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On the induction operation for shift subspaces and cellular automata as presentations of dynamical systems

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ABSTRACT

Consider the space of configurations from a finitely generated group to a finite alphabet. We look at the translation-invariant closed subsets of this space, and at their continuous transformations that commute with translations. It is well-known that such objects can be described "locally" via finite patterns and finitary functions; we are interested in re-using these descriptions with larger groups, a process that usually does not lead to objects isomorphic to the original ones. We first characterize, in terms of group actions, those dynamics that can be presented via structures like those above. We then prove that some properties of the "induced" entities can be deduced from those of the original ones, and vice versa. We finally show how to simulate the smaller structure into the larger one. Special attention is given to the class of sofic shifts.

1. Introduction

Cellular automata (briefly, CA) are presentations of global dynamics in local terms: the phase space is made of *configurations* on an underlying lattice structure, and the transition function is induced by a point-wise evolution rule, which changes the state at a node of the grid by only considering finitely many *neighboring* nodes. Originally, the only grids allowed were the hypercubic ones, identified with the group $\mathbb{Z}^d$ for some *dimension* $d > 0$, and all the alphabets were finite, though containing at least two elements; this shall be referred to as the *classical case* in the rest of the present paper.

With time, CA theory has borrowed concepts and tools from group theory, symbolic dynamics, and topology (cf. [4,5,8,14]). The lattice structure is provided by a *Cayley graph* of a finitely generated group: the "frames" of this class generalize those of the classical case, allowing more complicated grid geometries. Such broadening, however, preserves the requirement for finite neighborhoods, so that defining global evolution laws in local terms is still allowed. Moreover, the phase space can be a *subshift*, *i.e.*, it may leave out some configurations, but contains all of the *translates* of each of its elements, as well as the *limits* of sequences it contains. This choice of framework sets us farther from the use of CA as "parallel analogous to Turing machines"—the space of all configurations is uncountable, the subshift can be non-recursive, etc.—but simplifies reasoning about *simulations* between CA.

In this paper, which is an extended version of a work submitted to the LATA 2008 conference [2], we deal with two problems. The former, is to understand when a dynamical system can be described by a cellular automaton; the latter, is to study the phenomena which happen when a description for a subshift or a CA on a given group, is employed in the context provided by a *larger* group, in the sense that the old one is a *subgroup* of the new one. At the time of the conference, we were not aware of the paper by Ceccherini-Silberstein and Coornaert [3], which also deals with induction of CA on larger groups,

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and also considers a class of configuration spaces which is broader than the classical one. However, their work is focused on a broader class of *alphabets* without considering anything more general than the *full shift*; on the other hand, our own work is aimed towards the study of the most general subshifts, provided the alphabet remains finite.

For the first problem, a solution is found employing group theory: a dynamical system admits a CA presentation if and only if there exists a group action on it with special properties. It is also observed, on one hand how the new class of CA is strictly broader than the classical one; and on the other hand, how some key properties of classical CA are shared by the newer objects.

About the other one, we provide a lemma about mutual inclusion between images of shift subspaces via global CA functions, showing that it is preserved *either way* when switching between the smaller group and the larger one. This shall ensure that the operation of induction, performed by “recycling” the description of the old object (be it subshift or CA) in the new *context* given by the larger group, is not only well defined, but also independent on the specific description: in other words, the *induced* object only depends on the *inducing* object. We then show how several properties are transferred from the old objects to the new ones, some even either way as well; this is of interest, because the new spaces and dynamics are usually *richer* than the old one.

A simulation of the original automaton into the induced one is then explicitly constructed; this extends to the case of arbitrary, finitely generated groups the usual embedding of d-dimensional cellular automata into $(d+k)$-dimensional ones. This result, which goes in the same direction as Róka [14, Proposition 6], adds further details to the picture of CA presentation of dynamical systems: the class is not shrunk when the alphabet or the group are enlarged, even up to bijections (for alphabets) and isomorphisms (for groups). As a consequence of this fact, the free group on two generators contains enough “structure” to present any CA dynamics on any free group. Some remarks about *sofic shifts* are also made throughout the discussion.

The rest of the paper is organized as follows. Section 2 provides a background. Section 3 is about characterization for CA dynamics on subshifts. Section 4 contains the lemma of mutual inclusion which ensures that the induction operation is well defined. Section 5 deals with induced CA and how to embed the original CA into the induced one, together with several considerations for some special classes of subshifts. Conclusions and acknowledgements follow.

2. Background

A **dynamical system** (briefly, d.s.) is a pair (X, F) where the **phase space** X is a compact and metrizable topological space and the **evolution function** $F : X \to X$ is continuous. If $Y \subseteq X$ is closed and $F(Y) \subseteq Y$, then (Y, F) is a **subsystem** of (X, F). A **morphism** from a d.s. (X, F) to a d.s. (X', F') is a continuous $\vartheta : X \to X'$ such that $\vartheta \circ F = F' \circ \vartheta$; an **embedding** is an injective morphism, a **conjugacy** a bijective morphism.

Let G be a group. We write $H \leqslant G$ if H is a subgroup of G. If $H \leqslant G$ and $x\rho y$ iff $x^{-1}y \in H$, then ρ is an equivalence relation over G, whose classes are called the **left cosets** of H, one of them being H itself. If J is a **set of representatives** of the left cosets of H (one representative per coset) then $(j, h) \mapsto jh$ is a bijection between $J \times H$ and G.

A **(right) action** of G over a set X is a collection $\phi = \{\phi_g\}_{g\in G}$ of transformations of X (*i.e.*, $\phi_g : X \to X$ for every $g \in G$) such that $\phi_{gh} = \phi_h \circ \phi_g$ for all $g, h \in G$, and $\phi_{1_G} = \mathrm{id}_X$, the identity function of X. Observe that the ϕ_g's are invertible, with $(\phi_g)^{-1} = \phi_{(g^{-1})}$. When ϕ is clear from the context, $\phi_g(x)$ can be written x^g. Properties of functions (*e.g.*, continuity) are extended to actions by saying that ϕ has property P iff each ϕ_g has property P.

If G is a group and $S \subseteq G$, the **subgroup generated by** S is the set $\langle S\rangle$ of all $g \in G$ such that

$$g = s_1 s_2 \cdots s_n \tag{1}$$

for some $n \geqslant 0$, with $s_i \in S$ or $s_i^{-1} \in S$ for all i. S is a **set of generators** for G if $\langle S\rangle = G$; a group is **finitely generated** (briefly, f.g.) if it has a finite set of generators (briefly, f.s.o.g.). The **length** of $g \in G$ with respect to S is the least $n \geqslant 0$ such that (1) holds, and is indicated by $\|g\|_S$. The **distance** of g and h w.r.t. S is the length $d_S^G(g, h)$ of $g^{-1}h$; the **disk** of center g and radius R w.r.t. S is $D_{R,S}^G(g) = \{h \in G \mid d_S^G(g, h) \leqslant R\}$. In all such writings, G and/or S will be omitted if irrelevant or clear from the context; g, if equal to 1_G.

An **alphabet** is a finite set with two or more elements; all alphabets are given the discrete topology. A **configuration** is a map $c \in A^G$ where A is an alphabet and G is a f.g. group. Observe that the product topology on A^G is induced by any of the distances d_S defined by putting $d_S(c_1, c_2) = 2^{-r}$, r being the minimum length w.r.t. S of a $g \in G$ s.t. $c_1(g) \neq c_2(g)$. Moreover, $\lim_{n\to\infty} c_n = c$ in the product topology if and only if, for every $g \in G$, $c_n(g) = c(g)$ except for at most finitely many values of n.

The **natural action** σ^G of G over A^G is defined as

$$(\sigma_g^G(c))(h) = c(gh) \quad \forall c \in A^G \quad \forall g, h \in G. \tag{2}$$

The superscript G may be omitted if irrelevant or clear from the context. Observe that σ^G is continuous. A closed subset X of A^G that is invariant by σ^G is called a **shift subspace**, or briefly **subshift**; A^G itself is called the **full shift**. We use the notation $X \leqslant A^G$ meaning that X is a subshift of A^G. The restriction of σ^G to X is again called the natural action of G over X and indicated by σ^G. From now on, unless differently stated, we will write c^g for $\sigma_g^G(c)$.

