EXPOSITIONES MATHEMATICAE

Expo. Math. 21 (2003): 179–183

www.urbanfischer.de/journals/expomath

Mathematical Note

Hippocrates' Lunes and Transcendence

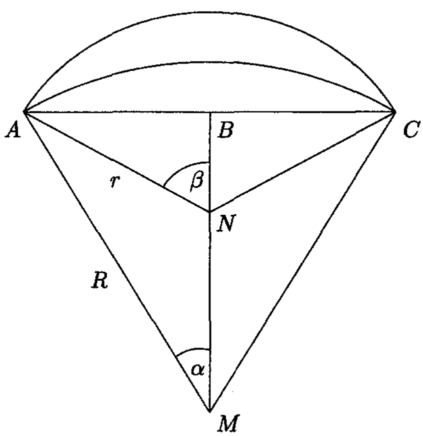
Kurt Girstmair

Institut für Mathematik, Universität Innsbruck, Technikerstr. 25/7, A-6020 Innsbruck, Austria

Abstract: Hippocrates gave the first examples of circular lunes that can be squared by means of ruler and compass only. By its nature, the problem of classifying all these lunes consists of two parts. The algebraic part was solved in 1947, whereas the transcendental part was neglected by most authors, with the effect that its final solution in 1966 remained unnoticed. The aim of this note is to fill this gap.

1. Introduction

Hippocrates of Chios lived in the second half of the fifth century B. C. His famous "lunes" are the first known examples of areas bounded by curved lines that can both be constructed and squared by means of ruler and compass only.



In a more modern terminology the *problem of squareable lunes* can be described as follows: The lune in the diagram is bounded by two circular arcs over the same chord AC of length $AC = 2AB$. Without loss of generality we may assume that AB has the given length $AB = 1$. The centers of the corresponding circles are M and N, their radii being $MA = R$

E-mail address: Kurt.Girstmair@uibk.ac.at

0732-0869/03/21/02-179 $15.00/0

and $NA = r$. By α and β we denote the angles AMB and ANB, respectively.

The area L of the lune is the difference of the two circular segments

$$r^2\beta - \Delta(ANC), \ \ R^2\alpha - \Delta(AMC).$$

Here the triangular areas $\Delta(ANC)$, $\Delta(AMC)$ take the form

$$\Delta(ANC) = \sqrt{r^2-1}, \ \ \Delta(AMC) = \sqrt{R^2-1},$$

by the theorem of Pythagoras.

Now the lune is called *algebraically squareable* or simply *algebraic*, if R, r and the area

$$L = r^2\beta - R^2\alpha + \sqrt{R^2-1} - \sqrt{r^2-1} \tag{1}$$

lie in the field $\overline{\mathbb{Q}}$ of (complex) algebraic numbers. Further, it is called *elementarily squareable* or simply *elementary*, if R, r and $\sqrt{L}$ can be constructed from the unit length $AB = 1$ by means of ruler and compass only — in other words, if one can construct both the lune and and a square with the same area in the sense of a construction from Euclid's *Elements*. In algebraic terms this means that each of R, r and L lies in a field $F \subseteq \overline{\mathbb{Q}}$ such that

$$\mathbb{Q} \subseteq F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n = F$$

is a tower of quadratic extensions (cf. [8], p. 210 ff.). From (1) it is clear that the number L can be replaced by the simpler number

$$c = r^2\beta - R^2\alpha \tag{2}$$

in both definitions. Obviously, every elementary lune is algebraic.

Formula (1) corresponds to the situation $\beta < \pi/2$. In the case $\beta > \pi/2$ this formula looks slightly different but the decisive quantity c may be defined in the same way.

Hippocrates detected three different elementary lunes, cf. [6], p. 183 ff., cf. also [11]. They are (uniquely) determined by the ratio $\beta : \alpha$, which is $2:1$, $3:1$, and $3:2$ in these cases. In the eighteenth century two additional elementary lunes were discovered with ratios $5:1$ and $5:3$. T. Heath's classical works of 1921 and 1931 leave the impression that there are no other elementary lunes besides these five: *"As it is, Hippocrates' achievement is remarkable, for he discovered and proved the possibility of quadrature in the case of three out of five species of lunes which alone can be squared by 'plane' methods"* ([7], p. 131; cf. also [6], p. 200).

One of the aims of this note consists in showing that this kind of optimism was not justified at Heath's time and even much later (Heath died in 1940). According to [11], the "final solution" of the task of finding all elementary lunes was given in 1947. However, in this solution an important link was still missing. This link was published in 1966 — but it seems that nobody noticed its importance for the problem of squareable lunes so far.

