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Existence of mild solutions for stochastic differential equations and semilinear equations with non-Gaussian Lévy noise

S. Albeverio[a,b], V. Mandrekar[c], B. Rüdiger[d,*]

[a] *Institut für Angewandte Mathematik der Universität Bonn, Wegelerstr. 6, D-53115 Bonn, Germany*
[b] *SFB 611, Germany*
[c] *Department of Statistics and Probability, Michigan State University, East Lansing, MI 48824, USA*
[d] *Mathematisches Institut, Universität Koblenz-Landau, Campus Koblenz, Universitätsstrasse 1, 56070 Koblenz, Germany*

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Abstract

Existence and uniqueness of the mild solutions for stochastic differential equations for Hilbert valued stochastic processes are discussed, with the multiplicative noise term given by an integral with respect to a general compensated Poisson random measure. Parts of the results allow for coefficients which can depend on the entire past path of the solution process. In the Markov case Yosida approximations are also discussed, as well as continuous dependence on initial data, and coefficients. The case of coefficients that besides the dependence on the solution process have also an additional random dependence is also included in our treatment. All results are proven for processes with values in separable Hilbert spaces. Differentiable dependence on the initial condition is proven by adapting a method of S. Cerrai.
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* Corresponding author.
E-mail address: ruediger@wiener.iam.uni-bonn.de (B. Rüdiger).

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1. Introduction

Let $\mathcal{A}$ be a (generally unbounded) linear operator on a domain $D(\mathcal{A}) \subset H$.

$$\mathcal{A} : D(\mathcal{A}) \subset H \to H.$$

Moreover, let $\mathcal{A}$ be the infinitesimal generator of a pseudo-contraction semigroup $(S_t)_{t\geq 0}$ (see Definition 2.1 in Section 2 or e.g. Appendix A of [15], or [21], for the definition and the description of properties of pseudo-contraction semigroups).

In this article we shall study existence and uniqueness of the mild solutions of the stochastic differential equation (SDE)

$$\mathrm{d}Z = \mathcal{A}Z\mathrm{d}t + A(t, Z)\mathrm{d}t + \int_{H\setminus\{0\}} F(s, u, Z)q(\mathrm{d}s\mathrm{d}u) \tag{1}$$

on each time interval $[0, T]$, $T > 0$. $Z := (Z_t(\omega))_{t\in[0,T]}$ is a process with values in a separable Hilbert space H with norm $\|\cdot\|_H$. $q(\mathrm{d}s\mathrm{d}u) := N(\mathrm{d}s\mathrm{d}u)(\omega) - \mathrm{d}s\beta(\mathrm{d}u)$ is a compensated Poisson random measure (cPrm) on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq +\infty}, P)$, satisfying the "usual hypothesis" (see Section 2). β is a Lévy measure (in the sense of e.g. [5,35,1]) on $\mathcal{B}(H\setminus\{0\})$, or, more generally, on $\mathcal{B}(E\setminus\{0\})$, where E is a separable Banach space. ($\mathcal{B}(Y)$ denotes the Borel σ-field on a topological space Y.) ds denotes as usual the Lebesgue measure on $\mathcal{B}(\mathbb{R}_+)$, and $N(\mathrm{d}s\mathrm{d}u)(\omega)$ is a Poisson distributed σ-finite measure on the σ-algebra $\mathcal{B}(\mathbb{R}_+ \times E \setminus \{0\})$, generated by the product semiring $\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(E \setminus \{0\})$ of the Borel σ-algebra $\mathcal{B}(\mathbb{R}_+)$ and the trace σ-algebra $\mathcal{B}(E \setminus \{0\})$. It is well known that a cPrm with compensator given by $\mathrm{d}s\beta(\mathrm{d}u)$ is associated to a Lévy process $(X_t)_{t\geq 0}$. (See e.g. [6] (or [37] Section 2), for the definition of trace σ-algebra, and e.g. [25,35,44,46,1,45,14] Section 2, [37] for the definition and properties of compensated Poisson random measures).

From Section 4 to Section 9 we assume $A(t, Z) = A(t, Z_t)$ and $F(t, u, Z) = F(t, u, Z_t)$.

