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On Generalized Fuzzy Relational Equations and Their Applications

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The paper provides an idea of generalization of fuzzy relational equations where t- and s-norms are introduced. The first part contains an extensive presentation of the resolution of fuzzy relational equations; next the solutions are specified for a list of several triangular norms. Moreover the dual equations are considered. The second part deals with the applicational aspects of these equations in systems analysis, decision-making, and arithmetic of fuzzy numbers. © 1985 Academic Press, Inc.

1. INTRODUCTION

Since the introduction of fuzzy set theory [14] a significant group of papers devoted to theoretical and applicational aspects of fuzzy relational equations (cf. [5, 9–11]) studied by Sanchez [12] and forming a generalization of well-known Boolean equations has appeared.

The aim of this paper is two-fold:

—to present a generalization of fuzzy relational equations with triangular norms and provide the method of their resolution;

—to discuss applicational aspects of this class of equations, mainly in systems analysis, decision-making, and fuzzy arithmetic.

The paper is organized as follows. Section 2 introduces notation and notions which will be useful in further discussion and presents problem formulation. In Section 3 the resolution of the equation is provided, and the results are extended for various structures of fuzzy relational equations, such as, composite equations and polynomial equations. The formulation of the problems appearing in the fields of interest of fuzzy set theory, such as systems analysis, decision-making, and fuzzy arithmetic, is shown and discussed in Section 4.

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2. Preliminary Remarks and Problem Formulation

Let X, Y, Z,... be fuzzy sets defined on the spaces (universes of discourse) $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$,..., respectively, and R be a fuzzy relation expressed on the cartesian product $\mathfrak{X} \times \mathfrak{Y} \times \mathfrak{Z} \times \cdots \times$. $F(\mathfrak{X})$ denotes a family of all fuzzy sets defined on the space $\mathfrak{X}$,

$$F(\mathfrak{X}) = \{X \subseteq \mathfrak{X} \mid X\colon \mathfrak{X} \to [0, 1]\}. \tag{1}$$

For simplicity of notation, we identify fuzzy sets and fuzzy relations with their membership functions. The triangular norms which play a significant role in further investigations are defined as follows.

Definition 1. The t-norm is a continuous two-place function

$$t\colon [0, 1] \times [0, 1] \to [0, 1] \tag{2}$$

satisfying a collection of the properties

(i)	$0tx = 0, \quad 1tx = x$			(3)
(ii)	$xty \leqslant ztw$	if $x \leqslant z$ and $y \leqslant w$	monotonicity	(4)
(iii)	$xty = ytx$		commutativity	(5)
(iv)	$(xty)\,tz = xt(ytz)$		associativity	(6)

$x, y, z, w \in [0, 1]$.

Definition 2. The s-norm is a continuous two-place function

$$s\colon [0, 1] \times [0, 1] \to [0, 1] \tag{7}$$

such that

(i)	$0sx = x, \quad 1sx = 1$			(8)
(ii)	$xsy \leqslant wsz$	if $x \leqslant w$ and $y \leqslant z$	monotonicity	(9)
(iii)	$xsy = ysx$		commutativity	(10)
(iv)	$(xsy)\,sz = xs(ysz)$		associativity	(11)

$x, y, z, w \in [0, 1]$.

Considering any t- or s-norm it can be easily noted that the following relationship holds true:

$$\underset{x, y \in [0,1]}{\forall} \quad xsy = 1 - (1 - x)\,t(1 - y). \tag{12}$$

Some t- and s-norms are listed below [6–8, 13].

t-norms

$$x t_1 y 1 - \min(1, ((1-x)^p + (1-y)^p)^{1/p}), \qquad p \geqslant 1 \tag{13}$$

$$x t_2 y = \log_s(1 + (s^x - 1)(s^y - 1)/(s-1)), \qquad 0 < s < \infty,\ s \neq 1 \tag{14}$$

$$x t_3 y = xy \tag{15}$$

$$x t_4 y = xy/(\gamma + (1-\gamma)(x + y - xy)), \qquad \gamma \geqslant 0. \tag{16}$$

s-norms

$$x s_1 y = \min(1, (x^p + y^p)^{1/p}), \qquad p \geqslant 1 \tag{17}$$

$$x s_2 y = 1 - \log_s(1 + (s^{1-x} - 1)(s^{1-y} - 1)/(s-1)), \qquad 0 < s < \infty,\ s \neq 1 \tag{18}$$

$$x s_3 y = x + y - xy \tag{19}$$

$$x s_4 y = (xy(\gamma - 2) + x + y)/(xy(\gamma - 1) + 1), \qquad \gamma \geqslant 0. \tag{20}$$

Moreover, specifying the values of the parameters standing in formulas given above, we get

$$\lim_{p \to \infty} (x t_1 y) = \lim_{s \to 0} (x t_2 y) = \min(x, y) \tag{21}$$

$$p = 1,\ x t_1 y = \lim_{s \to \infty} (x t_2 y) = \max(0, x + y - 1) \tag{22}$$

$$\gamma = 1,\ x t_4 y = \lim_{s \to 1} (x t_2 y) = xy \tag{23}$$

$$\lim_{s \to \infty} (x s_1 y) = \lim_{s \to 0} (x s_2 y) = \max(x, y) \tag{24}$$

$$\gamma = 1,\ x s_4 y = \lim_{s \to 1} (x s_2 y) = x + y - xy \tag{25}$$

$$\lim_{s \to \infty} (x s_2 y) = \min(x + y, 1), \tag{26}$$

$x, y \in [0, 1]$.