Let $E \subseteq G$, $|E| < \infty$. A **pattern** on A with **support** E is a map $p : E \to A$; we write $E = \mathrm{supp}\, p$. A pattern p **occurs** in a configuration c if there exists $g \in G$ such that $c^g|_{\mathrm{supp}\, p} = p$; p is **forbidden** otherwise. Given a set $\mathcal{F}$ of patterns, the set of all the configurations $c \in A^G$ for which all the patterns in $\mathcal{F}$ are forbidden is indicated as $\mathsf{X}_{\mathcal{F}}^{A,G}$; A and/or G will be omitted if irrelevant or clear from the context. It is well-known [5,8] that X is a subshift iff $X = \mathsf{X}_{\mathcal{F}}^{A,G}$ for some $\mathcal{F}$. X is a **shift of finite type** if $\mathcal{F}$ can be chosen finite; the full shift $A^G = \mathsf{X}_{\emptyset}^{A,G}$ is a shift of finite type. A pattern p is forbidden for $X \subseteq A^G$ if it is forbidden for all $c \in X$, *i.e.*, $c^g|_{\operatorname{supp} p} \neq p$ for all $c \in X, g \in G$; if X is a subshift, this is the same as $c|_{\operatorname{supp} p} \neq p$ for all $c \in X$.

A map $F : A^G \to A^G$ is **uniformly locally definable** (UL-definable) if there exist $\mathcal{N} \subseteq G$, $|\mathcal{N}| < \infty$, and $f : A^{\mathcal{N}} \to A$ such that

$$(F(c))(g) = f\left(c^g|_{\mathcal{N}}\right) \tag{3}$$

for all $c \in A^G, g \in G$; in this case, we write $F = F_f^{A,G}$. Observe that any UL-definable function F is continuous and commutes with the natural action of G on A^G; **Hedlund's theorem** [5,6] states that, if $X \subseteq A^G$ is a subshift and $F : X \to A^G$ is continuous and commutes with the natural action of G over X, then F is the restriction to X of a UL-definable function. Moreover, remark that, if X is a subshift and F is UL-definable, then $F(X)$ is a subshift too: if X is of finite type, we say that $F(X)$ is a **sofic shift**.

A **cellular automaton** (CA) with alphabet A and **tessellation group** G is a triple $\mathcal{A} = \langle X, \mathcal{N}, f \rangle$ where the **support** $X \subseteq A^G$ is a subshift, the **neighborhood index** $\mathcal{N} \subseteq G$ is finite, and the **local evolution function** $f : A^{\mathcal{N}} \to A$ satisfies $F_f^{A,G}(X) \subseteq X$. We may write $X_{\mathcal{A}}$ to specify the support X of $\mathcal{A}$. The restriction $F_{\mathcal{A}}$ of $F_f^{A,G}$ to $X_{\mathcal{A}}$ is the **global evolution function**, and $(X_{\mathcal{A}}, F_{\mathcal{A}})$ is the **associated dynamical system**. Observe that $(X_{\mathcal{A}}, F_{\mathcal{A}})$ is a subsystem of $(A^G, F_f^{A,G})$; when $X_{\mathcal{A}} = A^G$ is the full shift we say the CA is **full**. Also observe that, because of Hedlund's theorem, the class of CA with support X can be seen as a monoid w.r.t. function composition. When speaking of bijectivity, finiteness of type, etc. *of* $\mathcal{A}$, we simply "confuse" it with either $F_{\mathcal{A}}$ or $X_{\mathcal{A}}$. We say that $\mathcal{A}$ is **reversible** if there exists a CA $\mathcal{A}'$, *with same alphabet, tessellation group, and support as* $\mathcal{A}$, such that $F_{\mathcal{A}'} \circ F_{\mathcal{A}}$ and $F_{\mathcal{A}} \circ F_{\mathcal{A}'}$ both coincide with the identity function *of* X. Observe that every reversible CA is bijective *on its support*.

A pattern p is a **Garden of Eden** (briefly, GoE) for a CA $\mathcal{A} = \langle X, \mathcal{N}, f \rangle$ if it is allowed for X and forbidden for $F_{\mathcal{A}}(X)$. Any CA having a GoE pattern is non-surjective; compactness of X and continuity of $F_{\mathcal{A}}$ ensure that the vice versa holds as well [5,9]. $\mathcal{A}$ is **pre-injective** if $F_{\mathcal{A}}(c_1) \neq F_{\mathcal{A}}(c_2)$ for any two $c_1, c_2 \in X$ such that $\{g \in G \mid c_1(g) \neq c_2(g)\}$ is finite and non-empty. **Moore–Myhill's theorem** [11,12] states that every full CA with tessellation group $\mathbb{Z}^d$ is surjective iff it is pre-injective. This result has been extended to larger classes of full CA [4,9], but fails if the tessellation group has a free subgroup on two generators [4] or the support is not the full shift [5].

3. Characterization of CA dynamics via group actions

We have said in the introduction that cellular automata are presentations of dynamical systems. This nice concept remains vacuous until we specify what it *means*, for a CA, to be a presentation: intuitively, the meaning should be that the CA "describes well" the dynamics of the system. How well, is stated in

Definition 3.1. Let (X, F) be a d.s., $\mathcal{A}$ a CA. We say that $\mathcal{A}$ is a *presentation* of (X, F) if the latter and $(X_{\mathcal{A}}, F_{\mathcal{A}})$ are conjugate. We call $CA(A, G)$ the class of d.s. having a presentation as CA with alphabet A and tessellation group G. We call $FCA(A, G)$ the subclass of $CA(A, G)$ made of d.s. having a presentation as CA on the full shift A^G.

Example 3.2. The *HPP lattice gas automaton* [7,15] is not, strictly speaking, a CA because it has a many-to many (instead of many-to-one) local function. However, it is straightforward to construct a CA which is a presentation of the HPP dynamics, and whose alphabet is made of quadruples of boolean values. As such, HPP on the infinite grid belongs to $CA(\{0,1\}^4, \mathbb{Z}^2)$.

One can wonder whether the introduction of CA on "partial" subshifts is a factual extension of the concept. Why should it not be possible to rewrite a system using every possible configuration, instead of only a selected package? Why should we lose the feature of *computability* and step into a realm where even *recursive enumerability* is not ensured anymore? Once we sell our soul to the devil of uncomputability, we cannot get it back.

There may be several reasons to accept this kind of Faustian pact. The first one, is that the new model looks promising with respect to *embeddings*: it seems convenient to keep calling "cellular automaton" a local model of a subsystem of the associated dynamics. Indeed, as we will see in the next sections, the larger model actually behaves very well in this respect.

Another possible—and perhaps more compelling—reason, however, is that the *cardinality* of the phase space could be "wrong", in the sense that it may hamper a presentation as a full CA. However, if the system still displays the "correct" features, we may still want to get a presentation in local terms, and keep on calling it "cellular automaton".

And this is the content of

Proposition 3.3. *Let G be a f.g. group with $|G| \geqslant 3$.*

1. *If G is finite, then there exists $X \leqslant A^G$ such that $|X|$ is not a perfect power.*
2. *If G is infinite, then A^G has a countable subshift.*

Proof. Fix $a, b \in A$ with $a \neq b$.

If G is finite, the subset X of configurations such that

- $c(g) \in \{a, b\}$ for every $g \in G$, and
- there exist $g_a, g_b \in G$ s.t. $c(g_a) = a$ and $c(g_b) = b$

is closed and translation invariant, and has $2^{|G|} - 2 = 2 \cdot (2^{|G|-1} - 1)$ elements, which is not a perfect power since $|G| \geqslant 3$.

If G is infinite, then it is countable. Let X be the set of configurations such that

- $c(g) \in \{a, b\}$ for every $g \in G$, and
- $c(g) = b$ for at most one $g \in G$.

Then X is countable and translation invariant. It is closed as well, because if $\lim_{n\to\infty} c_n = c \notin X$, then all of the c_n's, except finitely many, either take value b in two points or take a value $q \notin \{a, b\}$. □

Corollary 3.4. *If $|G| \geqslant 3$ then $CA(A, G) \neq FCA(B, H)$ for any alphabet B and f.g. group H.*

Proof. By Proposition 3.3, there exists $X \leqslant A^G$ which is not in bijection with a full shift. Then no CA with support X can be conjugate to an element of $FCA(B, H)$.

An immediate example of such CA is the identical transformation of X. □

Example 3.5. With the notations and conventions of Corollary 3.4, a less trivial example of a CA whose associated d.s. admits no presentation as a full CA, can be constructed by fixing $\nu \in G \setminus \{1_G\}$, putting $f_\nu(\nu \mapsto x) = x$, and putting $\mathcal{A}_\nu = \langle X, \{\nu\}, f_\nu \rangle$, where X is as in the proof of Corollary 3.4.

