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On the Oscillatory Behavior of a Class of Linear Third Order Differential Equations

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1. INTRODUCTION

Our main purpose here is to answer a question which was raised in [3]. In [3] it was shown that many of the asymptotic and oscillatory properties of solutions of linear third order differential equations with constant coefficients are carried over to those with variable coefficients provided that the coefficients do not change sign.

The question answered here is motivated by the following situation: Consider the differential equation

$$y''' = ay'' + by' + cy, \tag{1}$$

where $a \geqslant 0$, $b \geqslant 0$, and $c > 0$ are constants. By the rule of signs, the characteristic polynomial $\lambda^3 - a\lambda^2 - b\lambda - c$ has one and only one positive real root γ and either two complex conjugate roots $\alpha + i\beta$, $\alpha - i\beta$, where

$$2\alpha = -\frac{b + \alpha^2 + \beta^2}{\gamma} < 0,$$

or two negative real roots counting multiplicities. In the first case every solution of (1) has the form

$$c_1 e^{\gamma t} + e^{\alpha t}(c_2 \cos \beta t + c_3 \sin \beta t).$$

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Consequently, a real nontrivial solution will have arbitrarily large zeros if and only if $c_1 = 0$ while if $c_1 \neq 0$ the solution and all of its derivatives will have the same sign from a certain point on. In the second case no nontrivial solution has infinitely many zeros and there exist solutions whose derivatives alternate in sign.

We consider real solutions of the differential equation

$$y''' = p(t)\, y'' + q(t)\, y' + r(t)\, y, \tag{2}$$

where p, q and r are continuous on a ray $a \leqslant t < \infty$ and

$$p(t) \geqslant 0, \qquad q(t) \geqslant 0, \qquad r(t) > 0, \qquad t \in [a, \infty). \tag{3}$$

A nontrivial solution of (2) is *oscillatory* if its set of zeros is not bounded above.

Our main result which is motivated by the constant coefficient case is the following:

THEOREM 1. *Under the assumptions* (3) *the following two conditions are equivalent:*

A. *There exists an oscillatory solution of* (2).

B. *If w is a nontrivial nonoscillatory solution of* (2), *then there exists a number $t_0 \geqslant a$ such that $w(t)\, w'(t)\, w''(t) \neq 0$ for $t \geqslant t_0$ and*

$$\operatorname{sgn} w(t) = \operatorname{sgn} w'(t) = \operatorname{sgn} w''(t), \qquad t \geqslant t_0 . \tag{4}$$

This result was conjectured in [3] and established under the additional restrictions $p(t) \equiv 0$, $2r(t) - q'(t) \geqslant 0$, and

$$\int_0^{\infty} t^4[2r(t) - q'(t)]\, dt = + \infty.$$

The following result will be a direct consequence of the proof of Theorem 1.

THEOREM 2. *If conditions* (3) *hold and there exists one oscillatory solution of* (2), *then there exist two linearly independent oscillatory solutions u and v of* (2) *such that any nontrivial linear combination of u and v is also oscillatory and the zeros of u and v separate,* i.e. *between every two consecutive zeros of u there is precisely one zero of v.*

If p is twice differentiable then the well known substitution

$$y(t) = z(t) \exp \left[- \tfrac{1}{3} \int_a^t p(s)\, ds \right]$$

transforms the differential equation (2) into the form

$$z''' = Q(t)\, z' + R(t)\, z. \tag{5}$$

Oscillatory properties remain invariant under this transformation. For the special differential equation (5) and with one more additional assumption we can give a more complete description of the oscillatory case.

THEOREM 3. *If*

$$Q(t) \geqslant 0, \qquad R(t) > 0, \qquad 2R(t) - Q'(t) \geqslant 0 \tag{6}$$

for $t \in [a, \infty)$ *and* (5) *has an oscillatory solution, then there exist two linearly independent oscillatory solutions* u *and* v *whose zeros separate and such that a solution of* (5) *is oscillatory if and only if it is a nontrivial linear combination of* u *and* v. *If* w *is a nontrivial solution of* (5) *which is not a linear combination of* u *and* v, *then*

$$\lim_{t\to\infty} | w(t) | = \lim_{t\to\infty} | w'(t) | = \infty.$$

It is still an open question whether or not the same type of behavior in the oscillatory case is true for the general differential equation (2) under assumptions (3).