In Section 3 we consider the case where the coefficients A and F, still being non-anticipating, depend on the path of the solution Z. For this case we assume that the cPrm is associated to a canonical Lévy process $(X_t)_{t\geq 0}$, defined on the associated filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ (see Section 2); i.e. for this case $\Omega = D(\mathbb{R}_+, H)$ is the space of càd-làg functions defined on $\mathbb{R}_+$ and with values in H, with the sup norm $\|\cdot\|_\infty := \sup_{t\in[0,T]} \|\cdot\|_H$. (When no misunderstanding is possible we write for a norm simply $\|\cdot\|$). $\mathcal{F} = \mathcal{F}_\infty$ and (for this case)

$$\mathcal{F}_t := \sigma\{Z := (Z_s)_{s\in[0,T]} \in D([0, T]; H) : \|Z_s\| \leq c, s \leq t, c \in \mathbb{R}\}. \tag{2}$$

Let $T > 0$ be fixed (and arbitrary). The stochastic process $Z := (Z_t(\omega))_{t\in[0,T]}$ is a mild solution of (1) with initial condition $Z_0 := Z_0(\omega)$, if it is a solution of the following convolution equation:

$$\begin{aligned} Z_t = {} & S_t Z_0 + \int_0^t S_{t-s}A(s, Z)\mathrm{d}s \\ & + \int_0^t \int_{E\setminus\{0\}} S_{t-s}F(s, u, Z)q(\mathrm{d}s\mathrm{d}u) \quad P\text{-a.s. } \forall t \in [0, T]. \end{aligned} \tag{3}$$

Let $\Lambda \in \mathcal{B}(E \setminus \{0\})$ and $t \in [0, T]$. Let $g(s, u, \omega)$ have values in a separable Hilbert space H. The stochastic integral

$$M_t := \int_0^t \int_\Lambda g q(\mathrm{d}u\mathrm{d}x) \tag{4}$$

is defined as the Ito integral w.r.t. the cPrm $q(\mathrm{d}s\mathrm{d}u) := N(\mathrm{d}s\mathrm{d}u)(\omega) - \mathrm{d}s\beta(\mathrm{d}u)$ (see Theorem 2.3 in Section 2). The Ito integral (4) exists if $g(s, u, \omega)$ is measurable in all the variables, satisfies adaptedness conditions with respect to the filtration and

$$\int_0^T \int_\Lambda \mathbf{E}[\|g(t, u)\|^2] \mathrm{d}t\beta(\mathrm{d}u) < \infty. \tag{5}$$

In [37] we proved that the Ito integrals (4) are càd-làg martingales. In the Appendix we show that the corresponding Meyer process is given by $\langle M\rangle_t := \int_0^T \int_\Lambda \|g(t, u, \cdot)\|^2 \mathrm{d}t\beta(\mathrm{d}u)$. We shall see in Section 2 that, as S_t is a pseudo-contraction semigroup, the Ito integrals $\int_0^t \int_\Lambda S_{t-s} g q(\mathrm{d}s\mathrm{d}x)$ exist under the same assumptions. Moreover we shall prove in Section 2 that the integrals on the r.h.s of (3) are càd-làg.

In Section 3 we shall prove existence and uniqueness of the solution of (3) for the case where the coefficients $A(s, Z)$ and $F(s, u, Z)$, as functions of $Z := (Z_t)_{t\in\mathbb{R}_+} \in D(\mathbb{R}_+, H)$, satisfy, for s, and resp. u fixed, a Lipschitz condition. The coefficients $A(s, Z)$ and $F(s, u, Z)$ are measurable and non-anticipating (see Section 3). Let us stress that the consideration of SDEs with coefficients depending on the entire past path of a process ("non-Markov case") is particularly important in certain applications, including the modelling of the dynamics of polymers moving in a random medium, see, e.g. [8,13,19].

In Section 4 we analyze the case $A(s, Z) = a(s, Z_s)$ and $F(s, u, Z) = f(s, u, Z_s)$ and assume however more general cPrms, which are not necessarily associated to canonical Lévy processes on the Skorohod space $D(\mathbb{R}_+, H)$. For this case, where the coefficients depend only on the process at time s, we can also analyze more properties of the solution $(Z_t)_{t\in[0,T]}$: in Section 5 we prove that a Yosida approximation theorem holds, in Section 6 we analyze Markov properties [20], in Section 8 we analyze the continuous dependence of the solution on initial data, drift and noise coefficients, in Section 9 we prove the differentiable dependence of the solution on the initial data. The results in Section 8 also hold for the case where the coefficients are random and of the form $a(s, Z_s, \omega)$ and $f(s, u, Z_s, \omega)$. For this case an existence result of a mild solution of (3) is proven in Section 7, but uniqueness holds only up to a version. The results obtained for the case of random coefficients in Section 7 are not only interesting on their own but they are also needed in Sections 8 and 9, where dependence on initial data is discussed.