Regarding Example 3.5, it must be noted (cf. [5]), that $F_{\mathcal{A}_\nu}$ is *not*, in general, the translation σ_ν. Actually, if $c \in A^G$, then

$$(F_{\mathcal{A}_\nu}(c))(g) = f_\nu(c^g|_{\{\nu\}}) = c(g\nu);$$

to have this coincide with $c(\nu g) = (\sigma_\nu(c))(g)$ *for every g and c*, we must have $g\nu = \nu g$ for every $g \in G$, that is, ν must belong to the *center* of G. This is a phenomenon already observed by Fiorenzi [5]; since it is useful to keep it in mind, we state it as

Proposition 3.6. *Let $g \in G$. Then σ_g^G, as a homeomorphism of A^G, is UL-definable iff g is central in G.*

Proof. If $\sigma_g^G : A^G \to A^G$ is UL-definable, then it commutes with σ_h^G for every $h \in G$ because of Hedlund's theorem. This implies, by evaluating the translates $\sigma_{gh}(c)$ and $\sigma_{hg}(c)$ in 1_G, that $c(gh) = c(hg)$ for all $c \in A^G$, $h \in G$: which is only possible if $gh = hg$ for all $h \in G$. □

We have thus given some reasons why to deal with the more general concept of cellular automaton over a subshift, instead of sticking to the classical one on the full shift. Now that we know what a CA presentation *is*, we must understand what a CA presentation *requires*.

It turns out that the key feature of the dynamical systems that admit some presentation as CA, is that they allow the tessellation group chosen for the CA to act on their phase space *as if it was acting on a subshift*. Here, "as if" means that some key properties of σ^G are shared by the action ϕ of the group G over the space X.

We therefore state

Definition 3.7. Let X be a set, A an alphabet, G a group, ϕ an action of G over X. X is *discernible on A by ϕ* if there exists a continuous function $\pi : X \to A$ such that, for any two distinct $x_1, x_2 \in X$, there exists $g \in G$ such that $\pi(\phi_g(x_1)) \neq \pi(\phi_g(x_2))$.

Continuity of π requires $\pi(x) = \pi(y)$ for x and y "near enough" in X. Therefore, discernibility means that distinct points may always be "displaced far enough".

Example 3.8. Let $\mathcal{A} = \langle X, \mathcal{N}, f \rangle$ be a CA, and let $(X, F_{\mathcal{A}})$ be its associated d.s. Then σ_G commutes with $F_{\mathcal{A}}$. Let $\pi(c) = c(1_G)$: then, for any two $c_1, c_2 \in X$, $c_1(g) \neq c_2(g)$ is the same as $\pi(\sigma_g^G(c_1)) \neq \pi(\sigma_g^G(c_2))$.

From Example 3.8 we know that any CA dynamics admits a "discerning action"—the natural action itself. Hence, the trail we are following seems to lead us to the properties that characterize CA dynamics.

And this is confirmed by (cf. [1])

Theorem 3.9. *Let A be an alphabet, G a f.g. group, (X, F) a d.s. The following are equivalent*:

1. $(X, F) \in CA(A, G)$.
2. *There exists a continuous action ϕ of G over X such that F commutes with ϕ and X is discernible on A by ϕ.*

Proof. We start with supposing that $\mathcal{A} = \langle X_{\mathcal{A}}, \mathcal{N}, f \rangle$ is a presentation of (X, F). Let $\theta : X \to X_{\mathcal{A}}$ be a conjugacy from (X, F) to $(X_{\mathcal{A}}, F_{\mathcal{A}})$; put

$$\phi_g = \theta^{-1} \circ \sigma_g^G \circ \theta$$

for all $g \in G$, and

$$\pi(x) = (\theta(x))(1_G).$$

Remark that $\phi = \{\phi_g\}_{g \in G}$ is an action of G over X and that $(\theta(x))(g) = (\theta(x))^g(1_G)$ for all x and g. Continuity of ϕ and commutation with F are straightforward to verify. If $x_1 \neq x_2$, then $(\theta(x_1))(g) \neq (\theta(x_2))(g)$ for some $g \in G$, thus

$$\pi(\phi_g(x_1)) = (\sigma_g^G(\theta(x_1)))(1_G) \neq (\sigma_g^G(\theta(x_2)))(1_G) = \pi(\phi_g(x_2)).$$

For the converse implication, let π as in Definition 3.7: then $\tau : X \to A^G$ defined by

$$(\tau(x))(g) = \pi(\phi_g(x))$$

is injective. Moreover, $(\tau(\phi_g(x))(h) = \pi(\phi_h(\phi_g(x))) = \pi(\phi_{gh}(x)) = (\tau(x))(gh)$ for every $x \in X$, $g, h \in G$: thus, $\tau \circ \phi_g = \sigma_g^G \circ \tau$ for all $g \in G$, and $X' = \tau(X)$ is invariant under σ^G.

We now prove that τ is continuous. Let $\lim_{n \in \mathbb{N}} x_n = x$ in X: by continuity of π and ϕ, $\lim_{n \in \mathbb{N}}(\tau(x_n))(g) = (\tau(x))(g)$ in A for all G. Since A is discrete, this implies $\pi(\phi_g(x_n)) = \pi(\phi_g(x))$ for all n except finitely many: which is the definition of convergence of $\tau(x_n)$ to $\tau(x)$ in the product topology of A^G.

Since X and A^G are compact and Hausdorff, X' is closed in A^G and a subshift, while τ is a homeomorphism between X and X'. Define $F' : X' \to X'$ by $F' = \tau \circ F \circ \tau^{-1}$: then (X', F') is a d.s. and τ is a conjugacy between (X, F) and (X', F'). But for every $g \in G$

$$\phi_g \circ \tau^{-1} = (\tau \circ \phi_{g^{-1}})^{-1} = (\sigma_{g^{-1}}^G \circ \tau)^{-1} = \tau^{-1} \circ \sigma_g^G,$$

thus

$$\sigma_g^G \circ F' = \tau \circ \phi_g \circ F \circ \tau^{-1} = \tau \circ F \circ \phi_g \circ \tau^{-1} = F' \circ \sigma_g^G;$$

hence, F' commutes with σ^G. By Hedlund's theorem, there exist a finite $\mathcal{N}' \subseteq G$ and a map $f' : A^{\mathcal{N}'} \to A$ such that $(F'(c))_g = f'(c^g|_{\mathcal{N}'})$ for all $c \in X', g \in G$: then $\langle X', \mathcal{N}', f' \rangle$ is a presentation of (X, F) as a cellular automaton. □

The meaning of Theorem 3.9 is that (X, F) has a CA presentation with alphabet A and tessellation group G, if and only if G can act on X as it would naturally do on A^G, and without interfering with F. This explains why the characterization works for $CA(A, G)$, and not for $FCA(A, G)$: the natural action, by itself, is incapable of telling the full shift from any other shift. Consequently, any action on X that "emulates" the natural action shall not be able to tell whether (X, F) has a presentation as a full CA or not.

Theorem 3.9 has two immediate consequences. The first one is a generalization, to our class of general CA, of a principle first discovered by Hedlund [6] in dimension 1, then extended by Richardson [13] to classical CA of arbitrary dimension.

Corollary 3.10. *Let $(X, F) \in CA(A, G)$. If F is bijective then $(X, F^{-1}) \in CA(A, G)$.*

Proof. Let ϕ be as in Theorem 3.9. Then X is discernible on A by ϕ, and

$$F^{-1} \circ \phi_g = (\phi_{g^{-1}} \circ F)^{-1} = (F \circ \phi_{g^{-1}})^{-1} = \phi_g \circ F^{-1}$$

for all $g \in G$. Apply Theorem 3.9. □

Corollary 3.10 can—and, in fact, has been (cf. [5])—proved by purely topological means. Yet our proof emphasizes the role of the tessellation group.

Corollary 3.11 (*Hedlund–Richardson's principle*)**.** *Every bijective CA is reversible.*

The second consequence of Theorem 3.9 is that existence of a presentation as CA actually depends on the *minimum number* of elements of the alphabet and the *isomorphism class* of the tessellation group. This is intuitively true, because on one hand, isomorphic groups have "isomorphic" actions on equal spaces; and on the other hand, if one has enough "letters" to be able to tell elements from each other via the action, then having even more letters cannot be a hindrance.

Proposition 3.12. *Let A and B be alphabets, and let G and Γ be f.g. groups.*

1. *If $|A| \leqslant |B|$ then $CA(A, G) \subseteq CA(B, G)$.*
2. *If G is isomorphic to Γ then $CA(A, G) = CA(A, \Gamma)$.*

Proof. To prove point 1, let $\iota : A \to B$ be injective. Let $(X, F) \in CA(A, G)$, and let ϕ satisfy point 2 of Theorem 3.9, π being the discerning map. Then X is discernible over B by ϕ, $\iota \circ \pi$ being the discerning map.

