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A new approach to fibrewise fibrations and cofibrations

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Abstract

We study the fibrewise (pointed) homotopy, fibrewise (pointed) fibration and fibrewise (pointed) cofibration in the category **MAP**. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

For a base space B, the category $\mathbf{TOP}_B$ is the fibrewise topology over B. For general topology of continuous maps or fibrewise general topology, see Pasynkov [6]. In $\mathbf{TOP}_B$, the fibrewise homotopy was studied by many mathematicians; for this, see [3,5]. In [1,2], Buhagiar studied fibrewise topology in the category of all continuous maps, called **MAP** by him (as a way of thinking of a category, **MAP** can be seen in earlier works, see, for example, [7]). The study of fibrewise topology in **MAP** is a generalization of it in the category $\mathbf{TOP}_B$. In this study, we clarify that in treating fibrewise homotopy and fibrewise pointed homotopy, we can freely consider $I \times B$ as base spaces, and therefore need not consider some complicated procedures ($(I \times B) \times_B X = I \times X$ and the reduced fibrewise cylinder $I \tilde{\times} X$ for constructing sections).

The objects of **MAP** are continuous maps from any topological space into any topological space. For two objects $p_1 : X_1 \to B_1$ and $p_2 : X_2 \to B_2$, a morphism from

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p_1 into p_2 is a pair (ϕ, α) of continuous maps $\phi : X_1 \to X_2, \alpha : B_1 \to B_2$ such that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\phi} & X_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p^2} \\ B_1 & \xrightarrow{\alpha} & B_2 \end{array}$$

is commutative. We note that this situation is a generalization of the category $\mathbf{TOP}_B$ since the category $\mathbf{TOP}_B$ is isomorphic to the particular case of **MAP** in which the spaces $B_1 = B_2 = B$ and $\alpha = \mathrm{id}_B$. We call an object $p : X \to B$ an **M**-*fibrewise space* and denote (X, p, B). Also, for two **M**-fibrewise spaces (X_1, p_1, B_1), (X_2, p_2, B_2), we call the morphism (ϕ, α) from p_1 into p_2 an **M**-*fibrewise map,* and denote $(\phi, \alpha) : (X_1, p_1, B_1) \to (X_2, p_2, B_2)$.

In this paper, we assume that all spaces are topological spaces, and all maps are continuous. The space $I = [0, 1]$ and id is the identity map of I into I. Moreover, we use the following notation: For any $t \in I$, the maps $\sigma_t : X \to I \times X$ and $\delta_t : B \to I \times B$ are defined by

$$\sigma_t(x) = (t, x), \quad \delta_t(b) = (t, b) \quad (x \in X,\ b \in B).$$

For other undefined terminology, see [4,5].

2. M-fibrewise homotopy

In this section, we shall define an **M**-fibrewise homotopy, which is an extended version of fibrewise homotopy [5, §18].

Definition 2.1. Let $(\phi, \alpha), (\theta, \beta) : (X_1, p_1, B_1) \to (X_2, p_2, B_2)$ be **M**-fibrewise maps. The **M**-*fibrewise homotopy of* (ϕ, α) *into* (θ, β) is an **M**-fibrewise map $(H, h) : (I \times X_1, \mathrm{id} \times p_1, I \times B_1) \to (X_2, p_2, B_2)$ such that $H\sigma_0 = \phi,\ H\sigma_1 = \theta,\ h\delta_0 = \alpha,\ h\delta_1 = \beta$.

If there exists an **M**-fibrewise homotopy of (ϕ, α) into (θ, β), we say (ϕ, α) is **M**-fibrewise homotopic to (θ, β) and denote $(\phi, \alpha) \simeq^{\mathbf{M}} (\theta, \beta)$.

Remark. For fibrewise maps $\phi, \theta : X \to Y$ over B, the fibrewise homotopy $f : I \times X \to Y$ is of course coincident with the **M**-fibrewise homotopy $(H, h) : (I \times X, \mathrm{id} \times p, I \times B) \to (Y, q, B)$ such that $H(t, x) = f(t, x)$ and $h(t, b) = b$. Therefore the concept of **M**-fibrewise homotopy is an extension one of the fibrewise homotopy.

Lemma 2.1. *The relation* $\simeq^{\mathbf{M}}$ *is an equivalence relation.*

The proof can be easy to see, and so is ommited.

Definition 2.2. An **M**-fibrewise map $(\theta, \alpha) : (X_1, p_1, B_1) \to (X_2, p_2, B_2)$ is called an **M**-*fibrewise homotopy equivalence* if there exists an **M**-fibrewise map $(\phi, \beta) : (X_2, p_2, B_2) \to$

(X_1, p_1, B_1) such that $(\phi\theta, \beta\alpha) \simeq^{\mathbf{M}} (\mathrm{id}_{X_1}, \mathrm{id}_{B_1})$, $(\theta\phi, \alpha\beta) \simeq^{\mathbf{M}} (\mathrm{id}_{X_2}, \mathrm{id}_{B_2})$. We call (ϕ, β) the **M**-fibrewise homotopy inverse of (θ, α).

If there exists an **M**-fibrewise homotopy equivalence $(\theta, \alpha) : (X_1, p_1, B_1) \to (X_2, p_2, B_2)$, we denote $(X_1, p_1, B_1) \cong^{\mathbf{M}} (X_2, p_2, B_2)$.

Lemma 2.2. *The relation $\cong^{\mathbf{M}}$ is an equivalence relation.*

The proof can be easily verified, and so is ommited.

Definition 2.3. Let (X, p, B) be an **M**-fibrewise space. If $A \subset X$, $B_0 \subset B$ and $p(A) \subset B_0$, we call $(A, p|A, B_0)$ an **M**-*fibrewise subspace of* (X, p, B). We sometimes use the notation (A, p_0, B_0) instead of $(A, p|A, B_0)$.

Definition 2.4. Let $(A, p_1|A, B_0)$ be an **M**-fibrewise subspace of (X_1, p_1, B_1), let $(\phi, \alpha), (\theta, \beta) : (X_1, p_1, B_1) \to (X_2, p_2, B_2)$ an **M**-fibrewise maps such that $\phi(x) = \theta(x)$ and $\alpha(b) = \beta(b)$ for $x \in A$ and $b \in B_0$. By an **M**-*fibrewise homotopy of* (ϕ, α) *into* (θ, β) *under* $(A, p_1|A, B_0)$ we mean an **M**-fibrewise homotopy (H, h) of (ϕ, α) into (θ, β) such that for fixed $x \in A$, $b \in B_0$, $H(t, x)$ and $h(t, b)$ are constant for any $t \in I$.

Definition 2.5. Let (X, p, B) an **M**-fibrewise space. An **M**-fibrewise subspace $(A, p_1|A, B_0)$ is an **M**-*fibrewise retract of* (X, p, B) if there exists an **M**-fibrewise map $(R, r) : (X, p, B) \to (A, p_1|A, B_0)$ such that $R(x) = x$ and $r(b) = b$ for any $x \in A$, $b \in B_0$. We call (R, r) an **M**-*fibrewise retraction.*

Definition 2.6. Let (X, p, B) be an **M**-fibrewise space. An **M**-fibrewise subspace $(A, p_1|A, B_0)$ is an **M**-*fibrewise deformation retract* of (X, p, B) if there exists an **M**-fibrewise homotopy $(H, h) : (I \times X, \mathrm{id} \times p, I \times B) \to (X, p, B)$ of $(\mathrm{id}_X, \mathrm{id}_B)$ into an **M**-fibrewise retraction (R, r) under $(A, p_1|A, B_0)$, where $(R, r) : (X, p, B) \to (A, p_1|A, B_0)$.

Theorem 2.3. *Let (X, p, B) be an **M**-fibrewise space and $(A, p_1|A, B_0)$ an **M**-fibrewise subspace of (X, p, B). If $(\{0\} \times X \cup I \times A, \mathrm{id} \times p, \{0\} \times B \cup I \times B_0)$ is an **M**-fibrewise retract of (X, p, B), then $(\{0\} \times X \cup I \times A, \mathrm{id} \times p, \{0\} \times B \cup I \times B_0)$ is an **M**-fibrewise deformation retract of $(I \times X, \mathrm{id} \times p, I \times B)$.*

Proof. Let $(R, r) : (I \times X, \mathrm{id} \times p, I \times B) \to (\{0\} \times X \cup I \times A, \mathrm{id} \times p, \{0\} \times B \cup I \times B_0)$ be an **M**-fibrewise retraction. We put

$$R(t, x) = \big(R_1(t, x), R_2(t, x)\big), \qquad r(t, b) = \big(r_1(t, b), r_2(t, b)\big).$$

We define $(H, h) : (I \times I \times X, \mathrm{id} \times \mathrm{id} \times p, I \times I \times B) \to (I \times X, \mathrm{id} \times p, I \times B)$ by

$$H(s, t, x) = \big((1 - s)t + sR_1(t, x), R_2(st, x)\big)$$

$$h(s, t, b) = \big((1 - s)t + sr_1(t, b), r_2(st, b)\big).$$

Then it is easy to see that (H,h) is an **M**-fibrewise homotopy of $(\mathrm{id}_X, \mathrm{id}_B)$ into (R,r) under $(\{0\}\times X \cup I\times A, \mathrm{id}\times p, \{0\}\times B\cup I\times B_0)$. □

3. M-fibrewise cofibrations

In this section, we consider an extended version of fibrewise cofibrations, and obtain some generalized theorems of fibrewise version [5, §20]. We begin with the following definition.

