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J. Math. Anal. Appl. 341 (2008) 540–547

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

# On the existence of $T$-direction of algebroid functions: A problem of J.H. Zheng ☆

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Received 25 May 2007

Available online 13 November 2007

Submitted by A.V. Isaev

**Abstract**

In this paper, we solve a problem of J.H. Zheng (see Problem 5.12 of [J.H. Zheng, On value distribution of meromorphic functions with respect to arguments, preprint]) by proving that for any $\nu$-valued algebroid function satisfying $\limsup_{r\to\infty}\frac{T(r,w)}{\log^2 r}=+\infty$, there exists a $T$-direction dealing with multiple values of $w(z)$.



*Keywords:* Algebroid function; $T$-direction; Multiple values

## 1. Introduction and main results

The value distribution theory of meromorphic functions due to R. Nevanlinna (see [3] for standard references) was extended to the corresponding theory of algebroid functions by H. Selberg [8], E. Ullrich [10] and G. Valiron [11] around 1930. The singular direction for $w(z)$ is one of the main objects studied in the theory of value distribution of algebroid functions. Several types of singular directions have been introduced in the literature. Their existence and some connections between them have also been established. G. Valiron [12] conjectured that there exists at least one Borel direction for any $\nu$-valued algebroid function of order $\rho$ $(0<\rho<+\infty)$. A. Rauch [7] proved that there exists a direction such that the corresponding Borel exceptional values form a set of linear measure zeros. N. Toda [9] proved that there exists a direction such that the set of corresponding Borel exceptional values is countable. Later Y.N. Lü and Y.X. Gu [5] proved that there exists a direction such that the number of Borel exceptional values is equal to $2\nu$ at most.

For a meromorphic function $f(z)$, J.H. Zheng [17] introduced a new singular direction, namely a $T$-direction, and conjectured that a transcendental meromorphic function $f(z)$ must have at least one $T$-direction, provided that

$$\limsup_{r\to\infty}\frac{T(r,f)}{\log^2 r}=+\infty.$$

☆ The work is supported in part by the National Key Basic Research Project of China Grant No. 2005CB321902.

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0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2007.10.032

This result was later proved by H. Guo, J.H. Zheng and T. Ng [2] by using Ahlfors–Shimizu character $T(r,\Omega)$ of a meromorphic function in an angular domain $\Omega$. A recent work of Q.D. Zhang [16] shows that the connection between $T$-direction and Borel direction. Thus a natural question is: *Are there similar results for algebroid functions* (this problem was raised by J.H. Zheng in [18])? In this paper we investigate this problem.

Let $w = w(z)$ $(z \in \mathbb{C})$ be the $\nu$-valued algebroid function defined by the irreducible equation

$$A_\nu(z)w^\nu + A_{\nu-1}(z)w^{\nu-1} + \cdots + A_0(z) = 0, \tag{1.1}$$

where $A_\nu(z), \ldots, A_0(z)$ are analytic functions without any common zeros. The single-valued domain $\widetilde{R}_z$ of definition of $w(z)$ is a $\nu$-valued covering of the $z$-plane and it is a Riemann surface.

A point in $\widetilde{R}_z$ is denoted by $\tilde{z}$ if its projection in the $z$-plane is $z$. The open set which lies over $|z| < r$ is denoted by $|\tilde{z}| < r$. Let $n(r,a)$ be the number of zeros, counted according to their multiplicities, of $w(z) - a$ in $|\tilde{z}| \leqslant r$, $\overline{n}^{l)}(r,a)$ be the number of distinct zeros with multiplicity $\leqslant l$ of $w(z) - a$ in $|\tilde{z}| \leqslant r$. Let

$$S(r,w) = \frac{1}{\pi}\iint\limits_{|\tilde{z}|\leqslant r}\left[\frac{|w'(z)|}{1+|w(z)|^2}\right]^2 d\omega = \frac{1}{\pi}\int_0^{2\pi}\int_0^r\left(\frac{|w'(re^{i\theta})|}{1+|w(re^{i\theta})|^2}\right)^2 r\,dr\,d\theta,$$