We refer e.g. also to [15,17,23,24,26,27,30–33] and references therein for other interesting existence results on mild solutions of infinite-dimensional SDEs with Gaussian or Poisson noise. In particular we mention results by Knoche [31–33] which concern SDEs of the type (1) on Sobolev spaces, where properties of the differential dependence of the resolvent on the initial data are discussed. Work by Filipović and Tappe [22], by Marinelli [39], and Peszat, Zabczyk [42] concern applications of SDEs of the type (1) to financial models. In these work an analysis of a Lévy driven Heath–Jarrow–Morton term structure equation is given. We also refer to [36] for applications to filtering, where Zakai's equation is derived in a general setting. References [9–11,26] treat SPDEs with state space including M-type-p separable Banach spaces. In particular in [26] Hausenblas considers on such spaces SDEs of the type (1), assuming however that the operator $\mathcal{A}$ is the generator of a compact analytic semigroup. Existence and uniqueness of solutions are analyzed in [26] on the domain of the fractional powers of the operator $\mathcal{A}$, in terms of which Lipschitz conditions for the drift and noise coefficients (depending only on the solution process) are assumed. The method uses interpolation inequalities for fractional powers generating analytic semigroups of contractions. We refer also to a recent book by Peszat, Zabczyk [43], which contains a description of several results obtained (also recently) by the scientific community on SDEs with non-Gaussian noise on infinite-dimensional spaces.

2. Properties of stochastic integrals w.r.t. cPrms

We assume that a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le+\infty}, P)$, satisfying the "usual hypothesis", is given:

(i) $\mathcal{F}_t$ contains all null sets of $\mathcal{F}$, for all t such that $0 \le t < +\infty$.
(ii) $\mathcal{F}_t = \mathcal{F}_t^+$, where $\mathcal{F}_t^+ = \cap_{u>t} \mathcal{F}_u$, for all t such that $0 \le t < +\infty$, i.e. the filtration is right continuous.
Moreover, we assume
(iii) the filtration $\mathcal{F}_0$ is independent of $(\mathcal{F}_t)_{0<t\le+\infty}$.

In this Section we assume that $q(\mathrm{d}s\mathrm{d}x) := N(\mathrm{d}s\mathrm{d}x)(\omega) - \mathrm{d}s\beta(\mathrm{d}x)$ is a compensated Poisson random measure (cPrm) on $(E, \mathcal{B}(E))$, where E is a separable Banach space, and is defined on the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le+\infty}, P)$ (see e.g. [46] for the definition of cPrm). We then analyze the properties of stochastic integrals obtained by convolution of pseudo-contraction semigroups $(S_t)_{t\ge0}$ on a separable Hilbert space H w.r.t $N(\mathrm{d}s\mathrm{d}x)(\omega) - \mathrm{d}s\beta(\mathrm{d}x)$. I.e. we analyze the integrals

$$\int_0^t \int_{E\setminus\{0\}} S_{t-s} f(s,u) q(\mathrm{d}s\mathrm{d}u) \tag{6}$$

where $(S_t)_{t\ge0}$ is a pseudo-contraction semigroup on a separable Hilbert space H.

Definition 2.1. A continuous semigroup $(S_t)_{t\ge0}$ on H, which has the property

$$|||S_t||| \le \exp(\alpha t) \quad \forall t \ge 0. \tag{7}$$

for some constant $\alpha > 0$, and with $|||\cdot|||$ denoting the operator norm on H, is called a pseudo-contraction semigroup on H.

Let us first consider the stochastic integrals (4). These are defined like Ito integrals w.r.t. cPrms. The latter have been introduced and analyzed e.g. in [3,7,47], for the case of $\mathbb{R}^d$-valued functions, and in [37,45] for the case of Banach spaces valued functions. Let us recall here the definition.