Definition 3.1. An **M**-fibrewise map $(u,\gamma):(A, p_0, B_0)\to(X_1, p_1, B_1)$ is an **M**-*fibrewise cofibration* if (u,γ) has the following **M**-fibrewise extension property. Let $(\phi,\alpha):(X_1, p_1, B_1)\to(X_2, p_2, B_2)$ be an **M**-fibrewise map and $(G,g):(I\times A, \mathrm{id}\times\gamma, I\times B_0)\to(X_2, p_2, B_2)$ an **M**-fibrewise homotopy such that diagrams

$$\begin{array}{ccc} A & \xrightarrow{\sigma_0} & I\times A \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle G} \\ X_1 & \xrightarrow{\phi} & X_2 \end{array}$$

$$\begin{array}{ccc} B_0 & \xrightarrow{\delta_0} & I\times B_0 \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle g} \\ B_1 & \xrightarrow{\alpha} & B_2 \end{array}$$

are commutative. Then there exists an **M**-fibrewise homotopy $(H,h):(I\times X_1, \mathrm{id}\times p_1, I\times B_1)\to(X_2, p_2, B_2)$ such that $H\kappa_0=\phi$, $H(\mathrm{id}\times u)=G$, $h\rho_0=\alpha$, $h(\mathrm{id}\times\gamma)=g$, where maps $\kappa_0:X_1\to I\times X_1$ and $\rho_0:B_1\to I\times B_1$ are defined by $\kappa_0(x)=(0,x)$ and $\rho_0(b)=(0,b)$ for any $x\in X_1, b\in B_1$.

For an **M**-fibrewise map $(u,\gamma):(A, p_0, B_0)\to(X_1, p_1, B_1)$, we can construct the **M**-fibrewise push out (M, p, B) of the cotraids

$$I\times A \xleftarrow{\sigma_0} A \xrightarrow{u} X_1$$
$$I\times B_0 \xleftarrow{\delta_0} B_0 \xrightarrow{\gamma} B_1$$

as follows: $M=(I\times A+X_1)/{\sim}$ and $B=(I\times B_0+B_1)/{\approx}$, where $(0,a)\sim u(a)$ for $a\in A$ and $(0,b)\approx\gamma(b)$ for $b\in B_0$, and $p:M\to B$ is defined by

$$p(x)=\begin{cases}[\gamma p_0(a)] & \text{if } x=[u(a)], a\in A,\\ [t, p_0(a)] & \text{if } x=[t,a], t\neq 0,\\ [p_1(x)] & \text{if } x\in X_1-u(A),\end{cases}$$

where $[*]$ is an equivalence class. Then it is easily verified that p is well defined and continuous.

Now we shall consider the case in which (A, p_0, B_0) is an **M**-fibrewise subspace of (X_1, p_1, B_1) with $p_0=p_1|A$ and $(u,\gamma):(A, p_0, B_0)\to(X_1, p_1, B_1)$ is the inclusion. We can define an **M**-fibrewise map $(e,\varepsilon):(M,p,B)\to(\{0\}\times X_1\cup I\times A,\mathrm{id}\times p_1,\{0\}\times B_1\cup I\times B_0)$ by

$$e(x)=\begin{cases}(0,a) & \text{if } x=[u(a)],\ a\in A,\\ (t,a) & \text{if } x=[t,a],\ t\neq 0,\\ (0,x) & \text{if } x\in X_1-u(A),\end{cases}$$

$\varepsilon(b)$ is defined by a similar way. Moreover if A is closed in X_1 and B_0 is closed in B_1, the maps e and ε are homeomorphisms and we may identity (M,p,B) with $(\{0\}\times X_1\cup I\times A,\mathrm{id}\times p_1,\{0\}\times B_1\cup I\times B_0)$. For each **M**-fibrewise map $(u,\gamma):(A,p_0,B_0)\to(X_1,p_1,B_1)$, we can define an **M**-fibrewise map $(k,\xi):(M,p,B)\to(I\times X_1,\mathrm{id}\times p_1,I\times B_1)$ by

$$k(x)=\begin{cases}(0,u(a)) & \text{if } x=[u(a)],\ a\in A,\\ (t,u(a)) & \text{if } x=[t,a],\ t\neq 0,\\ (0,x) & \text{if } x\in X_1-u(A),\end{cases}$$

$$\xi(b)=\begin{cases}(0,\gamma(b')) & \text{if } b=[\gamma(b')],\ b'\in B_0,\\ (t,\gamma(b')) & \text{if } x=[t,b'],\ t\neq 0,\\ (0,x) & \text{if } x\in B_1-\gamma(B_0).\end{cases}$$

Now we can obtain the following.

Theorem 3.1. *The* **M**-*fibrewise map* $(u,\gamma):(A,p_0,B_0)\to(X_1,p_1,B_1)$ *is an* **M**-*fibrewise cofibration if and only if there exists an* **M**-*fibrewise map* $(L,l):(I\times X_1,\mathrm{id}\times p_1,I\times B_1)\to(M,p,B)$ *such that* $Lk=\mathrm{id}_M, l\xi=\mathrm{id}_B$.

Proof. "If" part: Suppose that there exists an **M**-fibrewise map (L,l) satisfying the condition. Let $(\phi,\alpha):(X_1,p_1,B_1)\to(X_2,p_2,B_2)$ and $(G,g):(I\times A,\mathrm{id}\times p_0,I\times B_0)\to(X_2,p_2,B_2)$ be **M**-fibrewise maps such that those satisfy the condition of Definition 3.1. We can define an **M**-fibrewise map $(\bar{H},\bar{h}):(M,p,B)\to(X_2,p_2,X_2)$ by

$$\bar{H}(z)=\begin{cases}\phi(u(a)) & \text{if } z=[u(a)],\ a\in A,\\ \phi(x) & \text{if } z=[x],\ x\in X_1-u(A),\\ G(t,a) & \text{if } z=[t,a],\ a\in A,\ t\neq 0,\end{cases}$$

$$\bar{h}(b)=\begin{cases}\alpha(\gamma(b')) & \text{if } b=[\gamma(b')],\ b'\in B_0,\\ \alpha(b') & \text{if } b=[b'],\ b'\in B_1-\gamma(B_0),\\ g(t,b') & \text{if } b=[t,b'],\ b'\in B_0,\ t\neq 0.\end{cases}$$

We consider $H:I\times X_1\to X_2$ and $h:I\times B_1\to B_2$ such that $H=\bar{H}L, h=\bar{h}l$. Then it is easy to see that the diagram

$$\begin{array}{ccc} I\times X_1 & \xrightarrow{H} & X_2\\ {\scriptstyle \mathrm{id}\times p_1}\downarrow & & \downarrow{\scriptstyle p_2}\\ I\times B_1 & \xrightarrow{h} & B_2\end{array}$$

is commutative. Further we can show that $H(\mathrm{id}\times u)=G, H\kappa_0=\phi, h(\mathrm{id}\times\gamma)=g$ and $h\rho_0=\alpha$. In fact, for any $(t,a)\in I\times A$, $H(\mathrm{id}\times u)(t,a)=H(t,u(a))=\bar{H}L(t,u(a))=$

$\bar{H}Lk([t,a]) = \bar{H}([t,a]) = G(t,a)$, and for any $x \in X_1$, $H\kappa_0(x) = \bar{H}L(0,x) = \bar{H}([x]) = \phi(x)$. Also $h(\mathrm{id} \times \gamma) = g$, and $h\rho_0 = \alpha$ can be shown similarly.

"Only if" part: Suppose that (u,γ) is an **M**-fibrewise cofibration. Let $(G,g):(I \times A, \mathrm{id} \times p_0, I \times B_0) \to (M,p,B)$ and $(\phi,\alpha):(X_1,p_1,B_1) \to (M,p,B)$ be two **M**-fibrewise maps defined by

$$G(t,a) = \begin{cases} [t,a] & \text{if } t \neq 0, \\ [u(a)] & \text{if } t = 0, \end{cases}$$

$$g(t,b) = \begin{cases} [t,b] & \text{if } t \neq 0, \\ [\gamma(b)] & \text{if } t = 0. \end{cases}$$

Then the following two diagrams

$$\begin{array}{ccc} A & \xrightarrow{\sigma_0} & I \times A \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle G} \\ X_1 & \xrightarrow{\phi} & M \end{array}$$

$$\begin{array}{ccc} B_0 & \xrightarrow{\delta_0} & I \times B_0 \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle g} \\ B_1 & \xrightarrow{\alpha} & B \end{array}$$

are commutative. Since (u,γ) is an **M**-fibrewise cofibration, there exists an **M**-fibrewise homotopy $(H,h):(I \times X_1, \mathrm{id} \times p_1, I \times B_1) \to (M,p,B)$ such that $H(\mathrm{id} \times u) = G$, $H\kappa_0 = \phi$, $h(\mathrm{id} \times \gamma) = g$ and $h\rho_0 = \alpha$. Then we can show that $Hk = \mathrm{id}_M$, $h\xi = \mathrm{id}_B$. In fact, for the case $[x] \in M,\ x \in X_1$,

$$Hk\big([x]\big) = H(0,x) = H\kappa_0(x) = \phi(x) = [x]$$

and the case $[t,a] \in M,\ t \neq 0,\ a \in A$,

$$Hk\big([t,a]\big) = H\big(t,u(a)\big) = H(\mathrm{id} \times u)(t,a) = G(t,a) = [t,a].$$