To prove point 2, let $\psi : G \to \Gamma$ be a group isomorphism. Let $(X, F) \in CA(A, G)$ and let ϕ satisfy point 2 of Theorem 3.9, π being the discerning map. Define $\phi' = \{\phi'_\gamma\}_{\gamma \in \Gamma}$ as

$$\phi'_\gamma = \phi_{\psi^{-1}(\gamma)}.$$

It is straightforward to check that ϕ' is an action which commutes with F. Let $x_1 \neq x_2$: if $g \in G$ is such that $\pi(\phi_g(x_1)) \neq \pi(\phi_g(x_2))$, then $\pi(\phi'_{\psi(g)}(x_1)) \neq \pi(\phi'_{\psi(g)}(x_2))$ as well. Thus ϕ' satisfies condition 2 of Theorem 3.9, and $(X, F) \in CA(A, \Gamma)$. From the arbitrariness of (X, F) follows $CA(A, G) \subseteq CA(A, \Gamma)$: by swapping the roles of G and Γ and repeating the argument with ψ^{-1} in place of ψ we obtain the reverse inclusion. □

4. Induced subshifts

Let $X \leqslant A^G$ be a shift subspace. We know that $X = \mathsf{X}_{\mathcal{F}}^{A,G}$ for some set $\mathcal{F}$ of patterns, that is, X is completely *described* by $\mathcal{F}$ in the *context* provided by A and G.

Let now Γ be a group having G as a subgroup. We want to define a new subshift X' of A^Γ, which is "induced" by X, in the sense that X' can be completely described by X. But we had observed that X, in turn, can be completely described by $\mathcal{F}$, *provided we know to be dealing with a subshift of A^G*; the first idea that comes to our mind is that X' should then be completely described by $\mathcal{F}$ as well, *provided we know to be dealing with a subshift of A^Γ*.

This is precisely the content of

Definition 4.1. Let $X = \mathsf{X}_{\mathcal{F}}^{A,G}$ be a subshift, and let $G \leqslant \Gamma$. The *subshift induced by X on A^Γ* is $X' = \mathsf{X}_{\mathcal{F}}^{A,\Gamma}$.

Example 4.2. Consider $A = \{0, 1\}$, $G = \mathbb{Z}$, $\Gamma = \mathbb{Z}^2$, $\mathcal{F} = \{11\}$, where 11 is the pattern $p : \{0, 1\} \subseteq \mathbb{Z} \to A$ such that $p(0) = p(1) = 1$. Then $X = \mathsf{X}_{\mathcal{F}}^{A,G}$ is the *golden mean shift* (cf. [8]); a configuration $c : \mathbb{Z} \to \{0, 1\}$ belongs to X if and only if it does not contain two adjacent 1's. On the other hand, a configuration $\chi : \mathbb{Z}^2 \to \{0, 1\}$ belongs to $X' = \mathsf{X}_{\mathcal{F}}^{A,\Gamma}$ if and only if no point on the square grid containing a 1 has his immediate right neighbor containing a 1 as well.

According to Definition 4.1, X' is what we obtain instead of X, when we interpret $\mathcal{F}$ as a description of a subshift of A^Γ instead of A^G, that is, in the context provided by Γ instead of G.

At first glance, Definition 4.1 seems to be a good solution to our "subshift induction problem". However, we know from basic theory (cf. [8]) that different sets of patterns can define identical subshifts; and we want induction to depend on *the object and not the description*. We must then ensure that Definition 4.1 is well posed and X' only depends on X rather than $\mathcal{F}$, *i.e.*, $\mathsf{X}_{\mathcal{F}_1}^{A,G} = \mathsf{X}_{\mathcal{F}_2}^{A,G}$ must imply $\mathsf{X}_{\mathcal{F}_1}^{A,\Gamma} = \mathsf{X}_{\mathcal{F}_2}^{A,\Gamma}$.

In fact, we are going to discover much more. We had noticed in Section 2 that the image of a subshift via a UL-definable function is a subshift; thus, we neither add nor lose anything by considering as subshifts objects of the form

$$X = F_f^{A,G}\left(\mathsf{X}_{\mathcal{F}}^{A,G}\right), \tag{4}$$

with $f : A^{\mathcal{N}} \to A$, $\mathcal{N}$ finite subset of G, and $\mathcal{F}$ set of patterns with supports contained in G. However, as we can choose to consider $\mathcal{F}$ as a description of a subshift of either A^G or A^Γ, so we can choose to to consider f as a description of a UL-definable function on either A^G or A^Γ. Thus, a more general fact we can check is the preservation of *mutual inclusion*—instead of just equality—between objects of the form (4).

And this is precisely the content of

Lemma 4.3. *Let A be an alphabet, and let G and Γ be f.g. groups with $G \leqslant \Gamma$. For $i = 1, 2$, let $\mathcal{F}_i$ be a set of patterns on A with supports contained in G, let $\mathcal{N}_i$ be a finite non-empty subset of G, and let $f_i : A^{\mathcal{N}_i} \to A$. Then*

$$F_{f_1}^{A,G}\left(\mathsf{X}_{\mathcal{F}_1}^{A,G}\right) \subseteq F_{f_2}^{A,G}\left(\mathsf{X}_{\mathcal{F}_2}^{A,G}\right) \text{ iff } F_{f_1}^{A,\Gamma}\left(\mathsf{X}_{\mathcal{F}_1}^{A,\Gamma}\right) \subseteq F_{f_2}^{A,\Gamma}\left(\mathsf{X}_{\mathcal{F}_2}^{A,\Gamma}\right).$$

Proof. Let J be a set of representatives of the left cosets of G in Γ such that $1_G = 1_\Gamma \in J$. To simplify notation, we will write

$$X_i = \mathsf{X}_{\mathcal{F}_i}^{A,G},\ \Xi_i = \mathsf{X}_{\mathcal{F}_i}^{A,\Gamma},\ F_i = F_{f_i}^{A,G},\ \Phi_i = F_{f_i}^{A,\Gamma},$$

so that the thesis becomes

$$F_1(X_1) \subseteq F_2(X_2) \quad \text{iff} \quad \Phi_1(\Xi_1) \subseteq \Phi_2(\Xi_2).$$

For the “if” part, let $c \in F_1(X_1)$, and let $x_1 \in X_1$ satisfy $F_1(x_1) = c$. Define $\xi_1 \in A^\Gamma$ by $\xi_1(jg) = x_1(g)$ for all $j \in J, g \in G$: then for all $j \in J, g \in G, p \in \mathcal{F}_1$

$$\xi_1^{jg}\Big|_{\operatorname{supp} p} = x_1^g\Big|_{\operatorname{supp} p} \neq p,$$

hence $\xi_1 \in \Xi_1$. Put $\chi = \Phi_1(\xi_1)$: by hypothesis, there exists $\xi_2 \in \Xi_2$ such that $\Phi_2(\xi_2) = \chi$, and by construction,

$$\chi(g) = f_1\left(\xi_1^g\Big|_{\mathcal{N}_1}\right) = f_1\left(x_1^g\Big|_{\mathcal{N}_1}\right) = c(g) \quad \forall g \in G.$$

Let $x_2 = \xi_2|_G$: then $x_2 \in X_2$ by construction. But

$$f_2\left(x_2^g\Big|_{\mathcal{N}_2}\right) = f_2\left(\xi_2^g\Big|_{\mathcal{N}_2}\right) = \chi(g) = c(g) \quad \forall g \in G,$$

thus $c \in F_2(X_2)$.

For the “only if” part, let $\chi \in \Phi_1(\Xi_1)$, and let $\xi_1 \in \Xi_1$ satisfy $\Phi_1(\xi_1) = \chi$. For each $j \in J$, define $x_{1,j} \in A^G$ as $x_{1,j}(g) = \xi_1(jg)$ for all $g \in G$. It is straightforward to check that $x_{1,j} \in X_1$ for all $j \in J$: let $c_j = F_1(x_{1,j})$. By hypothesis, for all $j \in J$ there exists $x_{2,j} \in X_2$ such that $F_2(x_{2,j}) = c_j$: define $\xi_2 \in A^\Gamma$ by $\xi_2(jg) = x_{2,j}(g)$ for all $j \in J, g \in G$. It is straightforward to check that $\xi_2 \in \Xi_2$; but for all $j \in J, g \in G$

$$f_2\left(\xi_2^{jg}\Big|_{\mathcal{N}_2}\right) = f_2\left(x_{2,j}^g\Big|_{\mathcal{N}_2}\right) = c_j(g) = f_1\left(x_{1,j}^g\Big|_{\mathcal{N}_1}\right) = f_1\left(\xi_1^{jg}\Big|_{\mathcal{N}_1}\right) = \chi(jg),$$

thus $\chi \in \Phi_2(\Xi_2)$. □

The reason why Lemma 4.3 is true, is the following. Each left coset of G can be thought of as a “slice” of Γ “shaped” as G. If each pattern’s support is contained in G, then the constraint of not having a pattern in $\mathcal{F}_i$ can be applied either *slice by slice* or on the whole Γ at once, with the same results. Similarly, the neighbors of γ w.r.t. $\mathcal{N}_i$ will all belong to the same slice as γ, so the Φ_i’s can be made to operate either slice by slice or on the whole Γ at once, with the same results. This means, however, that the yes/no information about the mutual inclusion of the $\Phi_i(\Xi_i)$’s is deducible from the $\mathcal{F}_i$’s and the f_i’s alone, and cannot be different from that on the $F_i(X_i)$’s.