A good resumé of the current status of the oscillation theory of third order linear differential equations may be found either in the recent book by C. A. Swanson [4] or the lecture notes by J. H. Barret [1].

2. PROOFS OF THEOREMS 1 AND 2

For convenience in proving Theorem 1 we state two lemmas. The proof of the first is essentially the same as the proof of a similar lemma in ([3], p. 447).

LEMMA 1. *If conditions* (3) *hold and* w *is a solution of* (2) *with* $w(t_0) \geqslant 0$, $w'(t_0) \geqslant 0$ *and* $w''(t_0) > 0$ *for some* $t_0 \in [a, \infty)$, *then* $w(t) > 0$, $w'(t) > 0$, $w''(t) > 0$ *for* $t > t_0$. *Similarly if* $w(t_0) \leqslant 0$, $w'(t_0) \leqslant 0$, *and* $w''(t_0) < 0$, *then* $w(t) < 0$, $w'(t) < 0$, $w''(t) < 0$ *for* $t > t_0$.

LEMMA 2. *If conditions* (3) *hold and* z *is a nontrivial nonoscillatory solution of* (2), *then there exists a number* $t_1 \geqslant a$ *such that* $z'(t) \neq 0$ *for* $t \geqslant t_1$.

PROOF. If z is nonoscillatory, there exists a number t_0 such that $z(t) \neq 0$ for $t \geqslant t_0$. Since $-z$ is a solution of (2) we may assume without loss of generality that $z(t) > 0$ for $t \geqslant t_0$. If $\tau \geqslant t_0$ is a number such that $z'(\tau) = 0$ and $z''(\tau) \geqslant 0$, then

$$z'''(\tau) = p(\tau)\, z''(\tau) + r(\tau)\, z(\tau) > 0,$$

so $z(s) > 0$, $z'(s) > 0$, $z''(s) > 0$ for some $s > \tau$. Hence by Lemma 1, $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$ for $t > s$. This shows that $z'(t)$ can have at most two zeros on the interval $[t_0, \infty)$ and proves the lemma.

PROOF OF THEOREM 1. The proof that condition A implies condition B is essentially the same as that given for a special case in [3] but is repeated for completeness. Suppose w is a nonoscillatory solution of (2) and u is an oscillatory solution. By Lemma 2 there exists a number t_1 such that

$$w(t)\, w'(t) \neq 0 \qquad \text{for} \qquad t \geqslant t_1 . \tag{7}$$

Since

$$u'(t)\, w(t) - w'(t)\, u(t) = w(t)^2 \left[\frac{u(t)}{w(t)}\right]',$$

by Rolle's theorem there exists a number $s \geqslant t_1$ such that

$$u'(s)\, w(s) - w'(s)\, u(s) = 0.$$

Hence there exist numbers c_1 and c_2 such that

$$c_1 u(s) + c_2 w(s) = 0,$$
$$c_1 u'(s) + c_2 w'(s) = 0,$$
$$c_1^2 + c_2^2 = 1.$$

Let $z = c_1 u + c_2 w$. If $z''(s) = 0$, then since z is a solution of (2) it would follow by the uniqueness theorem for linear differential equations that $z(t) = 0$ for all t, contradicting the linear independence of the oscillatory solution u and the nonoscillatory solution w. Hence by changing the sign of c_1 and c_2, if necessary, we may assume $z''(s) > 0$. Since $z(s) = z'(s) = 0$ it follows by Lemma 1 that $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$ for $t > s$, and consequently,

$$z'''(t) = p(t)\, z''(t) + q(t)\, z'(t) + r(t)\, z(t) > 0$$

for $t > s$. Hence

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} z'(t) = +\infty. \tag{8}$$