Thus $Hk = \mathrm{id}_M$. Also we can show similarly that $h\xi = \mathrm{id}_B$ $\square$

For two **M**-fibrewise spaces (X_1,p_1,B_1) and (A,p_0,B_0), if $A \subset X_1$, $B_0 \subset B_1$ and $p_0 = p_1|A$, the pair $((X_1,p_1,B_1),(A,p_0,B_0))$ is called by an **M**-*fibrewise pair.* If A is closed in X_0 and B_0 is closed in B_1, it is called a *closed* **M**-*fibrewise pair.* For an **M**-fibrewise pair $((X_1,p_1,B_1),(A,p_0,B_0))$, if the inclusion map $(u,\gamma):(A,p_0,B_0) \to (X_1,p_1,B_1)$ is an **M**-fibrewise cofibration, we call the pair $((X_1,p_1,B_1),(A,p_0,B_0))$ an **M**-*fibrewise cofibred pair.*

Theorem 3.2. *A closed* **M**-*fibrewise pair* $((X_1,p_1,B_1),(A,p_0,B_0))$ *is an* **M**-*fibrewise cofibred pair if and only if there exists an* **M**-*fibrewise retraction* $(R,r):(I \times X_1, \mathrm{id} \times p_1, I \times B_1) \to (\{0\} \times X_1 \cup I \times A, \mathrm{id} \times p_1, \{0\} \times B_1 \cup I \times B_0)$.

Proof. "Only if" part: Let the inclusion $(u,\gamma):(A,p_0,B_0) \to (X_1,p_1,B_1)$ be an **M**-fibrewise cofibration. From Theorem 3.1, there exists an **M**-fibrewise map $(L,l):(I \times$

$X_1, \mathrm{id} \times p_1, I \times B_1) \to (M, p, B)$ such that $Lk = \mathrm{id}_M,\ l\xi = \mathrm{id}_B$. Since $((X_1, p_1, B_1), (A, p_0, B_0))$ is a closed **M**-fibrewise pair, (M, p, B) is identified with $(\{0\} \times X_1 \cup I \times A, \mathrm{id} \times p_1, \{0\} \times B_1 \cup I \times B_0)$. So there exist homeomorphisms $g : M \to \{0\} \times X_1 \cup I \times A$ and $\mu : B \to \{0\} \times B_1 \cup I \times B_0$ such that (g, μ) is an **M**-fibrewise map. Let $(R, r) : (I \times X_1, \mathrm{id} \times p_1, I \times B_1) \to (\{0\} \times X_1 \cup I \times A, \mathrm{id} \times p_1, \{0\} \times B_1 \cup I \times B_0)$ be $R = gL,\ r = \mu l$. Then it is easily to verified that (R, r) is an **M**-fibrewise retraction.

"If" part: Let (R, r) be an **M**-fibrewise retraction. Using the same notation (g, μ) in the above, $(g^{-1}, \mu^{-1}) : (\{0\} \times X_1 \cup I \times X_1, \mathrm{id} \times p_1, \{0\} \times B_1 \cup I \times B_0) \to (M, p, B)$ is a (homeomorphic) **M**-fibrewise map. Let $L = g^{-1}R$ and $l = \mu^{-1}r$. Then it is easy to see that $(L, l) : (I \times X_1, \mathrm{id} \times p_1, I \times B_1) \to (M, p, B)$ satisfies the condition of Theorem 3.1. Therefore $((X_1, p_1, B_1), (A, p_0, B_0))$ is an **M**-fibrewise cofibred pair. □

Corollary 3.3. *Let $((X_1, p_1, B_1), (A, p_0, B_0))$ be a closed* **M**-*fibrewise cofibred pair. Then so is*

$$\big((T \times X_1, \mathrm{id}_T \times p_1, T \times B_1), (T \times A, \mathrm{id}_T \times p_0, T \times B_0)\big)$$

for any topological space T.

Proof. Since $((X_1, p_1, B_1), (A, p_0, B_0))$ is a closed **M**-fibrewise cofibred pair, there exists an **M**-fibrewise retraction $(R, r) : (I \times X_1, \mathrm{id} \times p_1, I \times B_1) \to (\{0\} \times X_1 \cup I \times A, \mathrm{id} \times p_1, \{0\} \times B_1 \cup I \times B_0)$. We define $(f, \alpha) : (I \times (T \times X_1), \mathrm{id} \times (\mathrm{id}_T \times p_1), I \times (T \times B_0)) \to (T \times (I \times X_1), \mathrm{id}_T \times (\mathrm{id} \times p_1), T \times (I \times B_1))$ by

$$f\big(t, (z, x)\big) = \big(z, (t, x)\big), \qquad \alpha\big(t, (z, b)\big) = \big(z, (t, b)\big).$$

Further we define $(g, \beta) : (T \times (\{0\} \times X_1 \cup I \times A), \mathrm{id}_T \times (\mathrm{id} \times p_1), T \times (\{0\} \times B_1 \cup I \times B_0)) \to (\{0\} \times (T \times X_1) \cup I \times (T \times A), \mathrm{id} \times (\mathrm{id}_T \times p_1), \{0\} \times (T \times B_1) \cup I \times (T \times B_0))$ by

$$g\big(z, (t, x)\big) = \big(t, (z, x)\big), \qquad \beta\big(z, (t, b)\big) = \big(t, (z, b)\big).$$

Then it is easy to see that (f, α) and (g, β) are **M**-fibrewise maps. We can define an **M**-fibrewise map $(\bar{R}, \bar{r}) : (I \times (T \times X_1), \mathrm{id} \times (\mathrm{id}_T \times p_1), I \times (T \times B_1)) \to (\{0\} \times (T \times X_1) \cup I \times (T \times A)), \mathrm{id} \times (\mathrm{id}_T \times p_1), (\{0\} \times (T \times B_1) \cup I \times (T \times B_0))$ by

$$\bar{R} = g(\mathrm{id}_T \times R) f, \qquad \bar{r} = \beta(\mathrm{id}_T \times r)\alpha.$$

Then it is easily to verified that $(\bar{R}, \bar{r})$ is an **M**-fibrewise retraction, therefore this completes the proof by Theorem 3.2. □

Theorem 3.4. *Let $(u_1, \gamma_1) : (A, p_0, B_0) \to (X_1, p_1, B_1),\ (u_2, \gamma_2) : (A, p_0, B_0) \to (X_2, p_2, B_2)$ be* **M**-*fibrewise maps and $(\phi, \alpha) : (X_1, p_1, B_1) \to (X_2, p_2, B_2)$ an* **M**-*fibrewise map such that $(\phi u_1, \alpha\gamma_1) \simeq^{\mathbf{M}} (u_2, \gamma_2)$. If (u_1, γ_1) is an* **M**-*fibrewise cofibration, then there exists an* **M**-*fibrewise map $(\psi, \beta) : (X_1, p_1, B_1) \to (X_2, p_2, B_2)$ such that $(\phi, \alpha) \simeq^{\mathbf{M}} (\psi, \beta)$ and $\psi u_1 = u_2,\ \beta\gamma_1 = \gamma_2$.*

Proof. Let $(H, h) : (I \times A, \mathrm{id} \times p_0, I \times B_0) \to (X_2, p_2, B_2)$ be an **M**-fibrewise homotopy such that $H(0, a) = \phi u_1(a), H(1, a) = u_2(1, a), h(0, b) = \alpha\gamma_1(0, b), h(1, b) = \gamma_2(b)$ for

$a \in A$ and $b \in B_0$. Since (u_1, γ_1) is an **M**-fibrewise cofibration, there exists an **M**-fibrewise homotopy $(K, k) : (I \times X_1, \mathrm{id} \times p_1, I \times B_1) \to (X_2, p_2, B_2)$ such that $K(\mathrm{id} \times u_1) = H$, $K\kappa_0 = \phi$, $k(\mathrm{id} \times \gamma_1) = h$, $k\rho_0 = \alpha$, where $\kappa_t : X_1 \to I \times X_1$ and $\rho_t : B_1 \to I \times B_1$ are defined by $\kappa_t(x) = (t, x)$ and $\rho_t(b) = (t, b)$ for any $t \in I$. Take (ψ, β) be $\psi = K\kappa_1$, $\beta = k\rho_1$. Then it is easy to see that $(\phi, \alpha) \simeq^{\mathbf{M}} (\psi, \beta)$ by the definition of (ψ, β). Further for any $x \in X_1$, $b \in B_1$,

$$\psi u_1(a) = K\kappa_1\big(u_1(a)\big) = K\big(1, u_1(a)\big) = K(\mathrm{id} \times u_1)(1, a) = H(1, a) = u_2(a),$$

$$\beta\gamma_1(b) = k\rho_1\big(\gamma_1(b)\big) = k\big(1, \gamma_1(b)\big) = k(\mathrm{id} \times \gamma_1)(1, b) = h(1, b) = \gamma_2(b).$$

This completes the proof. □

Let $(u, \gamma) : (A, p_0, B_0) \to (X_1, p_1, B_1)$ be an **M**-fibrewise map. Let (ϕ, α), $(\psi, \beta) : (X_1, p_1, B_1) \to (X_2, p_2, B_2)$ be **M**-fibrewise maps such that $\phi u = \psi u$ and $\alpha\gamma = \beta\gamma$. By an **M**-*fibrewise homotopy* of (ϕ, α) into (ψ, β) *under* (A, p_0, B_0) we mean an **M**-fibrewise homotopy (H, h) of (ϕ, α) into (ψ, β) such that $H(\mathrm{id} \times u(t, a))$ and $h(\mathrm{id} \times \gamma)(t, b)$ are independent of $t \in I$. When such an **M**-fibrewise homotopy exists we say that (ϕ, α) and (ψ, β) are **M**-fibrewise homotopic *under* (A, p_0, B_0) and write $(\phi, \alpha) \simeq^{\mathbf{M}} (\psi, \beta)$ under (A, p_0, B_0). For the case (A, p_0, B_0) is an **M**-fibrewise subspace of (X_1, p_1, B_1), see Definitions 2.4 and 2.6.