For discussion of a general way of constructing the triangular norm by use of the theory of functional equations the reader is referred to Aczél's monograph [1, Chap. 6].

We consider a fuzzy relational equation of the type

$$Y = X \square R \tag{27}$$

and a dual fuzzy relational equation

$$Y = X \triangle R \tag{28}$$

where $X \in F(\mathfrak{X})$, $Y \in F(\mathfrak{Y})$, $R \in F(\mathfrak{X} \times \mathfrak{Y})$; "$\square$" and "$\triangle$" stand for sup-$t$ and inf-s compositions, respectively. Making use of the membership functions of X, Y and R Eqs. (27)–(28) are read as follows:

$$Y(y) = \sup_{x \in \mathfrak{X}} [X(x)\, t R(x, y)] \tag{29}$$

$$Y(y) = \inf_{x \in \mathfrak{X}} [X(x)\, s R(x, y)]. \tag{30}$$

The problem of the resolution of the fuzzy relational equations given above is formulated as follows:

—X, Y are given, determine R,

—Y, R are given, determine X,

and will be solved in Section 3.

3. Resolution of Fuzzy Relational Equations $X \square R = Y$ and $X \triangle R = Y$

In order to solve the problems formulated in the previous section let us introduce two operators corresponding to the t- and s-norm.

DEFINITION 3. The Ψ-operator is a two-place function

$$\Psi\colon [0, 1] \times [0, 1] \to [0, 1] \tag{31}$$

such that

$$\text{(i)}\quad x \Psi \max(y, z) \geq \max(x \Psi y, x \Psi z) \tag{32}$$

$$\text{(ii)}\quad x t (x \Psi y) \leq y \tag{33}$$

$$\text{(iii)}\quad x \Psi (x t y) \geq y, \tag{34}$$

$x, y, z \in [0, 1]$.

DEFINITION 4. The β-operator is a two-place function

$$\beta\colon [0, 1] \times [0, 1] \to [0, 1] \tag{35}$$

fulfilling the properties

$$\text{(i)}\quad x \beta \min(y, z) \leq \min(x \beta y, x \beta z) \tag{36}$$

$$\text{(ii)}\quad x s (x \beta y) \geq y \tag{37}$$

$$\text{(iii)}\quad x \beta (x s y) \leq y. \tag{38}$$

The φ- and β-compositions are defined as follows:

—the φ (β) composition of fuzzy sets $X \in F(\mathfrak{X})$, $Y \in F(\mathfrak{Y})$ is the fuzzy relation $X \varphi Y$, $(X \beta Y) \in F(\mathfrak{X} \times \mathfrak{Y})$ with the membership function

$$(X \varphi Y)(x, y) = X(x)\ \Psi Y(y) \tag{39}$$

$$(X \beta Y)(x, y) = X(x)\ \beta Y(y) \tag{40}$$

$x \in \mathfrak{X}$, $y \in \mathfrak{Y}$,

—the φ (β) composition of the fuzzy relation $R \in F(\mathfrak{X} \times \mathfrak{Y})$ and the fuzzy set $Y \in F(\mathfrak{Y})$ is the fuzzy set $R \varphi Y$ $(R \beta Y) \in F(\mathfrak{X})$ with the membership function

$$(R \varphi Y)(x) = \inf_{y \in \mathfrak{Y}} (R(x, y)\ \Psi Y(y)) \tag{41}$$

$$(R \beta Y)(x) = \sup_{y \in \mathfrak{Y}} (R(x, y)\ \beta Y(y)) \tag{42}$$

Let us prove the following lemmas.

LEMMA 1.

$$\underset{X \in F(\mathfrak{X})}{\forall} \quad \underset{R \in F(\mathfrak{X} \times \mathfrak{Y})}{\forall} \quad R \subseteq X \varphi (X \square R). \tag{43}$$

Proof. Rewriting the right side of Eq. (43) in terms of the membership functions we get

$$\begin{aligned}(X \varphi (X \square R))(x, y) &= X(x)\ \Psi(X \square R)(y) = X(x)\ \Psi[\sup_{z \in \mathfrak{X}}(X(z)\ tR(z, y))] \\ &= X(x)\ \Psi\{\max[\sup_{z \in \mathfrak{X}, z \neq x}(X(z)\ tR(z, y)), X(x)\ tR(x, y)]\} \\ &\geqslant X(x)\ \Psi(X(x)\ tR(x, y)).\end{aligned} \tag{44}$$

Applying the inequality (34) we get

$$(X \varphi (X \square R))(x, y) \geqslant R(x, y) \tag{45}$$

for every $x \in \mathfrak{X}$, $y \in \mathfrak{Y}$, which completes the proof.

LEMMA 2.

$$\underset{X \in F(\mathfrak{X})}{\forall} \quad \underset{Y \in F(\mathfrak{Y})}{\forall} \quad X \square (X \varphi Y) \subseteq Y. \tag{46}$$

Proof. The inequality (46) forms a simple consequence of Eq. (33).