$$T(r,w) = \frac{1}{\nu}\int_0^r \frac{S(t,w)}{t}\,dt,$$

$$N(r,a) = \frac{1}{\nu}\int_0^r \frac{n(t,a) - n(0,a)}{t}\,dt + \frac{n(0,a)}{\nu}\log r,$$

$$\overline{N}^{l)}(r,a) = \frac{1}{\nu}\int_0^r \frac{\overline{n}^{l)}(t,a) - \overline{n}^{l)}(0,a)}{t}\,dt + \frac{\overline{n}^{l)}(0,a)}{\nu}\log r,$$

$$m(r,w) = \frac{1}{2\pi\nu}\int\limits_{|\tilde{z}|=r}\log^+\bigl|w\bigl(re^{i\theta}\bigr)\bigr|\,d\theta, \quad z = re^{i\theta},$$

where $|\tilde{z}| = r$ is the boundary of $|\tilde{z}| \leqslant r$. Moreover, $S(r,w)$ is a conformal invariant and is called the mean covering number of $|\tilde{z}| \leqslant r$ into $w$-sphere. We call $T(r,w)$ the characteristic function of $w(z)$. It is known from [4, $3^o$, p. 84] that $T(r,w) = m(r,w) + N(r,\infty) + O(1)$.

Let $n_\chi(r,w)$ be the number of the branch points of $\widetilde{R}_z$ in $|\tilde{z}| \leqslant r$, counted with the order of branch. Write

$$N_\chi(r,w) = \frac{1}{\nu}\int_0^r \frac{n_\chi(t,w) - n_\chi(0,w)}{t}\,dt + \frac{n_\chi(0,w)}{\nu}\log r.$$

By [4, Lemma 2.4, p. 87] we have

$$N_\chi(r,w) \leqslant 2(\nu - 1)T(r,w) + O(1).$$

We denote $\{z\colon\ |z| < r,\ \varphi_1 < \arg z < \varphi_2\}$ by $\Omega(r,\varphi_1,\varphi_2)$ and write $\widetilde{\Omega}$ for the part of $\widetilde{R}_z$ on $\Omega(r,\varphi_1,\varphi_2)$. Let

$$S(r,\varphi_1,\varphi_2;w) = \frac{1}{\pi}\iint\limits_{\widetilde{\Omega}}\left[\frac{|w'(z)|}{1+|w(z)|^2}\right]^2 d\omega,$$

$$T(r,\varphi_1,\varphi_2;w) = \frac{1}{\nu}\int_0^r \frac{S(t,\varphi_1,\varphi_2;w)}{t}\,dt.$$

$\overline{n}^{l)}(r,\varphi_1,\varphi_2;w=a)$ denotes the numbers of $w(z) - a$ in $\widetilde{\Omega}$. $n_\chi(r,\varphi_1,\varphi_2)$ denotes the number of the branch points of $\widetilde{R}_z$ in $\widetilde{\Omega}$. Similarly, $\overline{N}^{l)}(r,\varphi_1,\varphi_2;w=a)$ denotes the counting function of zeros of $w(z) - a$. $N_\chi(r,\varphi_1,\varphi_2)$ denotes the counting function of the branch points of $\widetilde{R}_z$ in $\widetilde{\Omega}$.

We define an angular domain

$$\Delta(\theta_0, \varepsilon) = \{z \mid |\arg z - \theta_0| < \varepsilon\}, \quad 0 < \varepsilon < \frac{\pi}{2}.$$

$\overline{N}^{l)}(r, \Delta(\theta_0, \varepsilon), a)$ denotes the counting function of zeros of $w(z) - a$ in $\Delta(\theta_0, \varepsilon)$.