Observe that the proof of Lemma 4.3 does not depend on the choice of the set J of representatives of the left cosets of G in Γ.

Corollary 4.4. *In the hypotheses of Lemma 4.3,*

1. $\mathsf{X}_{\mathcal{F}_1}^{A,G} \subseteq F_{f_2}^{A,G}(\mathsf{X}_{\mathcal{F}_2}^{A,G})$ *iff* $\mathsf{X}_{\mathcal{F}_1}^{A,\Gamma} \subseteq F_{f_2}^{A,\Gamma}(\mathsf{X}_{\mathcal{F}_2}^{A,\Gamma})$,
2. $F_{f_1}^{A,G}(\mathsf{X}_{\mathcal{F}_1}^{A,G}) \subseteq \mathsf{X}_{\mathcal{F}_2}^{A,G}$ *iff* $F_{f_1}^{A,\Gamma}(\mathsf{X}_{\mathcal{F}_1}^{A,\Gamma}) \subseteq \mathsf{X}_{\mathcal{F}_2}^{A,\Gamma}$, *and*
3. $\mathsf{X}_{\mathcal{F}_1}^{A,G} \subseteq \mathsf{X}_{\mathcal{F}_2}^{A,G}$ *iff* $\mathsf{X}_{\mathcal{F}_1}^{A,\Gamma} \subseteq \mathsf{X}_{\mathcal{F}_2}^{A,\Gamma}$.

Proof. Consider the neighborhood index $\{1_G\}$ and the local evolution function $f(1_G \mapsto a) = a$. Apply Lemma 4.3. □

Corollary 4.5. *Let A be an alphabet, let G and Γ be f.g. groups with $G \leqslant \Gamma$, and let $\mathcal{F}$ be a set of patterns on A with supports contained in G. If $\mathsf{X}_{\mathcal{F}}^{A,G}$ is sofic then $\mathsf{X}_{\mathcal{F}}^{A,\Gamma}$ is sofic.*

Proof. By hypothesis, $\mathsf{X}_{\mathcal{F}}^{A,G} = F(\mathsf{X}_{\mathcal{F}'}^{A,G})$ for some UL-definable function F and finite set of patterns $\mathcal{F}'$. Apply points 1 and 2 of Corollary 4.4. □

Now, under the same hypotheses on G, Γ, A, and $\mathcal{F}$, suppose $X' = \mathsf{X}_{\mathcal{F}}^{A,\Gamma}$ is sofic. This means that there exist a finite set $\mathcal{F}'$ of patterns *over* Γ and a function $f' : A^{\mathcal{N}'} \to A$ with $\mathcal{N}'$ finite subset *of* Γ, such that $X' = F_{f'}^{A,\Gamma}\left(\mathsf{X}_{\mathcal{F}'}^{A,\Gamma}\right)$. Is it then possible for $X = \mathsf{X}_{\mathcal{F}}^{A,G}$ *not* to be sofic? In fact, the finitary description for X' provided by $\mathcal{F}'$ and f' takes advantage of the (at least, *a priori*) greater complexity of the group Γ w.r.t. the group G; however, it is also true that $\mathcal{F}$ alone yields enough information to describe X' in the context provided by Γ. It would not be surprising, then, if the information provided by $\mathcal{F}$ in the context provided by G yielded enough information to describe X; we state this as a conjecture.

Conjecture 4.6. *With the notation of Corollary 4.5, suppose* $\mathsf{X}_{\mathcal{F}}^{A,\Gamma}$ *is sofic. Then* $\mathsf{X}_{\mathcal{F}}^{A,G}$ *is sofic.*

5. Induced cellular automata

In the previous section, we have learned to construct a subshift on a group from a subshift on a subgroup; while doing this, we have received some insight on how the underlying mechanism can also work for UL-definable functions. It then comes to our mind that similar mechanics could be applied to another field where locality is the key factor, that is, the field of cellular automata. This time, we can give our definition after having already done the bulk of the work.

Definition 5.1. Let $\mathcal{A} = \langle X, \mathcal{N}, f \rangle$ be a CA with alphabet A and tessellation group G, and let Γ be a f.g. group such that $G \leqslant \Gamma$. The CA *induced by* $\mathcal{A}$ *on* Γ is the cellular automaton

$$\mathcal{A}' = \left\langle X', \mathcal{N}, f \right\rangle, \tag{5}$$

where X' is the subshift induced by X on A^{Γ}.

Observe how Lemma 4.3 ensures that $\mathcal{A}'$ is well defined.

Example 5.2. Let $A = \{0, 1\}$, $G = \mathbb{Z}$, $\Gamma = \mathbb{Z}^2$, $\mathcal{N} = \{-1, 1\}$, $f(-1 \mapsto x, 1 \mapsto y) = x + y - 2xy$: then $\mathcal{A} = \left\langle A^{\mathbb{Z}}, \mathcal{N}, f \right\rangle$ is *Wolfram's rule 90*, such that the next value of each point is the exclusive OR of the current values of its leftmost and rightmost neighbors. The same rule applies to $\mathcal{A}' = \left\langle A^{\mathbb{Z}^2}, \mathcal{N}, f \right\rangle$, which can be seen as the joining of infinitely many copies of $\mathcal{A}$ along a vertical line.

Definition 5.1 is similar to the one given in [4] for CA over the full shift; ours, however, works for the broader class of CA over subshifts. (We still have, however, the constraint on finite alphabets, which [3] tries to overcome at least for the full shift.) As in the case of the induced subshift—which, by the way, is the support of the induced CA—$\mathcal{A}'$ is what we obtain by interpreting the local descriptions given by $\mathcal{F}$, $\mathcal{N}$, and f, in the context provided by Γ instead of G.

It must be remarked that, in general, $\mathcal{A}'$ is not conjugate to $\mathcal{A}$. For instance, if G is a proper non-trivial subgroup of a finite group Γ, then there can be no bijection between A^G and A^{Γ}, let alone conjugacies of cellular automata.

On the other hand, it had already been shown in [4] that, in the case of CA over full shifts, some important properties—notably, surjectivity—are preserved in the passage from the original CA to the induced one; which is not surprising, because intuitively $F_f^{A,\Gamma}$ is going to operate "slice by slice" on A^{Γ}, each "slice" being "shaped" as A^G. The next statement extends the aforementioned result from the case of CA on the full shift to the general case when X is an arbitrary subshift.

Theorem 5.3. *Let* $\mathcal{A} = \langle X, \mathcal{N}, f \rangle$ *be a CA with alphabet A and tessellation group G, let* $G \leqslant \Gamma$*, and let* $\mathcal{A}'$ *be the CA induced by* $\mathcal{A}$ *on* Γ.

1. $\mathcal{A}$ *is surjective iff* $\mathcal{A}'$ *is surjective*.
2. $\mathcal{A}$ *is pre-injective iff* $\mathcal{A}'$ *is pre-injective*.
3. $\mathcal{A}$ *is injective iff* $\mathcal{A}'$ *is injective*.

Proof. Let $\mathcal{F}$ satisfy $X = \mathsf{X}_{\mathcal{F}}^{A,G}$ (and $X' = \mathsf{X}_{\mathcal{F}}^{A,\Gamma}$). Take J as in proof of Lemma 4.3.

To prove the "if" part of point 1, suppose $\mathcal{A}$ has a GoE pattern p. By contradiction, assume that there exists $\chi \in X'$ such that $F_{\mathcal{A}'}(\chi)|_{\operatorname{supp} p} = p$. Let c be the restriction of χ to G. Then, since both $\mathcal{N}$ and $\operatorname{supp} p$ are subsets of G by hypothesis,

$$(F_{\mathcal{A}}(c))(x) = f\left(c^x|_{\mathcal{N}}\right) = f\left(\chi^x|_{\mathcal{N}}\right) = (F_{\mathcal{A}'}(\chi))(x) = p(x)$$

for every $x \in \operatorname{supp} p$: this is a contradiction.