LEMMA 3.

$$\forall_{Y \in F(\mathfrak{Y})} \forall_{R \in F(\mathfrak{X} \times \mathfrak{Y})} (R \circledP Y) \square R \subseteq Y. \tag{47}$$

Proof. Rewriting the above-stated equation in terms of membership functions of R and Y we get

$$\sup_{x \in \mathfrak{X}} \{ \inf_{y \in \mathfrak{Y}} [R(x, y) \, \Psi Y(y)] \, tR(x, y) \} \tag{48}$$

and then

$$\begin{aligned} &\sup_{x \in \mathfrak{X}} \{ \inf_{y \in \mathfrak{Y}} [R(x, y) \, \Psi Y(y)] \, tR(x, y) \} \\ &\quad = \sup_{x \in \mathfrak{X}} \{ \min [\inf_{z \in \mathfrak{Y}, z \neq y} (R(x, z) \, \Psi Y(z)), R(x, y) \, \Psi Y(y)] \, tR(x, y) \} \\ &\quad \leqslant \sup_{x \in \mathfrak{X}} ((R(x, y) \, \Psi Y(y)) \, tR(x, y)), \end{aligned} \tag{49}$$

which, taking into account Eq. (33), involves an inequality

$$\sup_{x \in \mathfrak{X}} [(R(x, y) \, \Psi Y(y)) \, tR(x, y)] \leqslant Y(y) \tag{50}$$

for every $y \in \mathfrak{Y}$, which completes the proof.

LEMMA 4.

$$\forall_{X \in F(\mathfrak{X})} \forall_{R \in F(\mathfrak{X} \times \mathfrak{Y})} X \subseteq R \circledP (X \square R). \tag{51}$$

Proof. We have

$$\begin{aligned} &\inf_{y \in \mathfrak{Y}} [R(x, y) \, \Psi (\sup_{x \in \mathfrak{X}} (X(x) \, tR(x, y))] \\ &\quad = \inf_{y \in \mathfrak{Y}} [R(x, y) \, \Psi \max (\sup_{z \in \mathfrak{X}, z \neq x} (X(z) \, tR(z, y)), X(x) \, tR(x, y))] \end{aligned} \tag{52}$$

and Eq. (34) involves the inequality

$$\begin{aligned} &\inf_{y \in \mathfrak{Y}} [R(x, y) \, \Psi (\sup_{x \in \mathfrak{X}} (X(x) \, tR(x, y))] \\ &\quad \geqslant \inf_{y \in \mathfrak{Y}} [R(x, y) \, \Psi (X(x) \, tR(x, y))] \geqslant X(x) \end{aligned} \tag{53}$$

for every $x \in \mathfrak{X}$ and hence the proposition.

Now we can prove the following theorem.

THEOREM 1. (i) *If fuzzy sets* $X \in F(\mathfrak{X})$, $Y \in F(\mathfrak{Y})$ *fulfill* sup-t *fuzzy relational equation* $X \square R = Y$, *then the greatest fuzzy relation* $\hat{R} \in F(\mathfrak{X} \times \mathfrak{Y})$ *such that* $X \square \hat{R} = Y$ *holds true is provided by means of the formula*

$$\hat{R} = X \circledast Y. \tag{54}$$

(ii) *If fuzzy set* $Y \in F(\mathfrak{Y})$ *and fuzzy relation* $R \in F(\mathfrak{X} \times \mathfrak{Y})$ *satisfy equation* $X \square R = Y$, *then the greatest fuzzy set* $\hat{X} \in F(\mathfrak{X})$ $(\hat{X} \square R = Y)$ *is equal to*

$$\hat{X} = R \circledast Y. \tag{55}$$

Proof. (i) Lemma 1 involves an inequality $R \subseteq \hat{R}$, denoting $\hat{R} = X \circledast Y$. Moreover from the fact that $R \subseteq \hat{R}$ we get $X \square R \subseteq X \square \hat{R}$, viz., $Y \subseteq X \square \hat{R}$. Then bearing in mind Lemma 2, we have $X \square \hat{R} \subseteq Y$, so finally $X \square \hat{R} = Y$.

(ii) Denoting $\hat{X} = R \circledast Y$, from Lemma 3 we obtain the relationship $\hat{X} \square R \subseteq Y$. The inequality $X \subseteq \hat{X}$ involves $X \square R \subseteq \hat{X} \square R$ and then using Lemma 4 we have $X \subseteq R \circledast Y = \hat{X}$, so $\hat{X} \square R = Y$.

The similar collection of lemmas can be proved for inf-s fuzzy relational equations.

LEMMA 5.

$$\forall_{X \in F(\mathfrak{X})} \ \forall_{R \in F(\mathfrak{X} \times \mathfrak{Y})} \quad X \circledB (X \triangle R) \subseteq R. \tag{56}$$

LEMMA 6.

$$\forall_{X \in F(\mathfrak{X})} \ \forall_{Y \in F(\mathfrak{Y})} \quad X \triangle (X \circledB Y) \supseteq Y. \tag{57}$$

LEMMA 7.

$$\forall_{R \in F(\mathfrak{X} \times \mathfrak{Y})} \ \forall_{Y \in F(\mathfrak{Y})} \quad (R \circledB Y) \triangle R \supseteq Y. \tag{58}$$

LEMMA 8.

$$\forall_{X \in F(\mathfrak{X})} \ \forall_{R \in F(\mathfrak{X} \times \mathfrak{Y})} \quad (R \circledB (X \triangle R) \subseteq X. \tag{59}$$

The following theorem forms a straightforward consequence of these lemmas.