Theorem 3.5. *Let* $(u, \gamma) : (A, p_0, B_0) \to (X_1, p_1, B_1)$ *be an* **M**-*fibrewise cofibration. Let* $(\theta, \alpha) : (X_1, p_1, B_1) \to (X_1, p_1, B_1)$ *be an* **M**-*fibrewise map under* (A, p_0, B_0) *such that* $(\theta, \alpha) \simeq^{\mathbf{M}} (\mathrm{id}_{X_1}, \mathrm{id}_{B_1})$. *Then there exists an* **M**-*fibrewise map* $(\theta', \alpha') : (X_1, p_1, B_1) \to (X_1, p_1, B_1)$ *under* (A, p_0, B_0) *such that* $(\theta'\theta, \alpha'\alpha) \simeq^{\mathbf{M}} (\mathrm{id}_{X_1}, \mathrm{id}_{B_1})$ *under* (A, p_0, B_0).

Proof. Let $(H, h) : (I \times X_1, \mathrm{id} \times p_1, I \times B_1) \to (X_1, p_1, B_1)$ be an **M**-fibrewise homotopy of (θ, α) into $(\mathrm{id}_{X_1}, \mathrm{id}_{B_1})$. Then the following two diagrams

$$\begin{array}{ccc} A & \xrightarrow{\sigma_0} & I \times A \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle H(\mathrm{id}\times u)} \\ X_1 & \xrightarrow{\mathrm{id}_{X_1}} & X_1 \end{array}$$

$$\begin{array}{ccc} B_0 & \xrightarrow{\delta_0} & I \times B_0 \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle h(\mathrm{id}\times\gamma)} \\ B_1 & \xrightarrow{\mathrm{id}_{B_1}} & B_1 \end{array}$$

are commutative. Since (u, γ) is an **M**-fibrewise cofibration, there exists an **M**-fibrewise homotopy $(K, k) : (I \times X_1, \mathrm{id} \times p_1, I \times B_1) \to (X_1, p_1, B_1)$ such that

$$K(\mathrm{id} \times u) = H(\mathrm{id} \times u), \qquad K\kappa_0 = \mathrm{id}_{X_1},$$
$$k(\mathrm{id} \times \gamma) = h(\mathrm{id} \times \gamma), \qquad k\rho_0 = \mathrm{id}_{B_1},$$

where $\kappa_t : X_1 \to I \times X_1$, $\rho : B_1 \to I \times B_1$ are defined by $\kappa_t(x) = (t, x)$ and $\rho_t(b) = (t, b)$ for any $t \in I$, respectively. Let $\theta' = K\kappa_1$ and $\alpha' = k\rho_1$. We can define an **M**-fibrewise map $(G, g) : (I \times X_1, \mathrm{id} \times p_1, I \times B_1) \to (X_1, p_1, B_1)$ as follows:

$$G(s, x) = \begin{cases} K(1 - 2s, \theta(x)) & \text{if } 0 \leqslant s \leqslant \frac{1}{2}, \\ H(2s - 1, x) & \text{if } \frac{1}{2} \leqslant s \leqslant 1, \end{cases}$$

$$g(s, b) = \begin{cases} k(1 - 2s, \alpha(b)) & \text{if } 0 \leqslant s \leqslant \frac{1}{2}, \\ h(2s - 1, b) & \text{if } \frac{1}{2} \leqslant s \leqslant 1. \end{cases}$$

Then it is easy to see that $G\kappa_0 = \theta'\theta$, $G\kappa_1 = \mathrm{id}_{X_1}$, $g\rho_0 = \alpha'\alpha$, $g\rho_1 = \mathrm{id}_{B_1}$.

Now we shall prove that $(\theta'\theta, \alpha'\alpha) \simeq^{\mathbf{M}} (\mathrm{id}_{X_1}, \mathrm{id}_{B_1})$ under (A, p_0, B_0). We consider $(M, m) : (I \times I \times A, \mathrm{id} \times \mathrm{id} \times p_0, I \times I \times B_0) \to (X_1, p_1, B_1)$ such that

$$M(s, t, x) = \begin{cases} K(1 - 2s(1 - t), u(a)) & \text{if } 0 \leqslant s \leqslant \frac{1}{2}, \\ H(1 - 2(1 - s)(1 - t), u(a)) & \text{if } \frac{1}{2} \leqslant s \leqslant 1, \end{cases}$$

$$m(s, t, b) = \begin{cases} k(1 - 2s(1 - t), \gamma(b)) & \text{if } 0 \leqslant s \leqslant \frac{1}{2}, \\ h(1 - 2(1 - s)(1 - t), \gamma(b)) & \text{if } \frac{1}{2} \leqslant s \leqslant 1. \end{cases}$$

Let any $s \in I$ fix. Let $(M_s, m_s) : (I \times A, \mathrm{id} \times p_0, I \times B_0) \to (X_1, p_1, B_1)$ be an **M**-fibrewise map defined by

$$M_s(t, a) = M(s, t, a), \qquad m_s(t, b) = m(s, t, b)$$

and $(G_s, g_s) : (X_1, p_1, B_1) \to (X_1, p_1, B_1)$ an **M**-fibrewise map defined by

$$G_s(x) = G(s, x), \qquad g_s = g(s, b).$$

Then the following two diagrams

$$\begin{CD} A @>{\sigma_0}>> I \times A \\ @V{u}VV @VV{M_s}V \\ X_1 @>{G_s}>> X_1 \end{CD}$$

$$\begin{CD} B_0 @>{\delta_0}>> I \times B_0 \\ @V{u}VV @VV{m_s}V \\ B_1 @>{g_s}>> B_1 \end{CD}$$

are commutative. Since (u, γ) is an **M**-fibrewise cofibration, there exists an **M**-fibrewise homotopy $(N_s, n_s) : (I \times X_1, \mathrm{id} \times p_1, I \times B_1) \to (X_1, p_1, B_1)$ such that $N_s(\mathrm{id} \times u) = M_s$, $N_s\kappa_0 = G_s$, $n_s(\mathrm{id} \times \gamma) = m_s$, $n_s\rho_0 = g_s$. Then it is easily verified that

$$\begin{aligned} \bigl(\theta'\theta, \alpha'\alpha\bigr) &= (G_0, g_0) = (N_0\kappa_0, n_0\rho_0) \simeq^{\mathbf{M}} (N_0\kappa_1, n_0\rho_1) \simeq^{\mathbf{M}} (N_1\kappa_1, n_1\rho_1) \\ &\simeq^{\mathbf{M}} (N_1\kappa_0, n_1\rho_0) = (G_1, g_1) = (\mathrm{id}_{X_1}, \mathrm{id}_{B_1}), \end{aligned}$$

where each $\simeq^{\mathbf{M}}$ is **M**-fibrewise homotopic under (A, p_0, B_0). This completes the proof. □

Using this theorem, we shall prove the following.

Theorem 3.6. *Let* $(u_1,\gamma_1):(A,p_0,B_0)\to(X_1,p_1,B_1)$ *and* $(u_2,\gamma_2):(A,p_0,B_0)\to(X_2,p_2,B_2)$ *be* **M***-fibrewise cofibrations. Let* $(\phi,\alpha):(X_1,p_1,B_1)\to(X_2,p_2,B_2)$ *be an* **M***-fiberwise map such that* $(\phi u_1,\alpha\gamma_1)=(u_2,\gamma_2)$*. Suppose that* (ϕ,α) *is an* **M***-fibrewise homotopy equivalence. Then* (ϕ,α) *is an* **M***-fibrewise homotopy equivalence under* (A,p_0,B_0).