To prove the "only if" part of point 1, suppose $\mathcal{A}'$ has a GoE pattern π. By hypothesis, there exists $\chi \in X'$ such that $\chi|_{\operatorname{supp} \pi} = \pi$. For all $j \in J$ define $c_j \in A^G$ as

$$c_j(g) = \chi(jg) \quad \forall g \in G,$$

and for all $j \in J$ such that $jG \cap \operatorname{supp} \pi \neq \emptyset$ define the pattern p_j over G as

$$p_j(x) = \pi(jx) \quad \forall x \text{ s.t. } jx \in \operatorname{supp} \pi.$$

Observe that $c_j \in X$ for all j, and that $p_j = c_j|_{jG \cap \operatorname{supp} \pi}$ when defined. But at least one of the patterns p_j must be a GoE for $\mathcal{A}$: otherwise, for all $j \in J$, either $jG \cap \operatorname{supp} \pi = \emptyset$, or there would exist $k_j \in X'$ such that $F_{\mathcal{A}}(k_j)|_{\operatorname{supp} p_j} = p_j$. In this case, however, $\kappa \in A^{\Gamma}$ defined by $\kappa(jg) = k_j(g)$ for all $j \in J, g \in G$ would satisfy $\kappa \in X'$ and $F_{\mathcal{A}'}(\kappa)|_{\operatorname{supp} \pi} = \pi$, against π being a GoE for $\mathcal{A}'$.

For the "if" part of point 2, suppose $c_1, c_2 \in X$ differ on all and only the points of a finite non-empty $U \subseteq G$, but $F_{\mathcal{A}}(c_1) = F_{\mathcal{A}}(c_2)$. For all $j \in J, g \in G$, put $\chi_1(jg) = c_1(g)$, and set $\chi_2(jg)$ as $c_2(g)$ if $j = 1_{\Gamma}$, $c_1(g)$ otherwise. Then χ_1 and χ_2 belong to

X' and differ precisely on U. Moreover, for every $\gamma \in \Gamma$, either $\gamma \in G$ or $\gamma\mathcal{N} \cap G = \emptyset$, so either $(F_{\mathcal{A}'}(\chi_i))(\gamma) = (F_{\mathcal{A}}(c_i))(\gamma)$ or $(F_{\mathcal{A}'}(\chi_1))(\gamma) = (F_{\mathcal{A}'}(\chi_2))(\gamma)$.

For the "only if" part of point 2, suppose $\mathcal{A}$ is pre-injective. Let $\chi_1, \chi_2 \in X'$ differ on all and only the points of a finite non-empty $U' \subseteq \Gamma$. For $i \in \{1, 2\}$, $\gamma \in \Gamma$, let $c_{i,\gamma}$ be the restriction of χ_i^γ to G: these are all in X, because a pattern occurring in $c_{i,\gamma}$ also occurs in χ_i, and cannot belong to $\mathcal{F}$. Let $U_\gamma = \{g \in G \mid c_{1,\gamma}(g) \neq c_{2,\gamma}(g)\}$: then $|U_\gamma| \leqslant |U'|$ for all $\gamma \in \Gamma$, plus $U_\gamma \neq \emptyset$ for at least one γ. For such γ, there exists $g \in G$ such that $(F_{\mathcal{A}}(c_{1,\gamma}))(g) \neq (F_{\mathcal{A}}(c_{2,\gamma}))(g)$: then by construction $(F_{\mathcal{A}'}(\chi_1))(\gamma g) \neq (F_{\mathcal{A}'}(\chi_2))(\gamma g)$ as well.

The proof of point 3 is straightforward to see. For the "if" part, let $c_1 \neq c_2$, $F_{\mathcal{A}}(c_1) = F_{\mathcal{A}}(c_2)$, and consider $\chi_i(\gamma) = c_i(g)$ iff $\gamma = jg$. For the "only if" part, given $\chi_1 \neq \chi_2$, consider $c_{i,j}(g) = \chi_i(jg)$, and observe that $F_{\mathcal{A}}(c_{1,j}) \neq F_{\mathcal{A}}(c_{2,j})$ for at least one $j \in J$. □

The reason why Theorem 5.3 is true, is similar to the one given for Lemma 4.3: the global evolution function of the induced CA operates "slice by slice" on the support of the induced CA; this, however, is the induced subshift, and is already "sliced" suitably for $F_{\mathcal{A}'}$. Moreover, each of the listed global properties can be expressed in local terms: for instance, surjectivity is equivalent to absence of GoE patterns, even in our broader context (cf. [5]). Pay attention, however, that these properties are usually r.e. or co-r.e., but not computable.

Observe that, as in the proof of Lemma 4.3, the choice of J is arbitrary.

Example 5.4. Let $\mathcal{A}$ be as in Example 5.2. It is a good exercise in cellular automata theory to check that each configuration has exactly four predecessors according to $\mathcal{A}$, that is, for every $c : \mathbb{Z} \to A$ there exist four distinct $c_i : \mathbb{Z} \to A$ such that $F_{\mathcal{A}}(c_i) = c$. (Hint: fix four patterns 00, 01, 10, 11.) Thus $\mathcal{A}$ is surjective, but not injective; Theorem 5.3 then says that $\mathcal{A}'$ is also surjective and non-injective.

Surjectivity and pre-injectivity are always shared by $\mathcal{A}$ and $\mathcal{A}'$, even when these two properties are not equivalent. This fact was used in [4] to prove that none of the implications in Moore–Myhill's theorem holds for full CA with tessellation group containing a free subgroup on two generators, starting from suitable counterexamples on the free group $\mathbb{F}_2$.

Having observed that $\mathcal{A}$ and $\mathcal{A}'$ may well be non-conjugate, we are left with a different question: is it possible to *embed* the original CA into the induced one? After all, we have *kept* the same local descriptions, and *enlarged* the group, so we should expect the induced dynamics to be *richer* than the original. Moreover, since the global evolution function of $\mathcal{A}'$ operates slice by slice, we should expect that, after having *fixed a point on each slice*, we should be able to reproduce $\mathcal{A}$ into $\mathcal{A}'$.

And this is precisely the content of

Lemma 5.5. *Let A be an alphabet, and let G and Γ be f.g. groups with $G \leqslant \Gamma$; let $\mathcal{A} = \langle X, \mathcal{N}, f\rangle$ be a CA with alphabet A and tessellation group G, and let $\mathcal{A}' = \langle X', \mathcal{N}, f\rangle$ be the CA induced by $\mathcal{A}$ over Γ. Let J be a set of representatives of the left cosets of G in Γ, and let $\iota_J : A^G \to A^\Gamma$ be defined by*

$$(\iota_J(c))(\gamma) = c(g) \text{ iff } \exists j \in J : \gamma = jg. \tag{6}$$

Then ι_J is an embedding of $\mathcal{A}$ into $\mathcal{A}'$, so that

$$\iota_J(\mathcal{A}) = \langle \iota_J(X), \mathcal{N}, f\rangle \tag{7}$$

is a CA conjugate to $\mathcal{A}$. In particular, $CA(A, G) \subseteq CA(A, \Gamma)$.

Proof. First, we observe that ι_J is injective and $\iota_J(X) \subseteq X'$. In fact, if $c_1(g) \neq c_2(g)$, then $(\iota_J(c_1))(jg) \neq (\iota_J(c_2))(jg)$ for all $j \in J$. Moreover, should a pattern p exist such that $(\iota_J(c))(\gamma x) = p(x)$ for all $x \in \operatorname{supp} p \subseteq G$, by writing $\gamma = jg$ and applying (6) we would find $c(gx) = p(x)$ for all $x \in \operatorname{supp} p$.

Next, we show that ι_J is continuous. Let S be a f.s.o.g. for G, Σ a f.s.o.g. for Γ. Let $R \geqslant 0$, and let

$$E_R = \left\{g \in G \mid \exists j \in J \mid jg \in D^\Gamma_{R,\Sigma}\right\}.$$

Since the writings $\gamma = jg$ are unique and $D^\Gamma_{R,\Sigma}$ is finite, E_R is finite too. Let $E_R \subseteq D^G_{r,S}$: if $c_1|_{D^G_{r,S}} = c_2|_{D^G_{r,S}}$, then $\iota_J(c_1)|_{D^\Gamma_{R,\Sigma}} = \iota_J(c_2)|_{D^\Gamma_{R,\Sigma}}$.

Next, we show that ι_J is a morphism of d.s. For every $c \in A^G$, $\gamma = jg \in \Gamma$, $x \in \mathcal{N}$ we have $\gamma x \in jG$ and $(\iota_J(c))(\gamma x) = (\iota_J(c))(jgx) = c(gx)$. Thus,

$$((F_{\mathcal{A}'} \circ \iota_J)(c))(\gamma) = f\left(\iota_J(c)^\gamma|_{\mathcal{N}}\right) = f\left(c^g|_{\mathcal{N}}\right) = (F_{\mathcal{A}}(c))(g) = ((\iota_J \circ F_{\mathcal{A}})(c))(\gamma),$$

so that $F_{\mathcal{A}'} \circ \iota_J = \iota_J \circ F_{\mathcal{A}}$. Moreover, $F_{\mathcal{A}'}(\iota_J(X)) = \iota_J(F_{\mathcal{A}}(X)) \subseteq \iota_J(X)$ because $F_{\mathcal{A}}(X) \subseteq X$.