THEOREM 2. (i) *If fuzzy sets* $X \in F(\mathfrak{X})$, $Y \in F(\mathfrak{Y})$ *satisfy* inf-s *fuzzy relational equation* $X \triangle R = Y$, *then the least fuzzy relation* $\check{R} \in F(\mathfrak{X} \times \mathfrak{Y})$ *fulfilling equation* $X \triangle \check{R} = Y$ *is equal to*

$$\check{R} = X \textcircled{\beta} Y. \tag{60}$$

(ii) *If fuzzy set* $Y \in F(\mathfrak{Y})$ *and fuzzy relation* $R \in F(\mathfrak{X} \times \mathfrak{Y})$ *satisfy* inf-s *fuzzy relational equation* $X \triangle R = Y$, *then the least fuzzy set* $\check{X} \in F(\mathfrak{X})$ *such that* $\check{X} \triangle R = Y$ *holds true, is equal to*

$$X = R \textcircled{\beta} Y. \tag{61}$$

For the examples of the t- and s-norms specified before we have the following operators:

$$\begin{aligned} x \Psi_1 y &= 1, && \text{if } x \leqslant y \\ &= 1 - ((1-y)^p - (1-x)^p)^{1/p}, && \text{if } x > y,\ p \geqslant 1 \end{aligned} \tag{62}$$

$$\begin{aligned} x \Psi_2 y &= 1, && \text{if } x \leqslant y \\ &= \log_s(1 + (s^y - 1)(s-1)/(s^x - 1)), && \text{if } x > y,\ 0 < s < \infty,\ s \neq 1 \end{aligned} \tag{63}$$

$$\begin{aligned} x \Psi_3 y &= 1, && \text{if } x \leqslant y \\ &= y/x, && \text{if } x > y \end{aligned} \tag{64}$$

$$\begin{aligned} x \Psi_4 y &= 1, && \text{if } x \leqslant y \\ &= (y\gamma + xy(1-\gamma))/(x - y + y\gamma + xy(1-\gamma)), && \text{if } x > y,\ \gamma > 0 \end{aligned} \tag{65}$$

and

$$\begin{aligned} x \beta_1 y &= 0, && \text{if } x > y \\ &= (y^p - x^p)^{1/p}, && \text{if } x \leqslant y,\ p \geqslant 1 \end{aligned} \tag{66}$$

$$\begin{aligned} x \beta_2 y &= 0, && \text{if } x > y \\ &= 1 - \log_s(1 + (s^{1-y} - 1)(s-1)/(s^{1-x} - 1)), && \text{if } x \leqslant y, \\ & && 0 < s < \infty,\ s \neq 1 \end{aligned} \tag{67}$$

$$\begin{aligned} x \beta_3 y &= 0, && \text{if } x > y \\ &= (y-x)/(1-x), && \text{if } x \leqslant y \end{aligned} \tag{68}$$

$$\begin{aligned} x \beta_4 y &= 0, && \text{if } x > y \\ &= (y-x)/(x(\gamma-1) - y) + 1), && \text{if } x \leqslant y,\ \gamma > 0. \end{aligned} \tag{69}$$

The operators Ψ_1, Ψ_2, β_1, β_2 are also displayed in Figs. 1–4.

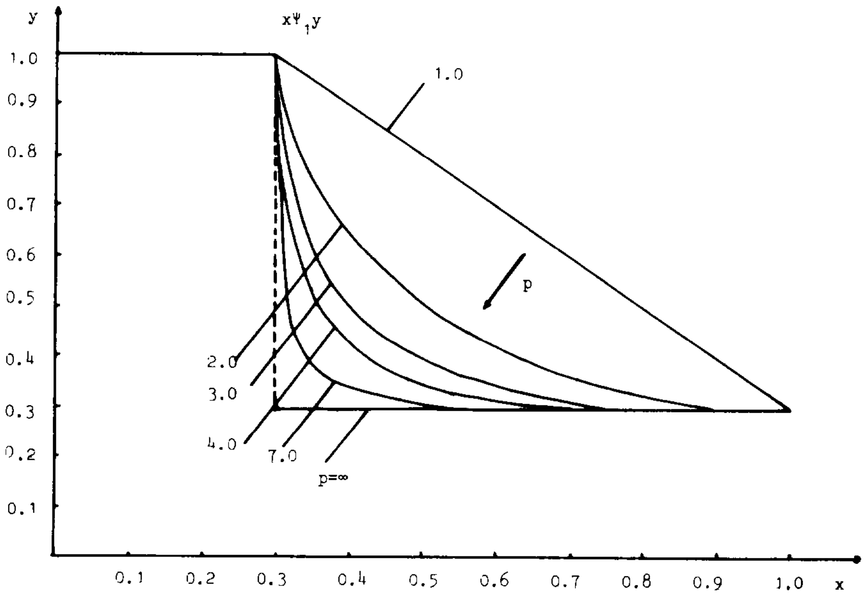

FIG. 1. $x\Psi_1 y$ vs x ($y = 0.3$ fixed).