**Definition.** Let $w = w(z)$ ($z \in \mathbb{C}$) be the $\nu$-valued algebroid function defined by (1.1) and $l$ ($\geqslant 2\nu + 1$) be a positive integer. For arbitrary $\varepsilon > 0$ ($0 < \varepsilon < \frac{\pi}{2}$), if

$$\limsup_{r\to\infty} \frac{\overline{N}^{l)}(r, \Delta(\theta_0, \varepsilon), a)}{T(r, w)} > 0$$

holds for any complex value $a$ except at most $2\nu$ possible exceptions, then the half line $B$: $\arg z = \theta_0$ ($0 \leqslant \theta_0 < 2\pi$) is called a *$T$-direction dealing with multiple values of $w(z)$*.

In this paper, by using Ahlfors' theory of covering surfaces, we give a positive answer by proving

**Theorem 1.** *Let $w = w(z)$ ($z \in \mathbb{C}$) be the $\nu$-valued algebroid function defined by* (1.1)*, $l$ ($\geqslant 2\nu + 1$) be a positive integer, satisfying that*

$$\limsup_{r\to\infty} \frac{T(r, w)}{\log^2 r} = +\infty,$$

*then there exists a $T$-direction dealing with multiple values of $w(z)$.*

**Remark.** It is clear that a $T$-direction must be a Julia direction since

$$\overline{N}^{l)}\big(r, \Delta(\theta_0, \varepsilon), a\big) \leqslant \bar{n}^{l)}\big(r, \Delta(\theta_0, \varepsilon), a\big) \cdot \log r$$

and

$$\limsup_{r\to\infty} \frac{\log r}{T(r, w)} = 0$$

by the fact that $w(z)$ is transcendental. N. Toda [9] gave a transcendental algebroid function without Julia directions, Lü [6] calculated the $T(r, w)$ of N. Toda's example and pointed that $T(r, w) = O(\log^2 r)$. Therefore this example shows that the growth condition is sharp.

The set $E = \{\theta\colon \theta$ is a $T$-direction of $w(z)\}$ is a non-empty closed subset of $[0, 2\pi)$.

**Theorem 2.** *Let $w = w(z)$ ($z \in \mathbb{C}$) be the $\nu$-valued algebroid function defined by* (1.1)*, $l$ ($\geqslant 2\nu + 1$) be a positive integer, satisfying that*

$$\limsup_{r\to\infty} \frac{T(2r, w)}{T(r, w)} > 1,$$

*then there exists a $T$-direction dealing with multiple values of $w(z)$.*

## 2. The proof of the theorems

In order to prove our theorems, we need three lemmas.

**Lemma 1.** *Let $F(r)$ be a positive nondecreasing function defined for $1 < r < +\infty$ and satisfies*

$$\limsup_{r\to\infty} \frac{F(r)}{\log^2 r} = +\infty. \tag{2.1}$$

*Then, for any subset $E \subset (1, +\infty)$ satisfying $\int_E \frac{dr}{r\log r} < \frac{1}{2}$,*

$$\limsup_{r\to\infty, r\in(1,+\infty)\setminus E} \frac{F(r)}{\log^2 r} = +\infty.$$

**Proof.** Otherwise, there exists some set $E \subset (1, +\infty)$ with $\int_E \frac{dr}{r\log r} := A < \frac{1}{2}$, such that

$$\limsup_{r\to\infty,\, r\in(1,+\infty)\setminus E} \frac{F(r)}{\log^2 r} < +\infty. \tag{2.2}$$

For any $\{r_n'\} \subset (1, +\infty)$, $\{r_n'\} \to \infty$, we have

$$\begin{aligned}
\int\limits_{[r_n', 2(r_n')^2]\setminus E} \frac{dr}{r\log r} &\geqslant \int\limits_{[r_n', 2(r_n')^2]} \frac{dr}{r\log r} - \int\limits_{E} \frac{dr}{r\log r} \\
&= \int\limits_{\log r_n'}^{\log 2 + 2\log r_n'} \frac{dr}{r} - A \\
&\geqslant \frac{\log 2 + \log r_n'}{\log 2 + 2\log r_n'} - A \\
&\geqslant \frac{1}{2} - A > 0.
\end{aligned}$$