Finally, we observe that $\iota_J(X)$ is a subshift. In fact, if $X = \mathsf{X}^{A,G}_{\mathcal{F}}$, then $\iota_J(X) = \mathsf{X}^{A,\Gamma}_{\mathcal{F}\cup\mathcal{F}'}$, where

$$\mathcal{F}' = \left\{p \in A^{\{j_1 g, j_2 g\}} \mid j_1, j_2 \in J, g \in G, j_1 \neq j_2, p(j_1 g) \neq p(j_2 g)\right\}. \tag{8}$$

It is straightforward that $\iota_J(X) \subseteq \mathsf{X}^{A,\Gamma}_{\mathcal{F}\cup\mathcal{F}'}$. Let $\chi \in \mathsf{X}^{A,\Gamma}_{\mathcal{F}\cup\mathcal{F}'}$: then $c(g) = \chi(jg)$ is well defined, and $\chi = \iota_J(c)$ by construction. Moreover, for every $g \in G$, $p \in F$, and any $j \in G$ $(c^g)_{\operatorname{supp} p} = (\chi^{jg})_{\operatorname{supp} p} \neq p$, so $c \in X$ and $\chi \in \iota_J(X)$. □

Observe that, in the hypotheses of Lemma 5.5, ι_J depends explicitly on J. Thus ι_J might, in general, show "better" or "worse" properties according to the choice of J; these, however, have no effect on the *abstract dynamics* of $\iota_J(\mathcal{A})$, which is always the same as $\mathcal{A}$'s. Moreover, we are *not* assuming $1_\Gamma \in J$; hence, in general, $E_R \not\subseteq D^G_{R,S}$, even if $S \subseteq \Sigma$.

Example 5.6. Let $\Gamma = \mathbb{Z}^2$, $G = \{(x, 0) \mid x \in \mathbb{Z}\}$, $S = \{(1, 0)\}$, $\Sigma = \{(1, 0), (0, 1)\}$, and

$$J = \{(1,0)\} \cup \{(0,y) \mid y \in \mathbb{Z}, y \neq 0\}.$$

Then $E_1 = \{(0, 0), (-1, 0), (-2, 0)\} \not\subseteq D^G_{1,S}$.

Lemma 5.5 says that growing the tessellation group does not shrink the class of presentable dynamics. This fact and Proposition 3.12 together yield

Theorem 5.7. *Let A, B be alphabets and G, Γ be f.g. groups. If $|A| \leqslant |B|$ and G is isomorphic to a subgroup of Γ, then $CA(A, G) \subseteq CA(B, \Gamma)$.*

Proof. Let $G \cong H \leqslant \Gamma$. Then $CA(A, G) = CA(A, H) \subseteq CA(A, \Gamma) \subseteq CA(B, \Gamma)$. □

Corollary 5.8. *Let $\mathbb{F}_n$ be the free group on $n < \infty$ generators. For every alphabet A and every $n > 1$, $CA(A, \mathbb{F}_n) = CA(A, \mathbb{F}_2)$.*

Proof. Clearly, every $\mathbb{F}_n$ with $n > 1$ has a free subgroup on two generators: because of Theorem 5.7, $CA(A, \mathbb{F}_2) \subseteq CA(A, \mathbb{F}_n)$. However, it is a well-known fact in group theory (cf. [10, Section 2.4, Problem 2]) that $\mathbb{F}_2$ has a free subgroup on infinitely many generators, thus also a free subgroup on n generators for every $n > 0$: because of Theorem 5.7, $CA(A, \mathbb{F}_n) \subseteq CA(A, \mathbb{F}_2)$. □

Corollary 5.8 extends Róka [14, Proposition 6], which can be re-stated as follows: $FCA(A, \mathbb{F}_{n_1}) \subseteq CA(A, \mathbb{F}_{n_2})$ for any $n_1, n_2 > 1$. Observe that the inclusion in one direction also works for $n = 1$, with $\mathbb{F}_1 = \mathbb{Z}$. Since Moore–Myhill's theorem does not hold for the latter class (cf. [4]) we know that $FCA(\{0, 1\}, \mathbb{Z}) \neq FCA(\{0, 1\}, \mathbb{F}_2)$; however, we do not know of a similar statement for the corresponding CA-classes. In fact, the structure of $\mathbb{F}_2$ is intrinsically much more complex than that of $\mathbb{Z}$, where the same cannot be said of the other $\mathbb{F}_n$'s, which somehow "contain each other".

Conjecture 5.9. $CA(A, \mathbb{Z}) \neq CA(A, \mathbb{F}_2)$.

Now, if we look at (8), the set of "additional constraints" $\mathcal{F}'$ seems a bit cumbersome. Why is it necessary to take note of all the pairs $(j_1 g, j_2 g)$? Should we only make the checks on the pairs (j_1, j_2), and use the smaller set

$$\mathcal{F}'' = \left\{ p \in A^{\{j_1 j_2\}} \mid j_1, j_2 \in J, p(j_1) \neq p(j_2) \right\},$$

why should not we retrieve the same subshift?

The problem with the idea of replacing $\mathcal{F}'$ with $\mathcal{F}''$ is that we are forgetting that Γ can be non-commutative. Thus, $j\gamma$ is not forced to equal γj, which is what we get when we try to check whether the configuration χ has a pattern with support $\{j_1, j_2\}$. On the other hand, if $\mathcal{F}''$ were always a good choice, then, for $\iota_J(X)$ to be of finite type, it would suffice to have X of finite type and G of finite index in Γ, *independently on the choice of J*. This seems just too good to be true; and is actually false.

Theorem 5.10. *Let Γ be the group of ordered pairs (i, k), $i \in \{0, 1\}$, $k \in \mathbb{Z}$ with the product*

$$(i_1, k_1)(i_2, k_2) = (i_1 + i_2 - 2i_1 i_2, (-1)^{i_2} k_1 + k_2).$$

Let $A = \{a, b\}$, $G = \{(0, k), k \in \mathbb{Z}\} \leqslant \Gamma$, and $J = \{(0, 0), (1, 0)\}$. Then $\iota_J(A^G)$ is not a shift of finite type.

Proof. Let $S = \{(1, 0), (0, 1)\}$: it is straightforward to check that $\langle S \rangle = \Gamma$.

By contradiction, assume that $\iota_J(A^G) = \mathsf{X}^{A,\Gamma}_{\mathcal{F}}$ with $|\mathcal{F}| < \infty$; it is not restrictive to choose $\mathcal{F}$ so that $\operatorname{supp} p = D^\Gamma_{M,S}$ for all $p \in \mathcal{F}$. Let $\delta \in A^\Gamma$ satisfy $\delta(x) = b$ iff $x = (0, 0)$: then $\delta \notin \iota_J(A^G)$, so there must exist $p \in \mathcal{F}$, $\eta \in \Gamma$ such that $\delta^\eta|_{\operatorname{supp} p} = p$. It is straightforward to check that there exists exactly one $y \in D^\Gamma_M$ such that $p(y) = b$, and that $y = \eta^{-1} = (i, (-1)^{1-i} x)$ if $\eta = (i, x)$.

Now, for all $k \in \mathbb{Z}$ we have $d^\Gamma_S((0, k), (1, k)) = \|(1, 2k)\|^\Gamma_S = 1 + 2|k|$. This can be checked by observing the following two facts. Firstly, $(1, 2k) = (1, 0)(0, t) \ldots (0, t)$, with $2|k|$ factors $(0, t)$, and $t = 1$ or $t = -1$ according to $k > 0$ or $k < 0$. Secondly, multiplying (i, x) on the right by $(0, 1)$ or $(0, -1)$ does not change the value of i, while multiplying (i, x) on the right by $(1, 0)$ does not change $|x|$: hence, at least one multiplication by $(1, 0)$ and $2|k|$ multiplications by either $(0, 1)$ or $(0, -1)$ are necessary to reach $(1, 2k)$ from $(0, 0)$.

For $i \in \{0, 1\}$ let $\gamma_i = (i, 2M + 1)$. Let $\chi \in A^\Gamma$ be such that $\chi(\gamma) = b$ iff $\gamma = \gamma_0$ or $\gamma = \gamma_1$: then $\chi \in \iota_J(A^G)$. However, since $\eta^{-1} \in D^\Gamma_{M,S}$, for all $x \in D^\Gamma_M(\eta^{-1})$ we have $\gamma_0\eta x \in D^\Gamma_{2M}(\gamma_0)$. Hence, either $x = \eta^{-1}$, $\gamma_0\eta x = \gamma_0$, and $\chi^{\gamma_0\eta}(x) = b$; or $x \neq \eta^{-1}$, $0 < d_S(\gamma_0, \gamma_0\eta x) \leqslant 2M < 4M + 3 = d_S(\gamma_0, \gamma_1)$, and $\chi^{\gamma_0\eta}(x) = a$. Thus, $\chi^{\gamma_0\eta}|_{\mathrm{supp}\, p} = p$: this is a contradiction. □

The reason why Theorem 5.10 is true is that, in general, one cannot get an upper bound on $d_S(j_1g, j_2g)$ only by looking at $d_S(j_1, j_2)$, because the product is made *on the wrong side*. Consequently, one should not expect to determine finitely many constraints on the jg's only from finitely many constraints on the j's.