Proof. Since (ϕ,α) is an **M**-fibrewise homotopy equivalence, there exists an **M**-fibrewise homotopy inverse $(\psi,\beta):(X_2,p_2,B_2)\to(X_1,p_1,B_1)$. Then $(\psi u_2,\beta\gamma_2)=(\psi\phi u_1,\beta\alpha\gamma_1)\simeq^{\mathbf{M}}(u_1,\gamma_1)$. From Theorem 3.4, there exists an **M**-fibrewise map $(\psi',\beta'):(X_2,p_2,B_2)\to(X_1,p_1,B_1)$ such that $(\psi,\beta)\simeq^{\mathbf{M}}(\psi',\beta')$ and $(\psi'u_2,\beta'\gamma_2)=(u_1,\gamma_1)$. Since $(\psi\phi,\beta\alpha)\simeq^{\mathbf{M}}(\mathrm{id}_{X_1},\mathrm{id}_{B_1})$ and $(\psi'\phi,\beta'\alpha)\simeq^{\mathbf{M}}(\mathrm{id}_{X_1},\mathrm{id}_{B_1})$, from Theorem 3.5 there exists an **M**-fibrewise map $(\psi'',\beta''):(X_1,p_1,B_1)\to(X_1,p_1,B_1)$ such that $(\psi'',\beta'')\simeq^{\mathbf{M}}(\psi'\phi,\beta'\alpha)$ and $(\psi''\psi'\phi,\beta''\beta'\alpha)\simeq^{\mathbf{M}}(\mathrm{id}_{X_1},\mathrm{id}_{B_1})$ under (A,p_0,B_0). Let $\bar\psi=\psi''\psi'$ and $\bar\beta=\beta''\beta'$. Then $(\bar\psi\phi,\bar\beta\alpha)\simeq^{\mathbf{M}}(\mathrm{id}_{X_1},\mathrm{id}_{B_1})$ under (A,p_0,B_0).

Now we shall prove that there exists an **M**-fibrewise map $(\bar\phi,\bar\alpha):(X_1,p_1,B_1)\to(X_2,p_2,B_2)$ such that $(\bar\phi\bar\psi,\bar\alpha\bar\beta)\simeq^{\mathbf{M}}(\mathrm{id}_{X_2},\mathrm{id}_{B_2})$. Since $(\phi\bar\psi,\alpha\bar\beta)=(\phi\psi''\psi',\alpha\beta''\beta')\simeq^{\mathbf{M}}(\phi(\psi'\phi)\psi',\alpha(\beta'\alpha)\beta')=((\phi\psi')(\phi\psi'),(\alpha\beta')(\alpha\beta'))\simeq^{\mathbf{M}}(\mathrm{id}_{X_2},\mathrm{id}_{B_2})$, from Theorem 3.5 there exists an **M**-fibrewise map $(\phi',\alpha'):(X_2,p_2,B_2)\to(X_2,p_2,B_2)$ such that $(\phi'\phi\bar\psi,\alpha'\alpha\bar\beta)\simeq^{\mathbf{M}}(\mathrm{id}_{X_2},\mathrm{id}_{B_2})$ under (A,p_0,B_0). Let $\bar\phi=\phi'\phi$ and $\bar\alpha=\alpha'\alpha$. Then $(\bar\phi\bar\psi,\bar\alpha\bar\beta)\simeq^{\mathbf{M}}(\mathrm{id}_{X_2},\mathrm{id}_{B_2})$. Since

$$(\phi,\alpha)\simeq^{\mathbf{M}}\big((\bar\phi\bar\psi)\phi,(\bar\alpha\bar\beta)\alpha\big)=\big(\bar\phi(\bar\psi\phi),\bar\alpha(\bar\beta\alpha)\big)\simeq^{\mathbf{M}}\big(\bar\phi,\bar\alpha\big),$$

$(\bar\psi,\bar\beta)$ is an **M**-fibrewise homotopy inverse of (ϕ,α) under (A,p_0,B_0). □

Definition 3.2. Let $((X_1,p_1,B_1),(A,p_0,B_0))$ be a closed **M**-fibrewise pair. An **M**-*fibrewise Strøm structure on* $((X_1,p_1,B_1),(A,p_0,B_0))$ is a pair $((\alpha,\beta),(H,h))$ consisting of maps $\alpha:X_1\to I$, $\beta:B_1\to I$ which satisfy $\beta p_1=\alpha$ and are zero throughout (A,p_0,B_0) and an **M**-fibrewise homotopy $(H,h):(I\times X_1,\mathrm{id}\times p_1,I\times B_1)\to(X_1,p_1,B_1)$ under (A,p_0,B_0) of $(\mathrm{id}_{X_1},\mathrm{id}_{B_1})$ such that $H(t,x)\in A$, $h(s,b)\in B_0$ for any $t\leqslant\alpha(x)$, $s\leqslant\beta(b)$.

We obtain the following theorems.

Theorem 3.7. *A closed* **M***-fibrewise pair* $((X_1,p_1,B_1),\ (A,p_0,B_0))$ *is* **M***-fibrewise cofibred if and only if there exists an* **M***-fibrewise Strøm structure on* $((X_1,p_1,B_1),\ (A,p_0,B_0))$.

Proof. “If” part: Let $((\alpha,\beta),(H,h))$ be an **M**-fibrewise Strøm structure on $((X_1,p_1,B_1),(A,p_0,B_0))$. We can define an **M**-fibrewise map

$$(R,r):(I\times X_1,\mathrm{id}\times p_1,I\times B_1)$$
$$\to\big(\{0\}\times X_1\cup I\times A,\mathrm{id}\times p_1,\{0\}\times B_1\cup I\times B_0\big)$$

by

$$R(t,x)=\begin{cases}(0,H(t,x)) & \text{if } t\leqslant\alpha(x),\\ (t-\alpha(x),H(t,x)) & \text{if } t\geqslant\alpha(x),\end{cases}$$

$$r(t,b)=\begin{cases}(0,h(t,b)) & \text{if } t\leqslant\beta(b),\\ (t-\beta(b),h(t,b)) & \text{if } t\geqslant\beta(b).\end{cases}$$

Then (R,r) is an **M**-fibrewise retraction. In fact, for any $(0,x)\in\{0\}\times X_1$, $R(0,x)=(0,H(0,x))=(0,x)$ since $0\leqslant\alpha(x)$. Next, for any $(t,a)\in I\times A$, $R(t,a)=(t-0,H(t,a))=(t,a)$ since $t\geqslant\alpha(a)=0$, and (H,h) is an **M**-fibrewise map under (A,p_0,B_0). Thus R is a retraction. By the same way, r is also a retraction.

"Only if" part: Suppose that $((X_1,p_1,B_1),(A,p_0,B_0))$ is a closed **M**-fibrewise cofibred pair. Then from Theorem 3.2 there exists an **M**-fibrewise retraction $(R,r):(I\times X_1,\mathrm{id}\times p_1,I\times B_1)\to(\{0\}\times X_1\cup I\times A,\mathrm{id}\times p_1,\{0\}\times B_1\cup I\times B_0)$. Let $R(t,x)=(R_1(t,x),R_2(t,x))$ and $r(t,b)=(r_1(t,b),r_2(t,b))$. Then we define maps $\alpha:X_1\to I$ and $\beta:B_1\to I$ by

$$\alpha(x)=\sup_{t\in I}\bigl|R_1(t,x)-t\bigr|\quad(x\in X_1),$$
$$\beta(b)=\sup_{t\in I}\bigl|r_1(t,b)-t\bigr|\quad(b\in B_1).$$

Then it is easily verified that $((\alpha,\beta),(R_2,r_2))$ constitutes an **M**-fibrewise Strøm structure on $((X_1,p_1,B_1),(A,p_0,B_0))$. □

Theorem 3.8. *Let $((X_1,p_1,B_1),(X_1',p_1,B_1'))$ and $((X_2,p_2,B_2),(X_2',p_2,B_2'))$ be a closed* **M***-fibrewise cofibred pair. Then*

$$\bigl((X_1\times X_2,p_1\times p_2,B_1\times B_2),$$
$$\bigl(X_1'\times X_2\cup X_1\times X_2',p_1\times p_2,B_1'\times B_2\cup B_1\times B_2'\bigr)\bigr)$$

is also an **M***-fibrewise cofibred pair.*

Proof. Let $((\alpha_1,\beta_1),(H_1,h_1))$ and $((\alpha_2,\beta_2),(H_2,h_2))$ be **M**-fibrewise Strøm structures on $((X_1,p_1,B_1),(X_1',p_1,B_1'))$ and $((X_2,p_2,B_2),(X_2',p_2,B_2'))$, respectively. Define $\gamma:X_1\times X_2\to I$ and $\eta:B_1\times B_2\to I$ by

$$\gamma(x,y)=\min\bigl(\alpha_1(x),\alpha_2(y)\bigr)\quad\bigl((x,y)\in X_1\times X_2\bigr)$$
$$\eta(b,c)=\min\bigl(\beta_1(b),\beta_2(c)\bigr)\quad\bigl((b,c)\in B_1\times B_2\bigr)$$

and define $(K,k):(I\times(X_1\times X_2),\mathrm{id}\times(p_1\times p_2),I\times(B_1\times B_2))\to(X_1\times X_2,p_1\times p_2,B_1\times B_2)$ by

$$K\bigl(t,(x,y)\bigr)=\bigl(H_1\bigl(\min\bigl(t,\alpha_2(y)\bigr),x\bigr),H_2\bigl(\min\bigl(t,\alpha_1(x)\bigr),y\bigr)\bigr)$$
$$k\bigl(t,(b,c)\bigr)=\bigl(h_1\bigl(\min\bigl(t,\beta_2(c)\bigr),b\bigr),k_2\bigl(\min\bigl(t,\beta_1(b)\bigr),c\bigr)\bigr),$$

where $(x,y)\in X_1\times X_2$ and $(b,c)\in B_1\times B_2$. Then it is easily verified that $((\gamma,\eta),(K,k))$ constitutes an **M**-fibrewise Strøm structure. Thus this completes the proof from Theorem 3.7. □

Definition 3.3. Let us describe an **M**-fibrewise Strøm structure $((\alpha,\beta),(H,h))$ on the closed **M**-fibrewise pair $((X_1, p_1, B_1),(A, p_0, B_0))$ as *strict* if $\alpha < 1$ throughout X_1 and $\beta < 1$ throughout B_1.