Then there exists $r_n'' \in [r_n', 2(r_n')^2]\setminus E$. By the nondecreasing property of $F(r)$ we have

$$\frac{F(r_n'')}{\log^2 r_n''} \geqslant \frac{F(r_n')}{\log^2 (2r_n')^2} = \frac{F(r_n')}{\log^2 r_n'} \cdot \frac{1}{(2 + \frac{\log 2}{\log r_n'})^2}.$$

Combing this with (2.2) we have

$$\begin{aligned}
\limsup_{r_n'\to\infty} \frac{F(r_n')}{\log^2 r_n'} &= 4\limsup_{r_n'\to\infty} \frac{F(r_n')}{\log^2 r_n'} \cdot \frac{1}{(2 + \frac{\log 2}{\log r_n'})^2} \\
&\leqslant 4\limsup_{r_n''\to\infty} \frac{F(r_n'')}{\log^2 r_n''} \\
&\leqslant 4 \limsup_{r\to\infty,\, r\in(1,+\infty)\setminus E} \frac{F(r)}{\log^2 r} < +\infty.
\end{aligned}$$

Since $\{r_n'\}$ is arbitrary, the above inequality contradicts (2.1). Lemma 1 follows. □

**Lemma 2.** *(See [14, Lemma 2].) Let $w(z)$ be the $\nu$-valued algebroid function defined by* (1.1) *in $|z| < +\infty$. If $a_1, a_2, \ldots, a_q$ ($q \geqslant 3$) are $q$ distinct complex numbers in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, then for any $\varphi$, $0 < \varphi < \delta$, we have*

$$\begin{aligned}
\left(q - 2 - \frac{2}{l}\right) S(r, \varphi_0 - \varphi, \varphi_0 + \varphi; w) \leqslant{}& \sum_{j=1}^{q} \overline{n}^{l)}(r, \varphi_0 - \delta, \varphi_0 + \delta; w = a_j) \\
&+ \frac{l+1}{l} n_\chi(r, \varphi_0 - \delta, \varphi_0 + \delta) + \frac{2h^2\nu\pi}{(q - 2 - \frac{2}{l})(\delta - \varphi)} \log r \\
&+ \left(q - 2 - \frac{2}{l}\right) S(1, \varphi_0 - \varphi, \varphi_0 + \varphi; w) \\
&+ hL(1, \varphi_0 - \delta, \varphi_0 + \delta) + hL(r, \varphi_0 - \delta, \varphi_0 + \delta)
\end{aligned}$$

*and*

$$\begin{aligned}\left(q-2-\frac{2}{l}\right)T(r,\varphi_0-\varphi,\varphi_0+\varphi;w) \leqslant{} & \sum_{j=1}^{q}\overline{N}^{l)}(r,\varphi_0-\delta,\varphi_0+\delta;w=a_j)\\ & +\frac{l+1}{l}N_\chi(r,\varphi_0-\delta,\varphi_0+\delta)+\frac{2h^2\pi}{(q-2-\frac{2}{l})(\delta-\varphi)}\log^2 r\\ & +\frac{(q-2-\frac{2}{l})}{\nu}T(1,\varphi_0-\varphi,\varphi_0+\varphi;w)\\ & +\frac{(q-2-\frac{2}{l})}{\nu}S(1,\varphi_0-\delta,\varphi_0+\delta;w)\log r\\ & +\frac{h}{\nu}L(1,\varphi_0-\delta,\varphi_0+\delta)\log r\\ & +X(r,\varphi_0-\delta,\varphi_0+\delta;w),\end{aligned}$$