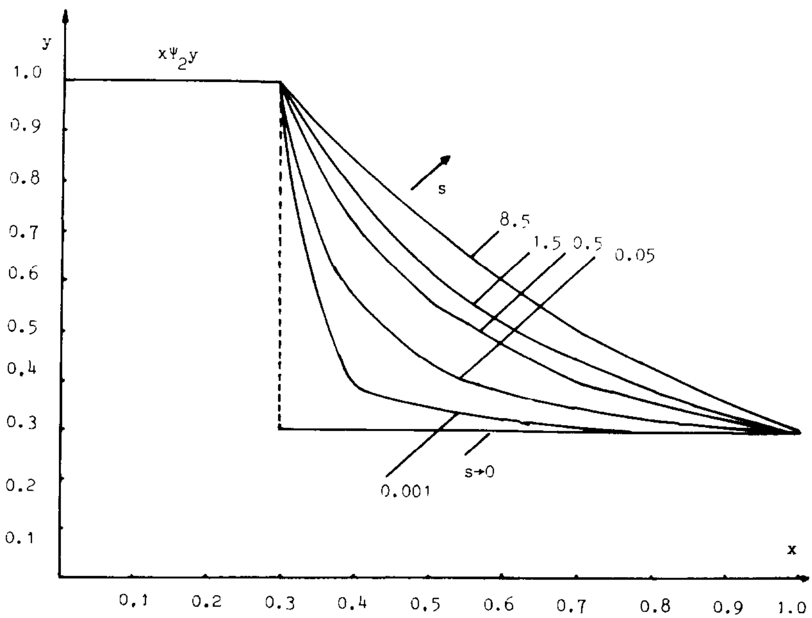

FIG. 2. $x\Psi_2 y$ vs x ($y = 0.3$ fixed).

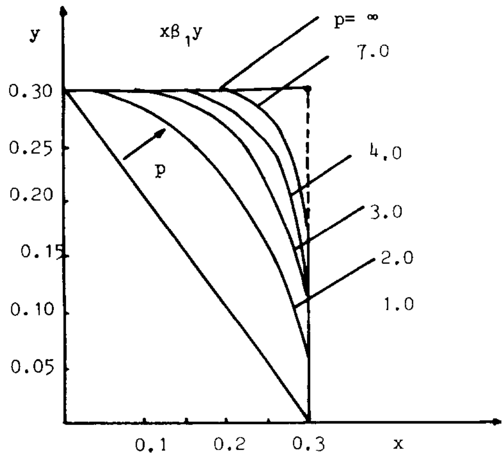

FIG. 3. $x\beta_1 y$ vs x ($y = 0.3$ fixed).

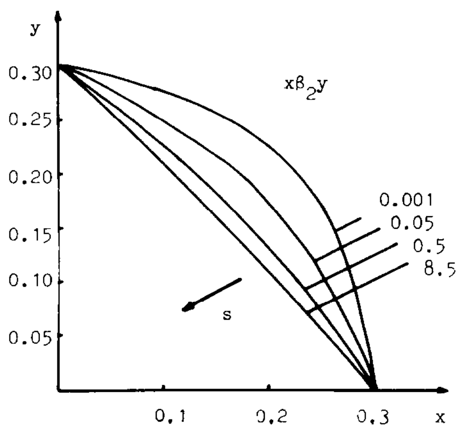

FIG. 4. $x\beta_2 y$ vs x ($y = 0.3$ fixed).

4. Resolution of Various Types of Generalized Fuzzy Relational Equations

In the previous section we were interested in the resolution of the equation of the simplest structure, viz., $Y=X\Box R$ (or $Y=X\triangle R$). We shall extend our results for equations possessing the structure

$$Y=X_1\Box X_2\Box\cdots\Box X_n\Box R \tag{70}$$

$$Y=X_1\triangle X_2\triangle\cdots\triangle X_n\triangle R \tag{71}$$

and composite fuzzy relational equations

$$T=S\Box W \tag{72}$$

$$T=S\triangle W \tag{73}$$

where now $X_i\in F(\mathfrak{X}_i)$, $i=1,2,\dots,n$, $Y\in F(\mathfrak{Y})$, $R\in F(\mathsf{X}_{i=1}^{n}\mathfrak{X}_i\times\mathfrak{Y})$ and $S\in F(\mathfrak{X}\times\mathfrak{Z})$, $W\in F(\mathfrak{Z}\times\mathfrak{Y})$, $T\in F(\mathfrak{X}\times\mathfrak{Y})$. Applying the membership functions of respective fuzzy sets and relations, Eqs. (70)–(73) are read as follows:

$$Y(y)=\sup_{\substack{x_1\in\mathfrak{X}_1\\x_2\in\mathfrak{X}_2\\\vdots\\x_n\in\mathfrak{X}_n}}(X_1(x_1)\,tX_2(x_2)\,t\cdots tX_n(x_n)\,tR(x_1,x_2,\dots,x_n,y)) \tag{74}$$

$$Y(y)=\inf_{\substack{x_1\in\mathfrak{X}_1\\x_2\in\mathfrak{X}_2\\\vdots\\x_n\in\mathfrak{X}_n}}(X_1(x_1)\,sX_2(x_2)\,s\cdots sX_n(x_n)\,sR(x_1,x_2,\dots,x_n,y)) \tag{75}$$