Let $T > 0$ and

$$\begin{aligned} M^T(E/H) := \quad & \{g : \mathbb{R}_+ \times E\setminus\{0\} \times \Omega \to H, \text{ such that } g \text{ is jointly measurable and} \\ & \mathcal{F}_t\text{-adapted } \forall u \in E\setminus\{0\}, t \in (0,T]\} \end{aligned} \tag{8}$$

$$M_\beta^{T,2}(E/H) := \left\{ g \in M^T(E/H) : \int_0^T \int \mathbf{E}[\|g(t,u)\|^2]\mathrm{d}t\beta(\mathrm{d}u) < \infty \right\} \tag{9}$$

where $\mathbf{E}[\cdot]$ denotes the expectation w.r.t. the probability P.

The following "simple functions" are dense in the Banach space $M_\beta^{T,2}(E/H)$ with norm $\|g\|_2 := \sqrt{\int_0^T \int \mathbf{E}[\|g(t,u)\|^2]}$, see [45] Theorem 4.2 (In [47] Chapter 2, Section 4 a bigger set of simple functions is considered).

Definition 2.2. A function g belongs to the set $\Sigma(E/H)$ of simple functions, if $g \in M^T(E/H)$, $T > 0$ and there exist $n \in \mathbb{N}$, $m \in \mathbb{N}$, such that

$$g(t,x,\omega) = \sum_{k=1}^{n-1} \sum_{l=1}^{m} \mathbf{1}_{A_{k,l}}(x) \mathbf{1}_{F_{k,l}}(\omega) \mathbf{1}_{(t_k,t_{k+1}]}(t) a_{k,l} \tag{10}$$

where $0 \notin \overline{A_{k,l}}$, $t_k \in (0, T]$, $t_k < t_{k+1}$, $F_{k,l} \in \mathcal{F}_{t_k}$, $a_{k,l} \in H$. For all $k \in 1, \ldots, n-1$ fixed, $A_{k,l_1} \times F_{k,l_1} \cap A_{k,l_2} \times F_{k,l_2} = \emptyset$ if $l_1 \neq l_2$.

A stochastic integral of simple functions is defined in a very natural way (see Chapter 3 in [45]): let $g \in \Sigma(E/H)$, the "natural stochastic integral" of g is

$$\int_0^T \int_A g(t, x, \omega) q(\mathrm{d}t\mathrm{d}x)(\omega)$$
$$= \sum_{k=1}^{n-1} \sum_{l=1}^{m} a_{k,l} \mathbf{1}_{F_{k,l}}(\omega) q((t_k, t_{k+1}] \cap (0, T] \times A_{k,l} \cap A)(\omega). \tag{11}$$

The "Ito integral" (4) is well defined by approximation through "simple functions", for all $g \in M_\beta^{T,2}(E/H)$. In fact the following property holds (see e.g. [47] Chapter 2, Section 4 [7,41,3], or Theorem 4.14 in [45]):

Theorem 2.3. *Given a sequence g_n of simple functions approximating g in $M_\beta^{T,2}(E/H)$, the sequences $\int_0^t \int_\Lambda g_n q(\mathrm{d}s\mathrm{d}x)$ converge in $L^2(\Omega, \mathcal{F}, P)$ and the limit does not depend on the sequence.*

The limit $\int_0^t \int_\Lambda g q(\mathrm{d}s\mathrm{d}x)$ is the "Ito integral of g on $(0, t] \times \Lambda$ w.r.t. $q(\mathrm{d}s\mathrm{d}x)$"

(The "Ito integral w.r.t. $q(\mathrm{d}s\mathrm{d}x)$" is called "strong-2-integral" in [45,37], to distinguish it from the "strong-1-integral obtained by convergence in L^1.)

Theorem 2.3 is proven by verifying for the simple functions $g \in \Sigma(E/H)$ that the following equality holds:

$$\mathbf{E}\left[\left\| \int_0^t \int_{E\setminus\{0\}} g(s, u) q(\mathrm{d}s\mathrm{d}u) \right\|^2\right] = \int_0^t \int_{E\setminus\{0\}} \mathbf{E}[\|g(s, u)\|^2] \mathrm{d}s\beta(\mathrm{d}u) \quad t \in [0, T]. \tag{12}$$

By density of the simple functions in $M_\beta^{T,2}(E/H)$ it follows the validity of the isometry (12) for all integrands $g \in M_\beta^{T,2}(E/H)$ (see e.g. [47] Chapter 2, Section 4, or [7,41,3], or Theorem 4.14 in [45]):

Here we prove the following inequality.