Theorem 3.9. *Let $((X_1, p_1, B_1),(A, p_0, B_0))$ be a closed* **M**-*fibrewise cofibred pair. Then there exists a strict* **M**-*fibrewise Strøm structure on this pair if and only if there exists an* **M**-*fibrewise deformation retraction of (X_1, p_1, B_1) onto $(A, p_0, B_0))$.*

Proof. "Only if" part: Let $((\alpha,\beta),(H,h))$ be an **M**-fibrewise Strøm structure on $((X_1, p_1, B_1),(A, p_0, B_0))$. Then we shall prove that (H,h) is an **M**-fibrewise deformation retraction. In fact, from the definition of an **M**-fibrewise Strøm structure, $H\kappa_0 = \mathrm{id}_{X_1},\ h\rho_0 = \mathrm{id}_{B_0}$, where $\kappa_t : X_1 \to I \times X_1$ and $\rho_t : B_1 \to I \times B_1$ are defined by $\kappa_t(x) = (t,x)$ and $\rho_t(b) = (t,b)$. Next for any $a \in A,\ a = H(1,a) = H\kappa_1(a) \in H\kappa_1(X_1)$. For any $x \in X_1$, since $1 \geqslant \alpha(x),\ H\kappa_1(x) = H(1,x) \in A$. $h\rho_1(B_1) = B_0$ is similarly proved.

"If" part: Let $(H,h) : (I \times X_1, \mathrm{id} \times p_1, I \times B_1) \to (X_1, p_1, B_1)$ be an **M**-fibrewise deformation retraction and $((\alpha,\beta),(K,k))$ an **M**-fibrewise Strøm structure. We can define maps $\alpha' : X_1 \to I$ and $\beta' : B_1 \to I$ by

$$\alpha'(x) = \min\bigl(\alpha(x), \tfrac{1}{2}\bigr), \quad \beta'(b) = \min\bigl(\beta(b), \tfrac{1}{2}\bigr) \quad (x \in X_1,\ b \in B_1).$$

Take $(H',h') : (I \times X_1, \mathrm{id} \times p_1, I \times B_1) \to (X_1, p_1, B_1)$ to be

$$H'(t,x) = H\bigl(\min(2t,1), K(t,x)\bigr), \qquad h'(t,b) = h\bigl(\min(2t,1), k(t,b)\bigr)$$

$(t \in I,\ x \in X_1,\ b \in B_1)$. Then it is easy to see that $((\alpha',\beta'),(H',h'))$ is a strict **M**-fibrewise Strøm structure. □

Returning to the proof of Theorem 3.8, we observe that if $\alpha_1 < 1,\ \beta_1 < 1$ or $\alpha_2 < 1,\ \beta_2 < 1$, then $\gamma < 1,\ \eta < 1$, so we obtain

Theorem 3.10. *Let $((X_1, p_1, B_1),(X_1', p_1, B_1'))$ and $((X_2, p_2, B_2),(X_2', p_2, B_2'))$ be an closed* **M**-*fibrewise cofibred pairs. If (X_1', p_1, B_1') or (X_2', p_2, B_2') is an* **M**-*fibrewise deformation retract of (X_1, p_1, B_1) or (X_2, p_2, B_2) respectively, then $(X_1' \times X_2 \cup X_1 \times X_2', p_1 \times p_2, B_1' \times B_2 \cup B_1 \times B_2')$ is an* **M**-*fibrewise deformation retract of $(X_1 \times X_2, p_1 \times p_2, B_1 \times B_2)$.*

4. M-fibrewise fibrations

In this section, we consider an extended version of fibrewise fibrations, and obtain some generalized theorems of fibrewise version [5, §23]. We begin with the following definition.

Definition 4.1. An **M**-fibrewise map $(\phi,\alpha) : (E, p_1, B_1) \to (F, p_2, B_2)$ is an **M**-*fibrewise fibration* if (ϕ,α) has the following property for any **M**-fibrewise space (X, p_0, B_0): Let

$(f,\beta):(X,p_0,B_0)\to(E,p_1,B_1)$ be an **M**-fibrewise map and $(H,h):(I\times X,\mathrm{id}\times p_0, I\times B_0)\to(F,p_2,B_2)$ an **M**-fibrewise homotopy such that following diagrams

$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ {\scriptstyle\sigma_0}\downarrow & & \downarrow{\scriptstyle\phi} \\ I\times X & \xrightarrow{H} & F \end{array}$$

$$\begin{array}{ccc} B_0 & \xrightarrow{\beta} & B_1 \\ {\scriptstyle\delta_0}\downarrow & & \downarrow{\scriptstyle\alpha} \\ I\times B_0 & \xrightarrow{h} & B_2 \end{array}$$

are commutative. Then there exists an **M**-fibrewise homotopy $(K,k):(I\times X,\mathrm{id}\times p_0, I\times B_0)\to(E,p_1,B_1)$ such that $\phi K=H$, $K\sigma_0=f$, $\alpha k=h$, $k\delta_0=\beta$.

The property involved here is called the **M**-*fibrewise homotopy lifting property*; the **M**-fibrewise homotopy (H,h) of $(\phi f,\alpha\beta)$ is lifted to an **M**-fibrewise homotopy (K,k) of (f,β) itself.

Theorem 4.1. *Let (X,p,B) be an* **M**-*fibrewise space, $\alpha:X\to I$, $\beta:B\to I$ maps such that $\alpha=\beta p$ and for $A=\alpha^{-1}(0)$, $B_0=\beta^{-1}(0)$, (A,p_0,B_0) an* **M**-*fibrewise deformation retract of (X,p,B), where $p_0=p|A$. Let $(\phi,\eta):(E_1,p_1,B_1)\to(E_2,p_2,B_2)$ be an* **M**-*fibrewise fibration. For two* **M**-*fibrewise maps $(f_1,\mu_1):(A,p_0,B_0)\to(E_1,p_1,B_1)$, $(f_2,\mu_2):(X,p,B)\to(E_2,p_2,B_2)$ such that $\phi f_1=f_2|A$ and $\eta\mu_1=\mu_2|B_0$, there exists an* **M**-*fibrewise map $(h,\zeta):(X,p,B)\to(E_1,p_1,B_1)$ such that $h|A=f_1$, $\phi h=f_2$, $\zeta|B_0=\mu_1$, $\eta\zeta=\mu_2$.*

Proof. Let $(R,r):(X,p,B)\to(A,p_0,B_0)$ be an **M**-fibrewise retraction and $(K,k):(I\times X,\mathrm{id}\times p, I\times B)\to(X,p,B)$ an **M**-fibrewise deformation retraction of (iR,jr) into $(\mathrm{id}_X,\mathrm{id}_B)$, where $i:A\to X$ and $j:B_0\to B$ are inclusions. Take $D:I\times X$ and $d:I\times B_1\to B_1$ to be

$$D(t,x)=\begin{cases} K\bigl(\min\bigl(1,\frac{t}{\alpha(x)}\bigr),x\bigr) & \text{if } x\notin A,\\ K(t,x) & \text{if } x\in A,\end{cases}$$

$$d(t,b)=\begin{cases} k\bigl(\min\bigl(1,\frac{t}{\beta(b)}\bigr),b\bigr) & \text{if } b\notin B_0,\\ k(t,b) & \text{if } b\in B_0.\end{cases}$$

Then following two diagrams

$$\begin{array}{ccc} X & \xrightarrow{f_1K\sigma_0} & E_1 \\ {\scriptstyle\sigma_0}\downarrow & & \downarrow{\scriptstyle\phi} \\ I\times X & \xrightarrow{f_2D} & E_2 \end{array}$$

$$\begin{array}{ccc} B & \xrightarrow{\mu_1 k\delta_0} & B_1 \\ {\scriptstyle\delta_0}\downarrow & & \downarrow{\scriptstyle\eta} \\ I\times B & \xrightarrow{\mu_2 d} & B_2 \end{array}$$

are commutative. Since (ϕ,η) is an **M**-fibrewise fibration, there exists an **M**-fibrewise homotopy $(G,g):(I\times X,\mathrm{id}\times p, I\times B)\to(E_1,p_1,B_1)$ such that $\phi G=f_2D,\ G\sigma_0=f_1K\sigma_0,\ \eta g=\mu_2 d,\ g\delta_0=\mu_1 k\delta_0$. Then take $h:X\to E_1$ and $\zeta:B\to B_1$ to be

$$h(x)=G\bigl(\alpha(x),x\bigr),\quad \zeta(b)=\bigl(\beta(b),b\bigr)\quad (x\in X,\ b\in B)$$