$y\in\mathfrak{Y}$,

$$T(x,y)=\sup_{z\in\mathfrak{Z}}(S(x,z)\,tW(z,y)) \tag{76}$$

$$T(x,y)=\inf_{z\in\mathfrak{Z}}(S(x,z)\,sW(z,y)), \tag{77}$$

$x\in\mathfrak{X}$, $y\in\mathfrak{Y}$.

Now the problem can be formulated as:

For Eqs. (70)–(71),

(a) let $X_1, X_2,\dots, X_n, Y$ be given, determine R,

(b) let $R, X_1, X_2,\dots, X_{j-1}, X_{j+1},\dots, X_n, Y$ be given, determine X_j,

(c) let R, Y be given, determine $X_1, X_2,\dots, X_n$.

Starting with (a) let us note that Eq. (74) can be rewritten as follows:

$$Y(y)=\sup_{\underline{x}\in\mathsf{X}_{i=1}^{n}\mathfrak{X}_i}(\underline{X}(\underline{x})\,tR(\underline{x},y)) \tag{78}$$

where $\underline{X}$ is the fuzzy relation defined on the cartesian product of $\mathfrak{X}_i$'s. Therefore the greatest fuzzy relation of the Eq. (74) is immediately expressed as

$$\hat{R} = \underline{X} \circledast Y, \tag{79}$$

viz.,

$$\hat{R}(x_1, x_2,..., x_n, y) = (X_1(x_1)\, tX_2(x_2)\, t \cdots tX_n(x_n))\, \Psi Y(y) \tag{80}$$

$x_i \in \mathfrak{X}_i$, $y \in \mathfrak{Y}$.

In (b) we obtain

$$Y = X_j \square (X_1 \square X_2 \square \cdots \square X_{j-1} \square X_{j+1} \square R) = X_j \square G \tag{81}$$

$$G \in F\left(\mathop{\mathsf{X}}_{\substack{i=1 \\ i \neq j}}^{n} \mathfrak{X}_i \times \mathfrak{Y} \right)$$

and then

$$X_j = G \circledast Y. \tag{82}$$

For (c) we put

$$Y(y) = (X_1 \square X_2 \square \cdots \square X_n) \square R \tag{83}$$

treating $(X_1 \square X_2 \square \cdots \square X_n)$ as the fuzzy relation defined on $\mathsf{X}_{i=1}^n \mathfrak{X}_i$ in the following way:

$$(X_1 \square X_2 \square \cdots \square X_n)(x_1, x_2,..., x_n) = X_1(x_1)\, tX_2(x_2)\, t \cdots tX_n(x_n), \tag{84}$$

$x_i \in \mathfrak{X}_i$. This fact yields

$$(X_1 \square X_2 \hat{\square} \cdots \square X_n) = R \circledast Y. \tag{85}$$

Now the respective fuzzy set $\hat{X}_i \in F(\mathfrak{X}_i)$ can be obtained by the projection of the relation $(X_1 \square X_2 \hat{\square} \cdots \square X_n)$ on the appropriate space, viz.

$$\hat{X}_i = \mathrm{Proj}_{\mathfrak{X}_i}(X_1 \square X_2 \hat{\square} \cdots \square X_n) \tag{86}$$

$$\hat{X}_i(x_i) = \sup_{\substack{x_1 \in \mathfrak{X}_1 \\ \vdots \\ x_{i-1} \in \mathfrak{X}_{i-1} \\ x_{i+1} \in \mathfrak{X}_{i+1} \\ \vdots \\ x_n \in \mathfrak{X}_n}} (X_1 \square X_2 \square \hat{\cdots} \square X_n)(x_1, x_2,..., x_n) \tag{87}$$

but we have to underline the fact that fuzzy sets $\hat{X}_i$, $i = 1, 2, \ldots, n$, computed by means of formula (87) fulfill Eq. (70) if the fuzzy relation $(X_1 \square X_2 \hat{\square} \cdots \square X_n)$ is separable (fuzzy sets X_i are weakly noninteractive [6]), we get

$$\hat{X}_1(x_1)\, t \hat{X}_2(x_2)\, t \cdots t \hat{X}_n(x_n) = (X_1 \square X_2 \hat{\square} \cdots \square X_n)(x_1, x_2, \ldots, x_n) \quad (88)$$

$x_i \in \mathfrak{X}_i$.

The results for inf-s fuzzy relational equations can be derived in an analogous way.

Considering a composite fuzzy relational equation $T = S \square W$ we state two problems:

(a) S, T are given, determine W,

(b) W, T are given, determine S.

Following the main stream of analysis presented in Section 3 we arrive at the following result (cf. [12]).