*where $h$ is a constant depending only on $\{a_1,a_2,\ldots,a_q\}$, $X(r,\varphi_0-\delta,\varphi_0+\delta;w)=\frac{h}{\nu}\int_1^r\frac{L(t,\varphi_0-\delta,\varphi_0+\delta)}{t}\,dt$, and satisfies that*

$$X(r,\varphi_0-\delta,\varphi_0+\delta;w)\leqslant h\sqrt{2\delta\pi T(r,\varphi_0-\delta,\varphi_0+\delta;w)}\log T(r,\varphi_0-\delta,\varphi_0+\delta;w)$$

*at most outside a set $E_\delta$ of $r$, where $E_\delta$ consists of a series of intervals and satisfies $\int_{E_\delta}\frac{1}{r\log r}\,dr<+\infty$.*

**Lemma 3.** *Let $w=w(z)$ $(z\in\mathbb{C})$ be the $\nu$-valued algebroid function defined by* (1.1)*, $l$ $(\geqslant 2\nu+1)$ be a positive integer and $m$ $(m>1)$ be a positive integer. Put $\varphi_0=0$, $\varphi_1=\frac{2\pi}{m},\ldots,\varphi_{m-1}=(m-1)\frac{2\pi}{m}$. Let*

$$\Delta(\varphi_i)=\left\{z\ \middle|\ |\arg z-\varphi_i|<\frac{3\pi}{m}\right\}\quad(0\leqslant i\leqslant m-1).$$

*Then there exists a $\Delta(\varphi_i)$ among $\Delta(\varphi_i)$ $(i=0,1,\ldots,m-1)$ such that*

$$\limsup_{r\to\infty}\frac{\overline{N}^{l)}(r,\Delta(\varphi_i),a)}{T(r,w)}>0$$

*for any value of $a$ with $2\nu$ possible exceptions.*

**Proof.** Suppose that the conclusion is false. Then for every $\Delta(\varphi_i)$ $(i=0,1,\ldots,m-1)$, there exists $q=2\nu+1$ exceptional values $\{a_i^j\}_{j=1}^q$ such that

$$\limsup_{r\to\infty}\frac{\overline{N}^{l)}(r,\Delta(\varphi_i),a_i^j)}{T(r,w)}=0. \tag{2.3}$$

Let $\beta$ be any positive integer. Put $\varphi_{i,k}=\frac{2\pi}{m}i+\frac{2k\pi}{\beta m}$, $0\leqslant i\leqslant m-1$, $0\leqslant k\leqslant\beta-1$. For any given numbers $r>1$, writing

$$\Delta_{i,k}=\bigl\{z\ \big|\ |z|<r,\ \varphi_{i,k}\leqslant\arg z<\varphi_{i,k+1}\bigr\}.$$

Then

$$\bigl\{|z|<r\bigr\}=\sum_{k=0}^{\beta-1}\sum_{i=0}^{m-1}\Delta_{i,k}.$$

There exists a $k_0$, without loss of generality, we may assume that $k_0=0$, such that

$$\sum_{i=0}^{m-1}n(\Delta_{i,0},\widetilde{R}_z)\leqslant\frac{1}{\beta}n_\chi(r,w).$$

Put

$$\begin{aligned}&\overline{\Delta}_i=\left\{z\ \middle|\ \frac{\varphi_{i,0}+\varphi_{i,1}}{2}\leqslant\arg z\leqslant\frac{\varphi_{i+1,0}+\varphi_{i+1,1}}{2}\right\},\\ &\Delta_i^0=\{z\mid\varphi_{i,0}<\arg z<\varphi_{i+1,1}\},\quad 0\leqslant i\leqslant m-1.\end{aligned}$$

Since $\Delta_i^0$ overlap $\Delta_{i,0}$ twice at most, then

$$\sum_{i=0}^{m-1} n\big(r, \Delta_i^0, \widetilde{R}_z\big) \leqslant \left(1+\frac{1}{\beta}\right) n_\chi(r, w).$$