By (7) either $w(t)\, w'(t) < 0$ or $w(t)\, w'(t) > 0$ for $t \geqslant t_1$. If $w(t)\, w'(t) < 0$ for $t \geqslant t_1$, then $w(t)$ would be bounded on $[t_1, \infty)$ and by (8)

$$\lim_{t\to\infty} c_1 u(t) = \lim_{t\to\infty} [z(t) - c_2 w(t)] = +\infty,$$

contradicting the assumption that u is oscillatory. Hence

$$\operatorname{sgn} w(t) = \operatorname{sgn} w'(t) \qquad \text{for} \qquad t \geqslant t_1 . \tag{9}$$

Now w'' can have at most one zero on the interval $[t_1, \infty)$. Indeed if $w''(t_2) = 0$ and $t_2 \geqslant t_1$ then since $q(t_2) \geqslant 0$ and $r(t_2) > 0$,

$$\operatorname{sgn} w'''(t_2) = \operatorname{sgn} [q(t_2)\, w'(t_2) + r(t_2)\, w(t_2)] = \operatorname{sgn} w(t_2).$$

Thus for some $t_3 > t_2$

$$\operatorname{sgn} w''(t_3) = \operatorname{sgn} w'(t_3) = \operatorname{sgn} w(t_3),$$

and by Lemma 1, $w''(t) \neq 0$ for $t > t_3$. Hence, there exists a number $\tau > t_1$ such that $w'(t)\, w''(t) \neq 0$ for $t \geqslant \tau$. If $w'(t)\, w''(t) < 0$ for $t > \tau$, then $w'(t)$ would be bounded on $[\tau, \infty)$ and consequently by (8)

$$\lim_{t\to\infty} c_1 u'(t) = \lim_{t\to\infty} [z'(t) - c_2 w'(t)] = +\infty,$$

contradicting the assumption that u is oscillatory. This shows that

$$\operatorname{sgn} w(t) = \operatorname{sgn} w'(t) = \operatorname{sgn} w''(t) \qquad \text{for} \qquad t \geqslant \tau$$

and completes the proof of the first half of Theorem 1.

REMARK. If w is a solution of (2) which satisfies the initial conditions

$$w(\tau) = w'(\tau) = 0, \qquad w''(\tau) > 0, \qquad \tau \in [a, \infty)$$

arbitrary, then by Lemma 1, $w(t) > 0$, $w'(t) > 0$, $w''(t) > 0$ for $t \geqslant t_0 > \tau$; so (2) always has nonoscillatory solutions satisfying (4), regardless of whether or not (2) has an oscillatory solution.

The proof that condition B implies condition A is much more interesting and employs a technique used by S. P. Hastings and the second author in [2]. Suppose condition B holds. Let z_0, z_1 and z_2 be the solutions of (2) defined by the initial conditions

$$z_k^{(j)}(a) = \delta_{jk} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases} \tag{10}$$

for $j, k = 0, 1, 2$.

For each positive integer $n > a$, let a_{0n}, a_{2n}, b_{1n}, b_{2n} be numbers such that

$$a_{0n} z_0(n) + a_{2n} z_2(n) = 0, \tag{11}$$

$$b_{1n} z_1(n) + b_{2n} z_2(n) = 0, \tag{12}$$

$$a_{0n}^2 + a_{2n}^2 = b_{1n}^2 + b_{2n}^2 = 1. \tag{13}$$

Define for each $n > a$ solutions

$$\begin{aligned} u_n &= a_{0n} z_0 + a_{2n} z_2, \\ v_n &= b_{1n} z_1 + b_{2n} z_2. \end{aligned} \tag{14}$$

By (13) there exists a sequence of integers $\{n_k\}$ such that

$$\lim_{n_k\to\infty} a_{0n_k} = a_0, \qquad \lim_{n_k\to\infty} a_{2n_k} = a_2,$$

$$\lim_{n_k\to\infty} b_{1n_k} = b_1, \qquad \lim_{n_k\to\infty} b_{2n_k} = b_2$$

and

$$a_0^2 + a_2^2 = b_1^2 + b_2^2 = 1. \tag{15}$$

If

$$u = a_0 z_0 + a_2 z_2, \qquad v = b_1 z_1 + b_2 z_2, \tag{16}$$

then for each fixed $t \in [a, \infty)$

$$\lim_{n_k\to\infty} u_{n_k}^{(j)}(t) = u^{(j)}(t), \qquad \lim_{n_k\to\infty} v_{n_k}^{(j)}(t) = v^{(j)}(t), \tag{17}$$