Then it is easy to see that (h,ζ) is the required one. □

Theorem 4.2. *Let $(\phi,\mu):(E,p_1,B_1)\to(F,p_2,B_2)$ be an* **M***-fibrewise fibration and $((F,p_2,B_2),(F',p_2,B_2'))$ an* **M***-fibrewise cofibred pair. Then $((E,p_1,B_1),(E',p_1,B_1'))$, where $E'=\phi^{-1}F',\ B_1'=\mu^{-1}B_2'$ is an* **M***-fibrewise cofibred pair.*

Proof. Let $((\alpha,\beta),(H,h))$ be an **M**-fibrewise Strøm structure on $((F,p_2,B_2),(F',p_2,B_2'))$. Then following two diagrams

$$\begin{array}{ccc} E & \xrightarrow{\mathrm{id}_E} & E \\ {\scriptstyle\sigma_0}\downarrow & & \downarrow{\scriptstyle\phi} \\ I\times E & \xrightarrow{H(\mathrm{id}\times\phi)} & F \end{array}$$

$$\begin{array}{ccc} B_1 & \xrightarrow{\mathrm{id}_{B_1}} & B_1 \\ {\scriptstyle\delta_0}\downarrow & & \downarrow{\scriptstyle\mu} \\ I\times B_1 & \xrightarrow{h(\mathrm{id}\times\mu)} & B_2 \end{array}$$

are commutative. Since (ϕ,μ) is an **M**-fibrewise fibration, there exists an **M**-fibrewise homotopy $(K,k):(I\times E,\mathrm{id}\times p_1,I\times B_1)\to(E,p_1,B_1)$ such that $\phi K=H(\mathrm{id}\times\phi),\ K\sigma_0=\mathrm{id}_E,\ \mu k=h(\mathrm{id}\times\mu),\ k\delta_0=\mathrm{id}_{B_1}$. Take $\gamma:E\to I$ and $\xi:B_1\to I$ to be

$$\gamma(x)=\min\bigl(2\alpha\bigl(\phi(x)\bigr),1\bigr),\quad \xi(b)=\min\bigl(2\beta\bigl(\mu(b)\bigr),1\bigr)\quad (x\in E,\ b\in B_1).$$

Then for any $e\in E'=\phi^{-1}F',\ \alpha\phi(e)=0$, so $\gamma(e)=0$. Similarly, for any $b\in B_1',\ \xi(b)=0$. Next take $(L,l):(I\times E,\mathrm{id}\times p_1,I\times B_1)\to(E,p_1,B_1)$ to be

$$\begin{aligned} L(t,x) &= K\bigl(\min\bigl(t,\alpha\phi(x)\bigr),x\bigr),\\ l(t,b) &= k\bigl(\min\bigl(t,\beta\mu(b)\bigr),b\bigr)\quad (t\in I,\ x\in E,\ b\in B_1). \end{aligned}$$

Then it is easy to see that $((\gamma,\xi),(L,l))$ is an **M**-fibrewise Strøm structure on $((E,p_1,B_1),(E',p_1,B_1'))$ so $((E,p_1,B_1),(E',p_1,B_1'))$ is an **M**-fibrewise cofibred pair by Theorem 3.7. □

Theorem 4.3. *Let $(\xi,\alpha):(X_1,p_1,B_1)\to(E,p,B),\ (\eta,\beta):(X_2,p_2,B_2)\to(E,p,B)$ be* **M***-fibrewise maps and $(\phi,\gamma):(X_1,p_1,B_1)\to(X_2,p_2,B_2)$ an* **M***-fibrewise map such that*

$(\eta\phi, \beta\gamma) \simeq^{\mathbf{M}} (\xi, \alpha)$. *If* (η, β) *is an* **M**-*fibrewise fibration, then there exists an* **M**-*fibrewise map* $(\psi, \varepsilon): (X_1, p_1, B_1) \to (X_2, p_2, B_2)$ *such that* $(\phi, \gamma) \simeq^{\mathbf{M}} (\psi, \varepsilon)$ *and* $(\eta\psi, \beta\varepsilon) = (\xi, \alpha)$.

Proof. From $(\eta\phi, \beta\gamma) \simeq^{\mathbf{M}} (\xi, \alpha)$, there exists an **M**-fibrewise homotopy $(G, g): (I \times X_1, \mathrm{id} \times p_1, I \times B_1) \to (E, p, B)$ such that $G\sigma_0 = \eta\phi$, $g\delta_0 = \beta\gamma$, $G\sigma_1 = \xi$, $g\delta_1 = \alpha$. Since (η, β) is an **M**-fibrewise fibration, there exists an **M**-fibrewise homotopy $(H, h): (I \times X_1, \mathrm{id} \times p_1, I \times B_1) \to (X_2, p_2, B_2)$ such that $H\sigma_0 = \phi$, $h\delta_0 = \gamma$, $\eta H = G$, $\beta h = g$. Put $\psi = H\sigma_1$ and $\varepsilon = h\delta_1$. Then it is easy to see that $(\phi, \gamma) = (H\sigma_0, h\delta_0) \simeq^{\mathbf{M}} (H\sigma_1, h\delta_1) = (\psi, \varepsilon)$ and $(\eta\psi, \beta\varepsilon) = (\eta H\sigma_1, \beta h\delta_1) = (G\sigma_1, g\delta_1) = (\xi, \alpha)$. □

5. M-fibrewise pointed homotopy

In this section, we consider an extended version of fibrewise pointed homotopy, and obtain some generalized results of fibrewise version [5, §19, 21]. The proofs of theorems of this section are very similar to those of the theorems of the previous Sections 3 and 4, so we omit the proofs.

When an **M**-fibrewise space (X, p, B) has a section $s: B \to X$, we call it an **M**-*fibrewise pointed space* and denote (X, p, B, s). For two **M**-fibrewise pointed spaces (X_1, p_1, B_1, s_1), (X_2, p_2, B_2, s_2), if an **M**-fibrewise map $(f, \alpha): (X_1, p_1, B_1) \to (X_2, p_2, B_2)$ satisfies $fs_1 = s_2\alpha$, we call it an **M**-*fibrewise pointed map* and denote $(f, \alpha): (X_1, p_1, B_1, s_1) \to (X_2, p_2, B_2, s_2)$.

Definition 5.1. Let $(\phi, \alpha), (\theta, \beta): (X_1, p_1, B_1, s_1) \to (X_2, p_2, B_2, s_2)$ be **M**-fibrewise pointed maps. If there exists an **M**-fibrewise pointed map $(H, h): (I \times X_1, \mathrm{id} \times p_1, I \times B_1, \mathrm{id} \times s_1) \to (X_2, p_2, B_2, s_2)$ such that (H, h) is an **M**-fibrewise homotopy of (ϕ, α) into (θ, β), we call it an **M**-*fibrewise pointed homotopy* of (ϕ, α) into (θ, β).

If there exists an **M**-fibrewise pointed homotopy of (ϕ, α) into (θ, β), we say (ϕ, α) is **M**-*fibrewise pointed homotopic* to (θ, β) and write $(\phi, \alpha) \simeq^{\mathbf{M}}_{(\mathbf{P})} (\theta, \beta)$.

Lemma 5.1. *The relation* $\simeq^{\mathbf{M}}_{(\mathbf{P})}$ *is an equivalence relation.*

Definition 5.2. An **M**-fibrewise pointed map $(\theta, \alpha): (X_1, p_1, B_1, s_1) \to (X_2, p_2, B_2, s_2)$ is called an **M**-*fibrewise pointed homotopy equivalence* if there exists an **M**-fibrewise pointed map $(\phi, \beta): (X_2, p_2, B_2, s_2) \to (X_1, p_1, B_1, s_1)$ such that $(\phi\theta, \beta\alpha) \simeq^{\mathbf{M}}_{(\mathbf{P})} (\mathrm{id}_{X_1}, \mathrm{id}_{B_1})$, $(\theta\phi, \alpha\beta) \simeq^{\mathbf{M}}_{(\mathbf{P})} (\mathrm{id}_{X_2}, \mathrm{id}_{B_2})$. Then we denote $(X_1, p_1, B_1, s_1) \cong^{\mathbf{M}}_{(\mathbf{P})} (X_2, p_2, B_2, s_2)$.

Lemma 5.2. *The relation* $\cong^{\mathbf{M}}_{(\mathbf{P})}$ *is an equivalence relation.*

Definition 5.3. Let $(\theta, \alpha), (\phi, \beta): (X_1, p_1, B_1, s_1) \to (X_2, p_2, B_2, s_2)$ be **M**-fibrewise pointed maps. Further let A be a subspace of X_1 and B_0 a subspace of B_1 such that

$p_1(A) = B_0$ and $\theta(x) = \phi(x)$ for any $x \in A$, $\alpha(b) = \beta(b)$ for any $b \in B_0$. By an **M**-*fibrewise pointed homotopy of* (θ, α) into (ϕ, β) *under* $(A, p_1|A, B_0, s_1|B_0)$ we mean an **M**-fibrewise pointed homotopy (H, h) of (θ, α) into (ϕ, β) such that for fixed $x \in A$ and $b \in B_0$, $H\sigma_t(x)$ and $h\delta_t(b)$ are constant for any $t \in I$. Moreover $(A, p_1|A, B_0, s_1|B_0)$ is called an **M**-*fibrewise pointed subspace* of (X_1, p_1, B_1, s_1).