By Lemma 2 we have

$$\left\{(2\nu+1)-2-\frac{2}{l}\right\} S(r, \overline{\Delta}_i, w) \leqslant \sum_{j=1}^{2\nu+1} \overline{n}^{l)}\big(r, \Delta_i^0, a_i^j\big) + \frac{l+1}{l} n_\chi\big(r, \Delta_i^0\big) + O(\log r) + h_i L(r, \varphi_{i,0}, \varphi_{i+1,1}).$$

Adding from $i=0$ to $m-1$, dividing both sides of this inequality by $r$ and then integrating both sides from 1 to $r$, we obtain the following inequality

$$\left\{(2\nu+1)-2-\frac{2}{l}\right\} T(r, w) \leqslant \sum_{j=1}^{2\nu+1} \sum_{i=0}^{m-1} \overline{N}^{l)}\big(r, \Delta_i^0, a_i^j\big) + \frac{l+1}{l}\left(1+\frac{1}{\beta}\right) N_\chi(r, w) + O\big(\log^2 r\big) + \sum_{i=0}^{m-1} X(r, \varphi_{i,0}, \varphi_{i+1,1}; w), \tag{2.4}$$

where $X(r, \varphi_{i,0}, \varphi_{i+1,1}; w) \leqslant h_i \sqrt{\frac{2\pi}{m}(1+\frac{1}{\beta})\pi T(r, \varphi_{i,0}, \varphi_{i+1,1}; w)} \log T(r, \varphi_{i,0}, \varphi_{i+1,1}; w)$ at most outside a set $E_i$ of $r$, where $E_i$ satisfies $\int_{E_i} \frac{1}{r \log r}\, dr < +\infty$ $(i = 0, 1, \ldots, m-1)$.

For any $i \in \{0, 1, \ldots, m-1\}$, we can choose $r_i > 0$ such that $T(r_i, \varphi_{i,0}, \varphi_{i+1,1}; w) > e^{2m}$. Then from the proof of Lemma 2 (see [14]) we have

$$\int_{E_i} \frac{1}{r \log r}\, dr \leqslant \frac{1}{\log T(r_i, \varphi_{i,0}, \varphi_{i+1,1}; w)} < \frac{1}{2m} < \frac{1}{2}.$$

Put $E = \bigcup_{i=0}^{m-1} E_i$, then

$$\begin{aligned}\int_E \frac{1}{r \log r}\, dr &\leqslant \sum_{i=0}^{m-1} \int_{E_i} \frac{1}{r \log r}\, dr \\ &\leqslant m \max_{0 \leqslant i \leqslant m-1} \int_{E_i} \frac{1}{r \log r}\, dr \\ &< m \cdot \frac{1}{2m} < \frac{1}{2}.\end{aligned}$$

Applying Lemma 1 to this set $E$ and $T(r, w)$, we obtain that

$$\limsup_{r \to \infty,\, r \in (1, +\infty) \setminus E} \frac{T(r, w)}{\log^2 r} = +\infty.$$

There exists $\{r_n\} \in (1, +\infty) \setminus E$,

$$\lim_{n \to \infty} \frac{T(r_n, w)}{\log^2 r_n} = +\infty.$$

For this sequence $\{r_n\}$, by (2.4) we have

$$\left\{(2\nu+1)-2-\frac{2}{l}\right\} T(r_n, w) \leqslant \sum_{j=1}^{2\nu+1} \sum_{i=0}^{m-1} \overline{N}^{l)}\big(r_n, \Delta_i^0, a_i^j\big) + \frac{l+1}{l}\left(1+\frac{1}{\beta}\right) N_\chi(r_n, w) + O\big(\log^2 r_n\big) + \sum_{i=0}^{m-1} X(r_n, \varphi_{i,0}, \varphi_{i+1,1}; w). \tag{2.5}$$

Note that $N_\chi(r_n, w) \leqslant 2(\nu - 1)T(r_n, w) + O(1)$ and from (2.3), dividing both sides of (2.5) by $T(r_n, w)$ and letting $n \to \infty$, we obtain

$$(2\nu + 1) - 2 - \frac{2}{l} \leqslant 2\frac{l+1}{l}(\nu - 1)\left(1 + \frac{1}{\beta}\right).$$

Letting $\beta \to \infty$ we get $l \leqslant 2\nu$. This contradicts $l \geqslant 2\nu + 1$ and Lemma 3 follows. □

Now we are in the position to prove our theorems.