$j = 0, 1, 2$. We assert that u and v are oscillatory solutions of (2). Assume to the contrary that u is nonoscillatory. By (13) and the independence of z_0 and z_2, u is nontrivial; so condition B implies the existence of a number $t_0 \geqslant a$ such that $u(t_0)\, u'(t_0)\, u''(t_0) \neq 0$ and

$$\operatorname{sgn} u(t_0) = \operatorname{sgn} u'(t_0) = \operatorname{sgn} u''(t_0).$$

Hence by (17) there exists an integer N such that $n_k \geqslant N$ implies

$$u_{n_k}(t_0)\, u'_{n_k}(t_0)\, u''_{n_k}(t_0) \neq 0,$$

$$\operatorname{sgn} u_{n_k}(t_0) = \operatorname{sgn} u'_{n_k}(t_0) = \operatorname{sgn} u''_{n_k}(t_0).$$

Consequently, by Lemma 1, $u_{n_k}(t) \neq 0$ for $t \geqslant t_0$ and $n_k \geqslant N$. On the other hand for all $n_k > \max[N, t_0]$ it follows by (11) that $u_{n_k}(n_k) = 0$ and we arrive at a contradiction. This proves that u is oscillatory and a similar argument shows that v is oscillatory.

If u and v were not linearly independent then by (15), (16), and the independence of z_0, z_1 and z_2 it would follow that

$$u = \pm z_2, \qquad v = \pm z_2.$$

But $z_2(a) = z_2'(a) = 0$, $z_2''(a) = 1$; so by Lemma 1, $z_2(t) \neq 0$ for $t > a$. This contradicts the fact that u and v are oscillatory and proves the independence of u and v.

We have thus shown that condition B implies the existence of two linearly independent oscillatory solutions u and v of (2) and hence condition A. This proves the second half of Theorem 1.

We now prove some further properties of the solutions u and v. We assert that any nontrivial linear combination of u and v is also oscillatory. Let $y = c_1 u + c_2 v$ with $c_1^2 + c_2^2 \neq 0$. Since u and v are independent, y is nontrivial; so if y were a nonoscillatory solution of (2) there would exist a number t_0 such that $y(t_0)\, y'(t_0) y''(t_0) \neq 0$ and

$$\operatorname{sgn} y(t_0) = \operatorname{sgn} y'(t_0) = \operatorname{sgn} y''(t_0).$$

If the sequences

$$\{u_{n_k}\}_{k=1}^{\infty} \qquad \text{and} \qquad \{v_{n_k}\}_{k=1}^{\infty}$$

are defined as previously and

$$y_{n_k} = c_1 u_{n_k} + c_2 v_{n_k},$$

then

$$\lim_{n_k \to \infty} y_{n_k}^{(j)}(t_0) = y^{(j)}(t_0).$$

Thus, since by (11) and (14), $y_{n_k}(n_k) = 0$, we arrive at a contradiction in exactly the same way we did in proving that u is oscillatory. Therefore y must be oscillatory.

Instead of proving directly that the zeros of u and v separate we will prove the stronger result that *if y_1 and y_2 are two nontrivial linear combinations of u and v which are independent then the zeros of y_1 and y_2 separate.* We first show that if

$$G(t) = u(t)\, v'(t) - v(t)\, u'(t),$$

then $G(t) \neq 0$ for $t \in [a, \infty)$. Assuming on the contrary that $G(s) = 0$ for some $s \in [a, \infty)$, there would exist constants c_1 and c_2 such that

$$c_1 u(s) + c_2 v(s) = 0,$$
$$c_1 u'(s) + c_2 v'(s) = 0,$$
$$c_1^2 + c_2^2 \neq 0.$$

If $y = c_1 u + c_2 v$ then by the independence of u and v, $y''(s) \neq 0$. But since $y(s) = y'(s) = 0$, by Lemma 1, $y(t) \neq 0$ for $t > s$ contradicting the previously established fact that y must be oscillatory. This proves that $G(t) \neq 0$ for all $t \in [a, \infty)$.

