i=1}^{k} x_{ij}/k, \quad (j = 1, \ldots, n_1), \quad \tilde{x}^{(2)} = \sum_{i=k+1}^{p} x_{ij}/(p-k), \quad (j = 1, \ldots, n_2). \]

**Theorem 3.1.** (1) \( s_c/\sigma^2_{c} \sim \chi^2_{c}, c = 1, 2; \)
(2) \( \tilde{x}^{(1)} - \tilde{x}^{(1)} 1_k, \tilde{x}^{(2)} - \tilde{x}^{(2)} 1_{p-k}, s_1 \text{ and } s_2 \text{ are independent.} \)
Proof. Let
\[ y_1 = (I_{n(1)} \otimes (I_k - \bar{J}_k)) \text{Vec}(X^{(1)}), \quad y_2 = (I_{n(2)} \otimes (I_{p-k} - \bar{J}_{p-k})) \text{Vec}(X^{(2)}). \]
It is easy to see that
\[
\begin{align*}
\tilde{x}_i^{(1)} - \bar{x}_{ij}^{(1)} 1_k &= \frac{1}{n(1)} (I_{n(1)}' \otimes I_k) y_1, \\
\tilde{x}_i^{(2)} - \bar{x}_{ij}^{(2)} 1_{p-k} &= \frac{1}{n(2)} (I_{n(2)}' \otimes I_{p-k}) y_2, \\
S_1 &= \sum_{i=1}^{n(1)} \sum_{j=1}^{n(2)} (x_{ij} - \tilde{x}_{ij}^{(1)} + \bar{x}_{ij}^{(1)})^2 = y_1' ((I_{n(1)} - \bar{J}_{n(1)}) \otimes I_k) y_1, \\
S_2 &= \sum_{i=k+1}^{p} \sum_{j=1}^{n(2)} (x_{ij} - \tilde{x}_{ij}^{(2)} + \bar{x}_{ij}^{(2)})^2 = y_2' ((I_{n(2)} - \bar{J}_{n(2)}) \otimes I_{p-k}) y_2,
\end{align*}
\]
which are linear forms and quadratic forms of normal variables \( y_1 \) and \( y_2 \), respectively.

Using the fact
\[
\text{Cov}(\text{Vec}(X^{(c)})) = \sigma^2 I_{n(c)} \otimes ((1 - \rho) I_{p(c)} + \rho J_{p(c)}), \quad c = 1, 2,
\]
\[
\text{Cov}(\text{Vec}(X^{(1)}), \text{Vec}(X^{(2)})) = \sigma^2 \rho \left( I_{n(2)} \otimes 1_k 1_{p-k} \right)_0,
\]
where zero matrix \( 0 \) is \((n^{(1)} - n^{(2)})k \times (n^{(2)}(p - k))\), we have
\[
\begin{align*}
\text{Cov}(y_1) &= \sigma^2 \left( I_{n(1)} \otimes (I_k - \bar{J}_k) \right), \\
\text{Cov}(y_2) &= \sigma^2 \left( I_{n(2)} \otimes (I_{p-k} - \bar{J}_{p-k}) \right), \\
\text{Cov}(y_1, y_2) &= 0.
\end{align*}
\]
Applying theorems on distribution of quadratic form and on the independence of a linear form and a quadratic form of normal variables (see, Corollary 3.4.3 and 3.5.1 of Wang and Chow [10]) to (3.5), we get Theorem 3.1 immediately. \( \square \)

Based on Theorem 3.1, we can construct exact test statistics for any contrasts of mean sub-vector \( u^{(c)} \), \( c = 1, 2 \). In the following, we consider the two simple hypotheses \( H_{01} : \mu_1 = \cdots = \mu_k \), and \( H_{02} : \mu_{k+1} = \cdots = \mu_p \).

An exact test statistic for \( H_{01} \) is given by
\[
F_{01} = \frac{n^{(1)} \sum_{i=1}^{k} (\tilde{x}_i - \bar{x}_i^{(1)})^2}{(k - 1) \hat{\sigma}_c^2},
\]
which has an \( F \) distribution with \( k - 1 \) and \( f \) degrees of freedom under hypothesis \( H_{01} \). Here, \( f = f_1 + f_2, \hat{\sigma}_c^2 = (s_1 + s_2)/f \).