Corollary 5.11. *For cellular automata on arbitrary f.g. groups, finiteness of type is not invariant by conjugacy. In particular, for subshifts on arbitrary f.g. groups, finiteness of type is not a topological property.*

The first statement in Corollary 5.11 seems to collide with [8, Theorem 2.1.10], stating that any two conjugate subshifts of $A^{\mathbb{Z}}$ are either both of finite type or both not of finite type. Actually, in the cited result, conjugacies are always intended as being between *shift dynamical systems*, which is a much more specialized situation than ours. Moreover, the tessellation group is always $\mathbb{Z}$, so that *the action is also the same*, while we have different groups and different actions. Last but not least, by Proposition 3.6, the only groups where *all* the translations are UL-definable are the abelian groups. On the other hand, the second statement remarks the well-known phenomenon that homeomorphisms do not preserve finiteness of type, even in symbolic dynamics over $\mathbb{Z}$.

Example 5.12. Let $\mathcal{F} = \{10^{2n+1}1 \mid n \in \mathbb{N}\}$: the subshift $X = \mathsf{X}^{\{0,1\},\mathbb{Z}}_{\mathcal{F}}$ is called the *even shift*. It can be proved (cf. [8, Section 3.1]) that X is not a shift of finite type. However, X is

1. non-empty—it contains the configuration with all 0's,
2. compact—as a subshift,
3. metrizable—with the distance inherited from the full shift,
4. totally disconnected—because the full shift is, and
5. perfect—every point of X can be seen as the limit point of some sequence of elements of X taking the value 1 only finitely many times.

By a theorem of Brouwer, the even shift is homeomorphic to the Cantor set, thus also to the full shift—which *is* of finite type.

In our attempt at finding a general criterion for finiteness of type, we have crashed against an apparently unsurmountable obstacle. We thus choose to switch our aim towards a more modest target. What if we add conditions on the *way G* is related to Γ, and are more careful in the choice of J?

A possible answer is given by

Theorem 5.13. *Let H and K be f.g. groups; let S be a finite set of generators for H such that $1_H \notin S$; let $\Gamma = H \times K, G = \{1_H\} \times K$, $J = H \times \{1_K\}$. Let A be an alphabet and let*

$$\mathcal{F}_S = \left\{ p \in A^{\{(1_H, 1_K), (s, 1_K)\}} \mid s \in S \cup S^{-1} \setminus \{1_H\}, p((1_H, 1_K)) \neq p((s, 1_K)) \right\}.$$

For every set $\mathcal{F}$ of patterns on A with supports contained in G, $\iota_J(\mathsf{X}^{A,G}_{\mathcal{F}}) = \mathsf{X}^{A,\Gamma}_{\mathcal{F}\cup\mathcal{F}_S}$. In particular, if $X \subseteq A^G$ is a shift of finite type, then $\iota_J(X)$ is also a shift of finite type.

Proof. First, observe that $\mathcal{F}_S \subseteq \mathcal{F}'$, where $\mathcal{F}'$ is given by (8), so that $\iota_J(\mathsf{X}^{A,G}_{\mathcal{F}}) = \mathsf{X}^{A,\Gamma}_{\mathcal{F}\cup\mathcal{F}'} \subseteq \mathsf{X}^{A,\Gamma}_{\mathcal{F}\cup\mathcal{F}_S}$. (Less restrictions means more objects.)

Let now $\chi \in A^\Gamma \setminus \iota_J(X)$; suppose that no $p \in \mathcal{F}$ occurs in χ. Let $h_1, h_2 \in H$, $k \in K$ satisfy $\chi((h_1, k)) \neq \chi((h_2, k))$, and let $h_1^{-1}h_2 = s_1s_2 \cdots s_N$ be a writing of minimal length of the form (1). For $i \in \{0, \ldots, N\}$ let $a_i = \chi(h_1s_1 \ldots s_i, k)$; for $i \in \{1, \ldots, N\}$ define $p_i : \{(1_H, 1_K), (s_i, 1_K)\} \to A$ by $p_i(1_H, 1_K) = a_{i-1}$ and $p_i(s_i, 1_K) = a_i$. Since $a_0 \neq a_N$, $a_{i-1} \neq a_i$ for some i: then $p_i \in \mathcal{F}_S$ and $\chi^{(h_1s_1\cdots s_{i-1},k)}\big|_{\mathrm{supp}\, p_i} = p_i$. Since χ is arbitrary, $\mathsf{X}^{A,\Gamma}_{\mathcal{F}\cup\mathcal{F}_S} \subseteq \iota_J(\mathsf{X}^{A,G}_{\mathcal{F}})$. □

The reason why Theorem 5.13 holds is that, though $\mathcal{F}_S$ sets less restraints than $\mathcal{F}'$, the components J and G of the direct product also set less restraints *on the products* by not "interfering" with each other. Thus, any $j_1g_1 \cdots j_ng_n$ can be rewritten as $j_1 \cdots j_ng_1 \cdots g_n$, and the result is still of the form $jg, j \in J, g \in G$.

Observe that, for Theorem 5.13 to hold, G needs not to be of finite index in Γ. However, the other hypotheses are quite strong, especially the ones on the structure of Γ as a direct product with G as a factor. Because of Theorem 5.10, where Γ is a *semi-direct product*, this result is unlikely to be improved easily.

Example 5.14. Let $A = \{0, 1\}$; let H and K be two distinct copies of $\mathbb{Z}$ with $S = \{1\}$. Identify $\Gamma = H \times K$ with $\mathbb{Z}^2$, $J = H \times \{0\}$ and $G = \{0\} \times K$ with $\mathbb{Z}$. Let $p : \{(0,0), (0,1)\} \to \{0, 1\}$ satisfy $p(0,0) = p(0,1) = 1$: then $X = \mathsf{X}^{A,G}_{\{p\}}$ can be identified with the golden mean shift. Let $p_{01}, p_{10} : \{(1,0), (1,1)\} \to \{0, 1\}$ be defined by

$$p_{01}(1,0) = 0,\ p_{01}(1,1) = 1,\ p_{10}(1,0) = 1,\ p_{10}(1,1) = 0.$$

Then $\iota_J(X) = \mathsf{X}^{A,\Gamma}_{\{p,p_{01},p_{10}\}}$.

We conclude with a statement about sofic shifts.

Theorem 5.15. *Let A, G, Γ, and J be as in Lemma 5.5. Suppose $\iota_J(X)$ is a shift of finite type for every shift of finite type $X \subseteq A^G$. Then $\iota_J(X)$ is a sofic shift for every sofic shift $X \subseteq A^G$.*

Proof. Let $X = F(Y)$ for some shift of finite type $Y \subseteq A^G$ and UL-definable function $F : A^G \to A^G$. Let $\mathcal{N} \subseteq G$, $|\mathcal{N}| < \infty$, and $f : A^{\mathcal{N}} \to A$ be such that $(F(c))_g = f(c^g|_{\mathcal{N}})$ for all $c \in A^G$, $g \in G$; let $\mathcal{A} = \langle A^G, \mathcal{N}, f \rangle$ and let F' be the global evolution function of $\iota_J(\mathcal{A})$. By Lemma 5.5, $F' \circ \iota_J = \iota_J \circ F$, so that $\iota_J(X) = \iota_J(F(Y)) = F'(\iota_J(Y))$ is the image of a shift of finite type via a UL-definable function. □

We suspect that the hypotheses in Theorem 5.15 are, in fact, redundant. Again, we state this as a conjecture.

Conjecture 5.16. *Let A, G, Γ, and J be as in Lemma 5.5. Suppose $X \leqslant A^G$ is a sofic shift. Then $\iota_J(X)$ is a sofic shift.*

6. Conclusions

At the end of our trek, we have seen how to get CA presentations of dynamical systems, and to construct new shift subspaces and cellular automata by enlarging their underlying groups. We have then remarked the properties of old objects inherited by the new ones, and taken note of some exceptions. Finally, we have observed how enlarging the group makes the class of presentable dynamics grow.

There is still much unfinished work to do. In particular, much to our shame, we were not able to either prove or disprove Conjectures 4.6 and 5.16, nor to determine whether they have found a solution. Also additional conditions on the discerning action ϕ in the proof of Theorem 3.9, such to get a characterization of full CA dynamics, has been painfully missed.

Aside of looking ourselves for the answers to such questions, our hope is that our modest work can be of interest, or even use, to researchers in the field.

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