**Proof of Theorem 1.** By Lemma 3, for any given positive integer $m$, there exists

$$\Delta_m = \left\{ z \;\middle|\; |\arg z - \theta_m| < \frac{3\pi}{m} \right\}$$

such that

$$\limsup_{r\to\infty} \frac{\overline{N}^{l)}(r, \Delta(\varphi_m), a)}{T(r, w)} > 0,$$

for any value of $a$ with $2\nu$ possible exceptions at most. By choosing a subsequence, we can assume that $\theta_m \to \theta_0$, when $m \to \infty$. Then $B$: $\arg z = \theta_0$ has the properties of Theorem 1. □

**Remark.** S.M. Wang and Z.S. Gao [13] proved the existence of $T$-direction under the condition

$$\limsup_{r\to\infty} \frac{T(r, w)}{\log^2 r} = +\infty, \qquad \liminf_{r\to\infty} \frac{\log T(r, w)}{\log r} = \mu < \infty.$$

In their paper, they mainly depend on their Lemma 1 (see also [15]) which needs the condition $\liminf_{r\to\infty} \frac{\log T(r,w)}{\log r} = \mu < \infty$. Our method here is essentially different from theirs.

**Proof of Theorem 2.** Since

$$T(r, w) = O\left(\log^2 r\right)$$

means

$$T(2r, w) \sim T(r, w),$$

so by hypothesis of Theorem 2, we have

$$\limsup_{r\to\infty} \frac{T(r, w)}{\log^2 r} = +\infty.$$

Thus we deduce Theorem 2 from Theorem 1 directly. □

**Remark.** It is not difficult to understand that $w(z)$ is a meromorphic function and $n_\chi(r, w) = 0$ if $\nu = 1$, thus $N_\chi(r, w) = 0$. We can get the $T$-direction of meromorphic functions dealing with multiple values. Using Lemma 2, we can find that we need not treat two cases like in [2, pp. 284–285]. This simplifies the proof of [2].

Next we give two meromorphic functions which have $T$-direction of meromorphic functions dealing with multiple values.

**Example 1.** Let $\Gamma(z)$ be the gamma function. From Proposition 7.3.6 of [1] we have

$$T(r, 1/\Gamma) = \left(1 + o(1)\right)\frac{1}{\pi} r \log r,$$

so that

$$T(2r, 1/\Gamma) > dT(r, 1/\Gamma), \quad d > 1.$$

By Theorem 2, we know that $1/\Gamma$ has at least one $T$-direction of meromorphic functions dealing with multiple values.

**Example 2.** From the proof of Corollary 6 of [19], we know that every transcendental meromorphic function $f(z)$ satisfying linear differential equation with rational coefficients must have at least one $T$-direction of meromorphic functions dealing with multiple values because of $T(2r, f) > dT(r, f)$, $d > 1$.

**Open problem.** In [14], the present author proved that for any $\nu$-valued algebroid function satisfying $\limsup_{r\to\infty} \frac{T(r,w)}{\log^2 r} = +\infty$, there exists a *Nevanlinna direction dealing with multiple values of* $w(z)$. Here we raise an interesting problem: *What is the relationship of these two singular directions*?

## Acknowledgments

The author expresses his sincere thanks to Professor J.H. Zheng for giving this problem and papers [2,17–19] to him in the course of Value Distribution and Complex Dynamics in Tsing Hua University in 2006. The author would also like to express his genuine gratitude to Professor Z.S. Gao and F.Z. Wang for their help.

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