An exact test statistic for \( H_{02} \) is given by
\[
F_{02} = \frac{n^{(2)} \sum_{i=k+1}^{p} (\tilde{x}_i - \bar{x}_i^{(2)})^2}{(p - k - 1) \hat{\sigma}_c^2},
\]
which has an \( F \) distribution with \( p - k - 1 \) and \( f \) degrees of freedom under hypothesis \( H_{02} \).
Furthermore, we can obtain an exact test statistic for simultaneously testing $H_{01}$ and $H_{02}$, which is

$$
F_0 = \frac{n^{(1)} \sum_{i=1}^{k} (\bar{x}_{i} - \bar{x}^{(1)}_{i})^2 + n^{(2)} \sum_{i=k+1}^{p} (\bar{x}_{i} - \bar{x}^{(2)}_{i})^2}{(p - 2)\hat{\sigma}^2}.
$$

(3.9)

Clearly, $F_0$ has an $F$ distribution with $p - 2$ and $f$ degrees of freedom under $H_0 = H_{01} + H_{02}$.

According to Theorem 3.1, we can construct exact confidence intervals of $a' \mathbf{u}^{(1)}$ and $b' \mathbf{u}^{(2)}$ for any non-null vectors $a$ and $b$ such that $a' \mathbf{1}_k = 0, b' \mathbf{1}_{p-k} = 0$, which are given by

(i) Exact confidence intervals for $a' \mathbf{u}^{(1)}$ and $b' \mathbf{u}^{(2)}$

$$
a' \mathbf{u}^{(1)} \in \left[a' \bar{x}^{(1)} + \hat{\sigma} \sqrt{\frac{k}{f}} t_{f, \frac{\alpha}{2}} \sqrt{a'a/n^{(1)}} \right],
$$

$$
b' \mathbf{u}^{(2)} \in \left[b' \bar{x}^{(2)} + \hat{\sigma} \sqrt{\frac{k}{f}} t_{f, \frac{\alpha}{2}} \sqrt{b'b/n^{(2)}} \right],
$$

(3.10)

respectively, where $t_{f, \frac{\alpha}{2}}$ is the upper $100\alpha/2\%$ of a $t$ distribution with $f$ degree of freedom.

(ii) Scheffé type of simultaneous confidence intervals for all $a' \mathbf{u}^{(1)}$ and all $b' \mathbf{u}^{(2)}$

$$
a' \mathbf{u}^{(1)} \in \left[a' \bar{x}^{(1)} + \hat{\sigma} \sqrt{\frac{(k - 1)f_{k-1, f, \alpha}}{n^{(1)}}} \right],
$$

$$
b' \mathbf{u}^{(2)} \in \left[b' \bar{x}^{(2)} + \hat{\sigma} \sqrt{\frac{(p - k - 1)f_{p-k-1, f, \alpha}}{n^{(2)}}} \right],
$$

(3.11)

respectively, where $F_{f, \frac{\alpha}{2}}$ is the upper $100\alpha\%$ of an $F$ distribution with $p^{(c)}$ and $f$ degrees of freedom.

Clearly, we can obtain Bonferroni type of simultaneous confidence intervals for $l$ linear contrasts $a'_1 \mathbf{u}^{(1)}, \ldots, a'_l \mathbf{u}^{(1)}$ and $b'_1 \mathbf{u}^{(2)}, \ldots, b'_l \mathbf{u}^{(2)}$ by replacing $t_{f, \frac{\alpha}{2}}$ in (3.10) with $t_{f, \frac{\alpha}{2}}$.

Remark 3.1. Based on between estimator $\mathbf{x} = (\bar{x}_1, \bar{x}_2')$, we can not give exact test statistics and exact confidence intervals for any contrast $\mu_i - \mu_{k+1}$ when $0 < i < k$ and $0 < l < p - k$.

The method above can be generalized to the case: $s > 2$. According to Remark 3.1, the larger $s$ is, the less $w$ is, where $w$ is the number of contrasts $\{\mu_i - \mu_j\}$ that we can make statistical inference based on between estimator $\mathbf{x}$. However, if contrast $\mu_i - \mu_j$ of interest belongs to one subset of complete data, we can adopt the methods given in this section.

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