2. The algebraic part of the problem

Suppose, for the time being, that we are looking for *elementary* lunes only. For this purpose we assume that the ratio $\theta = \beta/\alpha$ of the angles is *rational*, so

$$\theta \in \mathbb{Q}. \tag{3}$$

All authors dealing with this problem had to make this strong assumption, together with another one, namely

$$c = 0, \tag{4}$$

with c as in (2). By (3), we may write $\beta = m\gamma$, $\alpha = n\gamma$ with positive integers $m > n$ and $\gamma > 0$. Condition (4) implies

$$R^2/r^2 = \beta/\alpha = m/n,$$

and because of

$$R/r = \sin\beta/\sin\alpha = \sin m\gamma/\sin n\gamma$$

we arrive at the equation

$$\left(\frac{\sin m\gamma}{\sin n\gamma}\right)^2 = \frac{m}{n}, \tag{5}$$

which can be rewritten as

$$n\,(\sin m\gamma)^2 = m\,(\sin n\gamma)^2. \tag{6}$$

However, both sides of (6) are polynomials (with integer coefficients) in $x = \sin\gamma$ of respective degrees $2m$, $2n$. Hence (6) is, essentially, equivalent to an algebraic equation

$$f(x) = 0$$

of degree $2m$ over $\mathbb{Q}$ (in fact, the degree becomes $2m - 2$ if one takes the trivial solution $x = 0$ into account). So our problem is of a purely algebraic nature now: We have to figure out those pairs (m, n) for which x lies in a field $F \subseteq \overline{\mathbb{Q}}$ which is the top of a tower of quadratic extensions (cf. Section 1). In terms of Galois theory this means that we have to determine all pairs for which the order of the Galois group G of $f(x) = 0$ is a power of 2. It is not hard to check that this is the case for the five pairs (m, n) of Section 1. Mathematicians engaged in this problem believed that there are *no further* pairs giving rise to elementary lunes. Accordingly, their general strategy in the remaining cases can be described as follows: Look for sufficiently much information about G in order to exclude $|G| = 2^k$ (k a nonnegative integer). As far as we can judge, this information was obtained in several steps by Landau (1902/03), Chakalov (1929), Chebotarëv (1933/34) and Dorodnov (1947), cf. [9], [12], [3], [13], [5].

3. The transcendental part of the problem

It is clear that the strong assumptions (3) and (4) of the foregoing section have not been justified so far — so there might exist elementary lunes quite different from those described by equation (5). This justification requires transcendence results and will be given now. We also highlight its history or, strictly speaking, *what might have been* its history.

In what follows it suffices to assume that the lune is *algebraic*, so R, r and c are in $\overline{\mathbb{Q}}$. E. Landau (cf. [9]) made the first successful attempt to get rid of the said assumptions, inasmuch as he showed that (3) implies (4). He even showed more: Suppose that the ratio $\theta = \beta/\gamma$ is *algebraic* (so it need not be rational) and that $c \neq 0$. An easy calculation gives

$$\alpha = \frac{c}{R^2 - r^2\theta},$$

which is an algebraic number $\neq 0$. But then $\sin\alpha = 1/R$ is transcendental, by the theorem of Hermite and Lindemann, cf. [2], p. 259. Hence the lune is *not* algebraic, a contradiction. In other words, one can dispense with (4) even if (3) is replaced by the weaker assumption

$$\theta = \beta/\alpha \in \overline{\mathbb{Q}}. \tag{7}$$

The next important contributor to the solution of the algebraic part of the problem was the Bulgarian mathematician L. Chakalov. He was well aware of the fact that a stronger transcendence result than Hermite-Lindemann is needed in order to justify (3). In fact, he writes in the summary of [3]: *"The problem of squareable lunes presumably belongs to those transcendental problems whose most general solution must be considered as inaccessible by the present state of knowledge. Therefore I restrict myself, as all previous authors ... did, to the case where the central angles are commensurable ... "* (translated from the German).

Some years later N. Chebotarëv introduced the main technique for the solution of the algebraic part of the problem (in the sense of Section 2). But he did not care about the justification of condition (3), and so he wrote: *"Herr E. Landau reduced the problem of squareable lunes to the solution of the equation* $(\sin m\theta/\sin n\theta)^2 = m/n$*, where* m *and* n *are relatively prime integers"* ([13], translated from the German). Of course, the equation in question is the same as (5). But Landau had *not* pretended that it would be sufficient to study this equation. It seems that Chebotarëv's restriction to the algebraic part was accepted as the *state of the art* by all later writers on the subject (cf. [5], [11], [4], [10]).