THEOREM 3. (i) *If fuzzy relations $S \in F(\mathfrak{X} \times \mathfrak{Z})$ and $T \in F(\mathfrak{X} \times \mathfrak{Y})$ fulfill Eq. (72), then the greatest fuzzy relation $\hat{W} \in F(\mathfrak{Z} \times \mathfrak{Y})$ satisfying $T = S \square \hat{W}$ is equal to*

$$\hat{W} = S^T \circledast T \quad (89)$$

$$\hat{W}(z, y) = \inf_{x \in \mathfrak{X}} (S^T(z, x)\, \Psi T(x, y)) \quad (90)$$

(S^T stands for the transpose of S, $S^T(z, x) = S(x, z)$).

(ii) *If fuzzy relations $W \in F(\mathfrak{Z} \times \mathfrak{Y})$ and $T \in F(\mathfrak{X} \times \mathfrak{Y})$ satisfy Eq. (72) then the greatest fuzzy relation $\hat{S} \in F(\mathfrak{X} \times \mathfrak{Z})$ satisfying equation $T = \hat{S} \square W$ is provided by the formula*

$$\hat{S} = (W \circledast T^T)^T \quad (91)$$

$$\hat{S}(x, z) = (\inf_{y \in \mathfrak{Y}} (W(z, y)\, \Psi T^T(y, x)))^T. \quad (92)$$

For the dual fuzzy relational equation (73) we obtain formulas:

—S, T given, the least fuzzy relation $\check{W}$ is equal to

$$\check{W} = S^T \circledcirc T. \quad (93)$$

—W, T given, the least fuzzy relation $\check{S}$ is equal to

$$\check{S} = (W \circledcirc T^T)^T. \quad (94)$$

We can also consider polynomial fuzzy relational equations (cf. [9]) of the type

$$\bigcup_{i=1}^{I} (A^{(i)} \square X \square B^{(i)}) = Y, \tag{95}$$

where $A^{(i)} \in F\,(\mathfrak{X} \times \mathfrak{Z})$, $X \in F\,(\mathfrak{Z} \times \mathfrak{W})$, $B^{(i)} \in F\,(\mathfrak{W} \times \mathfrak{Y})$, $Y \in F\,(\mathfrak{X} \times \mathfrak{Y})$ and all universes of discourse consist of a finite number of elements, $\mathfrak{X} = \{x_1, x_2,..., x_n\}$, $\mathfrak{Z} = \{z_1, z_2,..., z_r\}$, $\mathfrak{W} = \{w_1, w_2,..., w_p\}$, $\mathfrak{Y} = \{y_1, y_2,..., y_m\}$. Then Eq. (96) can be expressed in the form

$$\mathbf{Y} = \mathbf{X} \square \mathbf{A} \tag{96}$$

where **A** is the fuzzy matrix constructed on the base of the matrices A$^{(i)}$ *and* $B^{(i)}$, $i = 1, 2,..., I$, and is equal to

$$\mathbf{A} = \bigcup_{i=1}^{I} (A^{(i)} \, ⓣ \, (B^{(i)})^T)^T \tag{97}$$

$$A^{(i)} \, ⓣ \, (B^{(i)})^T$$

$$= \begin{bmatrix} A^{(i)}(x_1, z_1)\, tB^{(i)})^T & A^{(i)}(x_1, z_2)(B^{(i)})^T & \cdots & A^{(i)}(x_1, z_r)\, t(B^{(i)})^T \\ \vdots & & & \\ A^{(i)}(x_n, z_1)\, t(B^{(i)})^T & A^{(i)}(x_n, z_2)\, t(B^{(i)})^T & \cdots & A^{(i)}(x_s, z_v)\, t(B^{(i)})^T \end{bmatrix} \tag{98}$$

$$A^{(i)}(x_s, z_v)\, t(B^{(i)})^T$$

$$= A^{(i)}(x_s, z_v)\, t \begin{bmatrix} B^{(i)}(w_1, y_1) & B^{(i)}(w_2, y_1) & \cdots & B^{(i)}(w_p, y_1) \\ B^{(i)}(w_2, y_1) & \cdots & & \\ \vdots & & & \\ B^{(i)}(w_1, y_m) & \cdots & & B^{(i)}(w_p, y_m) \end{bmatrix}$$

$$= \begin{bmatrix} A^{(i)}(x_s, z_v)\, tB^{(i)}(w_1, y_1), & A^{(i)}(x_s, z_v)\, tB^{(i)}(w_2, y_1) & \cdots \\ \vdots & & \\ A^{(i)}(x_s, z_v)\, tB^{(i)}(w_1, y_m), & A^{(i)}(x_s, z_v)\, tB^{(i)}(w_2, y_m) & \cdots \end{bmatrix}, \tag{99}$$

$s = 1, 2,..., n$, $v = 1, 2,..., r$.

X and **Y** are fuzzy sets resulting from vector representation of fuzzy relations X and Y.

$$\mathbf{X} = \underbrace{[X(z_1, w_1)\, X(z_1, w_2) \cdots X(z_1, w_p)\, X(z_2, w_1) \cdots X(z_r, w_p)]}_{rp\text{-elements}} \tag{100}$$

$$\mathbf{Y} = \underbrace{[Y(x_1, y_1)\, Y(x_1, y_2) \cdots Y(x_1, y_m)\, Y(x_2, y_1) \cdots Y(x_n, y_m)]}_{nm\text{-elements}}. \tag{101}$$

The fuzzy relational equation (96) can be solved using the formulas derived in Section 3.