If y is any linear combination of u and v, then

$$\begin{vmatrix} y(t) & y'(t) & y''(t) \\ u(t) & u'(t) & u''(t) \\ v(t) & v'(t) & v''(t) \end{vmatrix} = 0$$

for all t; so if

$$H(t) = u'(t)\, v''(t) - v'(t)\, u''(t)$$

then

$$G(t)\, y'' - G'(t)\, y' + H(t)\, y = 0,$$

or equivalently

$$\left[\frac{y'}{G(t)}\right]' + \frac{H(t)}{G(t)^2}\, y = 0.$$

Thus any linear combination of u and v is a solution of a nonsingular second order differential equation of the Sturm type. It therefore follows by a well known theorem that if y_1 and y_2 are any two linear combinations of u and v which are linearly independent, then the zeros of y_1 and y_2 must separate.

Theorem 2 now follows by simple logic. Suppose assumptions (3) are satisfied and (2) has an oscillatory solution, or equivalently condition A holds. Condition B therefore holds and, as shown above, condition B implies the existence of solutions u and v which have the properties stated in Theorem 2.

We conclude this section with the following characterization of the nonoscillatory case.

COROLLARY TO THEOREM 1. *If assumptions* (3) *hold then a necessary and sufficient condition that* (2) *has no oscillatory solutions is that there exist a solution* z *of* (2) *such that either*

$$z(t)\, z'(t) < 0, \qquad t \geqslant \tau \tag{18}$$

or,

$$z'(t)\, z''(t) < 0, \qquad t \geqslant \tau \tag{19}$$

for some $\tau \in [a, \infty)$.

PROOF. The sufficiency is a direct consequence of Theorem 1. Indeed if such a solution exists, condition B does not hold and hence condition A does not hold.

To see that the existence of such a solution is necessary we observe that if there exists no oscillatory solution of (2), there must exists a nonoscillatory solution z which does not satisfy (4). By Lemma 2 there exists a number t_0 such that $z'(t)\, z(t) \neq 0$ for $t \geqslant t_0$. If $z'(t)\, z(t) > 0$ for $t \geqslant t_0$, then referring back to the proof of Theorem 1 we see that there exists a number $\tau \geqslant t_0$ such that $z'(t)\, z''(t) \neq 0$ for $t \geqslant \tau$. Therefore, since z does not satisfy (4), $z'(t)\, z''(t) < 0$ for $t \geqslant t_0$ and we have a solution which satisfies (19). This proves the corollary.

3. Proof of Theorem 3

To prove Theorem 3 we make use of the following result established in ([3], p. 451).

Lemma 3. *If* $Q(t) > 0$, $R(t) \geqslant 0$, $2R(t) - Q'(t) \geqslant 0$ *for* $t \in [a, \infty)$, *then the derivative of an oscillatory solution of*

$$y''' = Q(t)\, y' + R(t)\, y \tag{20}$$

is bounded on $[a, \infty)$.

Suppose now that the conditions of Lemma 3 are satisfied and (20) has an oscillatory solution. Since the conditions of Theorem 2 are satisfied there exist two linearly independent oscillatory solutions u and v of (20) such that the zeros of u and v separate and such that any nontrivial linear combination of u and v is also oscillatory. Moreover u' and v' are bounded on $[a, \infty)$.

Let z be the solution of (20) which satisfies the initial conditions $z(a) = z'(a) = 0$, $z''(a) = 1$. By Lemma 1, $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, and hence by (20), $z'''(t) > 0$ for $t > a$. Consequently

$$\lim_{t \to +\infty} z(t) = \lim_{t \to +\infty} z'(t) = +\infty. \tag{21}$$

Since z is nonoscillatory, z is not a linear combination of u and v which implies that u, v and z are linearly independent. Hence every solution of (20) is a linear combination of u, v, and z.

If w is a solution such that $w = c_1 u + c_2 v + c_3 z$ with $c_3 \neq 0$, then by (21) and the boundedness of u' and v',

$$\lim_{t \to +\infty} | w(t) | = \lim_{t \to +\infty} | w'(t) | = +\infty$$

so w is nonoscillatory. This completes the proof of Theorem 3.

References

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