It is remarkable that in the year 1934, when Chebotarëv's paper [13] appeared, Gelfond and Schneider published a result that can be considered as a further important step towards the final justification of condition (3) (and thus, (4)): Let $\eta \in \mathbb{C} \smallsetminus \{0,1\}$ and $\theta \in \mathbb{C} \smallsetminus \mathbb{Q}$. Then the Gelfond-Schneider theorem says that one of η, θ and $\eta^\theta = e^{\theta \log \eta}$ must be transcendental (cf. [2], p. 272; here log is an arbitrary value of the complex logarithm). In our setting, $\sin\alpha = 1/R$ is in $\overline{\mathbb{Q}}$, and so is $\eta = e^{i\alpha}$. Further, $0 < \alpha < \pi$, so $\eta \notin \{0,1\}$. Suppose $\theta = \beta/\alpha$ is algebraic but irrational. Then

$$e^{\theta \log \eta} = e^{\theta \cdot i\alpha} = e^{i\beta}$$

is transcendental, which contradicts $\sin\beta = 1/r \in \overline{\mathbb{Q}}$. In other words, Landau's weaker assumption (7) actually implies (3) (and not only (4)).

It seems that this application of the Gelfond-Schneider theorem remained unnoticed, not only by Chakalov or Chebotarëv but even up to now. The same holds, *mutatis mutandis*, for a fundamental theorem of Baker (cf. [1], [2], p. 280) published in 1966. It yields the

Theorem. *If the lune is algebraically squareable, then the ratio* $\theta = \beta/\alpha$ *is rational and* $c = 0$. *Accordingly, equation* (5) *actually describes all algebraic lunes.*

Proof. If θ is in $\mathbb{Q}$, then $c = 0$ by Landau's above argument. Suppose, therefore, that θ is irrational. Then the numbers $i\alpha$, $i\beta$ are $\mathbb{Q}$-linearly independent. By assumption, $e^{i\alpha}$, $e^{i\beta}$ are algebraic. Baker's theorem says that 1, $i\alpha$, and $i\beta$ are $\overline{\mathbb{Q}}$-linearly independent, hence equation (2) can hold only with $c = R = r = 0$, which is impossible.

Remark. One may also consider "circular lenses" which are bounded by two circular arcs over the same chord, the arcs, however, lying on *different* sides of the chord. Here c equals

$r^2\beta + R^2\alpha$, which is always > 0. In [9] Landau observed that under his assumption, i.e., $\beta/\alpha \in \overline{\mathbb{Q}}$, such a lens is never algebraically squareable. Baker's theorem shows that this is true without Landau's assumption.

Acknowledgement: The author should like to thank Gerhard Auer for his invaluable help in retrieving "exotic" literature.

References

[1] A. Baker, *Linear forms in logarithms of algebraic numbers,* Mathematika 13 (1966), 204–216

[2] P. Bundschuh, *Einführung in die Zahlentheorie* (4th ed.), Springer, Berlin 1998

[3] L. Chakalov, *Vtori prinosă kămă problemata za kvadriruemitě lunichki,* Spis. Bălgarsk. Akad. Nauk. 41 (1929), 1–47; German summary, 48–53

[4] R. Church, *Review of* [5], Math. Reviews 9, 264 b

[5] A. V. Dorodnov, *O krugovykh lunochkakh kvadriruemykh pri pomoshchi tsirkulya i lineĭki,* Dokl. Akad. Nauk SSSR 58 (1947), 965–968

[6] T. L. Heath, *A History of Greek Mathematics,* vol. 1, Clarendon Press, Oxford 1921 (repr. Dover, New York 1981)

[7] T. L. Heath, *A Manual of Greek Mathematics,* Oxford University Press, Oxford 1931 (repr. Dover, New York 1963)

[8] N. Jacobson, *Basic Algebra I,* Freeman, San Francisco 1974

[9] E. Landau, *Über quadrierbare Kreisbogenzweiecke,* Sitzungsber. Berlin. Math. Ges. 1902, in: Archiv Math. Phys. (3rd ser.) 4 (1903), append., 1–6

[10] M. M. Postnikov, *The problem of squarable lunes,* Amer. Math. Monthly 107 (2000), 645–651

[11] C. J. Scriba, *Welche Kreismonde sind elementar quadrierbar? Die 2400jährige Geschichte eines Problems bis zur endgültigen Lösung 1933/1947,* Mitt. Math. Ges. Hamburg 11 (1988), 517–539

[12] L. Tschakaloff [= L. Chakalov], *Beitrag zum Problem der quadrierbaren Kreisbogenzweiecke,* Math. Z. 30 (1929), 552–559

[13] N. Tschebotaröw [= N. Chebotarëv], *Über quadrierbare Kreisbogenzweiecke I,* Math. Z. 39 (1934), 161–175

Received: 21. 10. 2002
Revised: 16. 12. 2002