5. Applicational Aspects of Fuzzy Relational Equations

In this section we shall present applications of the fuzzy relational equations with triangular norms as a useful tool for handling various problems of fuzzy set theory in a unique and compact manner, pointing out several ways leading to their resolutions.

One of the well-known areas of applications of fuzzy relational equations can be shown in systems analysis. Applying the concept of the state approach, any ill-defined, complex system with the fuzzy input, state, and output can be represented by means of a set of equations

$$\begin{cases} X_{k+1} = U_k \square X_k \square R & (102) \\ Y_{k+1} = X_{k+1} \square G & (103) \end{cases}$$

where $X_k, X_{k+1} \in F(\mathfrak{X})$, $U_k \in F(\mathfrak{U})$, $Y_k \in F(\mathfrak{Y})$ are fuzzy states, input, and output for discrete-time moments; $R \in F\ (\mathfrak{U} \times \mathfrak{X} \times \mathfrak{X})$, $G \in F\ (\mathfrak{X} \times \mathfrak{Y})$ are time-invariant fuzzy relations of the state and output. System equations (102)–(103) form a generalization of any deterministic system described by the use of the difference equation widely used in control theory. Identification and control problems can be formulated, for instance, in the following manner.

—Given a collection of fuzzy data–fuzzy sets of input and output, determine the structure and relations of the system.

—Given the desired state (output) of the system, determine a control policy (sequence of input fuzzy sets) which makes it possible to achieve the goal.

These problems are treated, e.g., in [3, 10, 11]. Analogously, as in non-fuzzy system analysis, we can consider the notion of the fuzzy system of higher order; the fuzzy system of the rth order is represented as

$$\begin{cases} X_{k+r} = U_k \square X_k \square X_{k+1} \square \cdots \square X_{k+r-1} \square R & (104) \\ Y_{k+r} = X_{k+r} \square G & (105) \end{cases}$$

$X_k, X_{k+1}, \ldots, X_{k+r} \in F(\mathfrak{X})$, $Y_{k+r} \in F(\mathfrak{Y})$,

$$R \in F(\mathfrak{U} \times \underbrace{\mathfrak{X} \times \mathfrak{X} \times \cdots \times \mathfrak{X}}_{(r+1)\text{-times}}),$$

$G \in F(\mathfrak{X} \times \mathfrak{Y})$.

Now, we formulate a decision-making process in terms of a fuzzy relational equation. Usually in such a situation we deal with a collection of goals and constraints expressed as fuzzy sets defined on respective spaces of goals $\mathfrak{G}_i$ and constraints $\mathfrak{C}_j$, viz., $G_i \in F(\mathfrak{G}_i)$, $C_j \in F(\mathfrak{C}_j)$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. A fuzzy decision D defined on the decision space $\mathfrak{D}$ ($D \in F(\mathfrak{D})$) depends on G_i and C_j and results from their aggregation; we can say there exists a relationship between G_i, C_j and D, which yields the equation

$$D = C_1 \Box C_2 \Box \cdots \Box C_m \Box G_1 \Box G_2 \Box \cdots \Box G_n \Box R \tag{106}$$

where R is a fuzzy relation defined on the cartesian product of the spaces $\mathsf{X}_{i=1}^{n} \mathfrak{G}_i \times \mathsf{X}_{j=1}^{m} \mathfrak{C}_j \times \mathfrak{D}$. Assuming that $\mathfrak{G}_i = \mathfrak{C}_j = \mathfrak{D}$ for all i, j, and specifying "$\Box$" as sup–min composition, we put R as the diagonal relation with the membership function

$$\begin{aligned} R(x_1, x_2, \ldots, x_{n+m+1}) &= 1, \qquad \text{if} \quad x_1 = x_2 = \cdots = x_{n+m+1} \\ &= 0, \qquad \text{otherwise}. \end{aligned} \tag{107}$$

Equation (106) is read as

$$D(x) = \min(C_1(x), C_2(x), \ldots, C_m(x), G_1(x), G_2(x), \ldots, G_n(x)) \tag{108}$$

so we obtain the well-known result given by Bellman and Zadeh [2].

The fuzzy relational equations discussed here can be also helpful in solving various problems in fuzzy arithmetic [5, 6]. Consider an equation for two fuzzy numbers

$$B = A + X \tag{109}$$

where A, B are known fuzzy numbers and the fuzzy number X has to be calculated. The formulation of the problem (108) in the form of a fuzzy relational equation leads immediately to the needed result. We have

$$B(b) = \sup_{a, x : b = x + a} (X(x) \, t A(a)) = \sup_x (X(x) \, t A(b - x)), \tag{110}$$

$a, b, x \in \mathfrak{R}$ $(= (-\infty, +\infty))$, and treating $A(b - x)$ as the fuzzy relation $R(x, b)$

$$R(x, b) = A(a - x) \tag{111}$$

for all $x, b \in \mathfrak{R}$, it yields

$$B = X \Box R \tag{112}$$

so finally the unknown fuzzy number is equal to

$$\hat{X} = R \circledP B. \tag{113}$$

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