Definition 5.4. Let (X_1, p_1, B_1, s_1) be an **M**-fibrewise pointed space, and (A, p_0, B_0, s_0) a subspace of (X_1, p_1, B_1, s_1) such that $p_1(A) = B_0$, where $p_0 = p_1|A$ and $s_0 = s_1|B_0$. An **M**-*fibrewise pointed retraction* we mean an **M**-fibrewise pointed map $(R, r) : (X_1, p_1, B_1, s_1) \to (A, p_0, B_0, s_0)$ such that (R, r) is an **M**-*fibrewise retraction.*

Definition 5.5. Let (X_1, p_1, B_1, s_1) be an **M**-fibrewise pointed space. An **M**-fibrewise pointed subspace $(A, p_1|A, B_0, s_1|B_0)$ of (X_1, p_1, B_1, s_1) is an **M**-*fibrewise pointed deformation retract* of (X_1, p_1, B_1, s_1) if there exists an **M**-fibrewise pointed homotopy $(H, h) : (I \times X_1, \mathrm{id} \times p_1, I \times B_1, \mathrm{id} \times s_1) \to (X_1, p_1, B_1, s_1)$ of $(\mathrm{id}_{X_1}, \mathrm{id}_{B_1})$ into (R, r) which is an **M**-fibrewise deformation retraction, where $(R, r) : (X_1, p_1, B_1, s_1) \to (A, p_0, B_0, s_0)$ is an **M**-fibrewise pointed retraction.

Theorem 5.3. *Let* (X_1, p_1, B_1, s_1) *be an* **M**-*fibrewise pointed space and* $(A, p_1|A, B_0, s_1|B_0)$ *an* **M**-*fibrewise pointed subspace of* (X_1, p_1, B_1, s_1). *If* $(\{0\} \times X_1 \cup I \times A, \mathrm{id} \times p_1, \{0\} \times B_1 \cup I \times B_0, \mathrm{id} \times s_1)$ *is an* **M**-*fibrewise pointed retract of* $(I \times X_1, \mathrm{id} \times p_1, I \times B_1, \mathrm{id} \times s_1)$, *then* $(\{0\} \times X_1 \cup I \times A, \mathrm{id} \times p_1, \{0\} \times B_1 \cup I \times B_0, \mathrm{id} \times s_1)$ *is an* **M**-*fibrewise pointed deformation retract of* $(I \times X_1, \mathrm{id} \times p_1, I \times B_1, \mathrm{id} \times s_1)$.

Definition 5.6. An **M**-fibrewise pointed map $(u, \gamma) : (A, p_0, B_0, s_0) \to (X_1, p_1, B_1, s_1)$ is an **M**-*fibrewise pointed cofibration* if (u, γ) has the following property: Let $(\phi, \alpha) : (X_1, p_1, B_1, s_1) \to (X_2, p_2, B_2, s_2)$ be an **M**-fibrewise pointed map and $(H, h) : (I \times A, \mathrm{id} \times \gamma, I \times B_0, \mathrm{id} \times s_0) \to (X_2, p_2, X_2, s_2)$ an **M**-fibrewise pointed homotopy such that the following two diagrams

$$\begin{array}{ccc} A & \xrightarrow{\sigma_0} & I \times A \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle H} \\ X_1 & \xrightarrow{\phi} & X_2 \end{array}$$

$$\begin{array}{ccc} B_0 & \xrightarrow{\delta_0} & I \times B_0 \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle h} \\ B_1 & \xrightarrow{\alpha} & B_2 \end{array}$$

are commutative. Then there exists an **M**-fibrewise pointed homotopy $(K, k) : (I \times X_1, \mathrm{id} \times p_1, I \times B_1, \mathrm{id} \times s_1) \to (X_2, p_2, B_2, s_2)$ such that $K\kappa_0 = \phi$, $K(\mathrm{id} \times u) = H$, $k\rho_0 = \alpha$, $k(\mathrm{id} \times \gamma) = h$, where $\kappa_0 : X_1 \to I \times X_1$ and $\rho_0 : B_1 \to I \times B_1$ are defined by $\kappa_0(x) = (0, x)$ and $\rho_0(b) = (0, b)$ for $x \in X_1$, $b \in B_1$.

For an **M**-fibrewise pointed map $(u,\gamma):(A,p_0,B_0,s_0)\to(X_1,p_1,X_1,s_1)$, we can construct the **M**-fibrewise push out (M,p,B,s) of the cotraids

$$I\times A \xleftarrow{\sigma_0} A \xrightarrow{u} X_1$$

$$I\times B_0 \xleftarrow{\delta_0} B_0 \xrightarrow{\gamma} B_1$$

by the same methods in Section 3. In this case, it is enough to add that $s:B\to M$ is defined by

$$s(b)=\begin{cases}[s_0(b')] & \text{if } b=[\gamma(b')],\ b'\in B_0,\\ [t,s_0(b')] & \text{if } b=[t,b'],\ t\neq 0,\ b'\in B_0,\\ [s_1(b)] & \text{if } b\in B_1-\gamma(B_0).\end{cases}$$

Lemma 5.4. *The map s is a section. So (M,p,B,s) is an* **M***-fibrewise pointed space.*

In the case in which (A,p_0,B_0,s_0) is an **M**-fibrewise pointed subspace of (X_1,p_1,B_1,s_1) such that $p_1(A)\subset B_0$ and (u,γ) is inclusion, by the same methods as Section 3, we can define an **M**-fibrewise pointed map $(e,\varepsilon):(M,p,B,s)\to(\{0\}\times X_1\cup I\times A,\mathrm{id}\times p_1,\{0\}\times B_1\cup I\times B_0,\mathrm{id}\times s_1)$. Further it is obvious that e and ε are homeomorphisms.

We use the same notation as Section 3.

Theorem 5.5. *An* **M***-fibrewise pointed map $(u,\gamma):(A,p_0,B_0,s_0)\to(X_1,p_1,B_1,s_1)$ is an* **M***-fibrewise pointed cofibration if and only if there exists an* **M***-fibrewise pointed map $(L,l):(I\times X_1,\mathrm{id}\times p_1,I\times B_1,\mathrm{id}\times s_1)\to(M,p,B,s)$ such that $Lk=\mathrm{id}_M$, $l\xi=\mathrm{id}_B$.*

Corollary 5.6. *Let $((X_1,p_1,B_1,s_1),(A,p_0,B_0,s_0))$ be a closed* **M***-fibrewise pointed cofibred pair. Then so is*

$$\big((T\times X_1,\mathrm{id}\times p_1,T\times B_1,\mathrm{id}\times s_1),(T\times A,\mathrm{id}\times p_0,T\times B_0,\mathrm{id}\times s_0)\big)$$

for any topological space T.

Theorem 5.7. *Let $(u_0,\gamma_0):(A,p_0,B_0,s_0)\to(X_1,p_1,B_1,s_1)$, $(u_1,\gamma_1):(A,p_0,B_0,s_0)\to(X_2,p_2,B_2,s_2)$ be* **M***-fibrewise pointed maps and $(\phi,\alpha):(X_1,p_1,B_1,s_1)\to(X_2,p_2,B_2,s_2)$ an* **M***-fibrewise pointed map such that $(\phi u_1,\alpha\gamma_1)\simeq^{\mathbf{M}}_{(\mathbf{P})}(u_2,\gamma_2)$. If (u_1,γ_1) is an* **M***-fibrewise pointed cofibration, then there exists an* **M***-fibrewise pointed map $(\psi,\beta):(X_1,p_1,B_1,s_1)\to(X_2,p_2,B_2,s_2)$ such that $\psi u_1=u_2$, $\beta\gamma_1=\gamma_2$.*

Theorem 5.8. *Let $(u,\gamma):(A,p_0,B_0,s_0)\to(X_1,p_1,B_1,s_1)$ be an* **M***-fibrewise pointed cofibration. Let $(\theta,\alpha):(X_1,p_1,B_1,s_1)\to(X_1,p_1,B_1,s_1)$ be an* **M***-fibrewise pointed map under (A,p_0,B_0,s_0) such that $(\theta,\alpha)\simeq^{\mathbf{M}}_{(\mathbf{P})}(\mathrm{id}_{X_1},\mathrm{id}_{B_1})$. Then there exists an* **M***-fibrewise pointed map $(\theta',\alpha'):(X_1,p_1,B_1,s_1)\to(X_1,p_1,B_1,s_1)$ under (A,p_0,B_0,s_0) such that $(\theta'\theta,\alpha'\alpha)\simeq^{\mathbf{M}}_{(\mathbf{P})}(\mathrm{id}_{X_1},\mathrm{id}_{B_1})$ under (A,p_0,B_0,s_0).*

Theorem 5.9. *Let $(u_1,\gamma_1):(A,p_0,B_0,s_0)\to(X_1,p_1,B_1,s_1)$ and $(u_2,\gamma_2):(A,p_0,B_0,s_0)\to(X_2,p_2,B_2,s_2)$ be* **M***-fibrewise pointed cofibrations. Let $(\phi,\alpha):(X_1,p_1,B_1,s_1)\to$*

(X_2, p_2, B_2, s_2) be an **M***-fibrewise pointed map such that $(\phi u_1, \alpha\gamma_1) = (u_2, \gamma_2)$. Suppose that (ϕ, α) is an* **M***-fibrewise pointed homotopy equivalence. Then (ϕ, α) is an* **M***-fibrewise pointed homotopy equivalence under (A, p_0, B_0, s_0).*

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