Lemma 2.4. *Let $f(s, u, \omega) \in M_\beta^{T,2}(E/H)$. Then for any $t \in [0, T]$, $0 \leq s \leq t$, $S_{t-s} f(s, u, \omega) \in M_\beta^{T,2}(E/H)$ and*

$$\mathbf{E}\left[\left\| \int_0^t \int_{E\setminus\{0\}} S_{t-s} f(s, u) q(\mathrm{d}s\mathrm{d}u) \right\|^2\right] \leq \mathrm{e}^{2\alpha t} \int_0^t \int_{E\setminus\{0\}} \mathbf{E}[\|f(s, u)\|^2] \mathrm{d}s\beta(\mathrm{d}u) \tag{13}$$

(with α as in (7)).

Proof of Lemma. From the assumption on S_t and (12) it follows that $S_{t-s} f(s, u, \omega) \in M_\beta^{T,2}(E/H)$ for all $t \in [0, T]$, $0 \leq s \leq t$ and

$$\mathbf{E}\left[\left\| \int_0^r \int_{E\setminus\{0\}} S_{t-s} f(s, u) q(\mathrm{d}s\mathrm{d}u) \right\|^2\right] \leq \mathrm{e}^{2\alpha t} \int_0^r \int_{E\setminus\{0\}} \mathbf{E}[\|f(s, u)\|^2] \mathrm{d}s\beta(\mathrm{d}u) \tag{14}$$

$\forall r \in [0, T]$. The result is then obtained by putting $r = t$. ■

Proposition 2.5. *Let $f(s,u,\omega) \in M_\beta^{T,2}(E/H)$, then $\int_0^t \int_{E\setminus\{0\}} S_{t-s} f q(\mathrm{d}s\mathrm{d}u)$ is càd-làg.*

Proof. In [37] it is proven that

$$M_t := \int_0^t \int_{E\setminus\{0\}} f q(\mathrm{d}s\mathrm{d}u) \tag{15}$$

is an $\mathcal{F}_t$-square integrable martingale and is càd-làg. In Theorem A.1 in the Appendix we shall prove that the corresponding Meyer process is given by

$$\langle M\rangle_t := \int_0^t \int_{E\setminus\{0\}} \|f\|^2 \mathrm{d}s\beta(\mathrm{d}u). \tag{16}$$

From Lemma 2.4 it follows that $S_{t-s} f(s,u,\omega) \in M_\beta^{T,2}(E/H)$ and that (6) is well defined as the Ito integral. As, moreover, S_{t-s} are linear bounded operators acting on H, it follows from Proposition 3.3 of [38] that the Ito integral $\int_0^t S_{t-s}\mathrm{d}M_s(\omega)$, with $(M_t)_{0\le t\le T}$ given through (15), is also well defined, and the following equality holds

$$Y_t := \int_0^t S_{t-s}\mathrm{d}M_s = \int_0^t \int_{E\setminus\{0\}} S_{t-s} f q(\mathrm{d}s\mathrm{d}u). \tag{17}$$

From Lemma 5 in [28] we have

$$P\left(\sup_{0\le t\le T} \|Y_t\| > \epsilon\right) \le 4\frac{\mathrm{e}^{2\alpha T}}{\epsilon^2}\mathbf{E}[\langle M\rangle_T]. \tag{18}$$

From [45] it follows that there is a sequence of simple functions $\{f_n\}_{n\in\mathbb{N}}$, which is L^2-approximating f, i.e. such that

$$\lim_{n\to\infty} \int_0^T \int_{E\setminus\{0\}} \mathbf{E}[\|f_n(t,u) - f(t,u)\|^2]\mathrm{d}t\beta(\mathrm{d}u) = 0. \tag{19}$$

Let

$$\begin{aligned} Y_t^n &:= \int_0^t \int_{E\setminus\{0\}} S_{t-s} f_n q(\mathrm{d}s\mathrm{d}u) \\ &= \int_0^t S_{t-s}\mathrm{d}M_s^n \end{aligned} \tag{20}$$

$$M_s^n := \int_0^t \int_{E\setminus\{0\}} f_n q(\mathrm{d}s\mathrm{d}u). \tag{21}$$

As $S_{t-s} f_n(s,u,\omega)$ is of the form (10), Y_t^n is a martingale and is càd-làg [37].

From Lemma 5 in [28] we have

$$\begin{aligned} P\left(\sup_{0\le t\le T} \|Y_t^n - Y_t^m\| > \epsilon\right) &\le 4\frac{\mathrm{e}^{2\alpha T}}{\epsilon^2}\mathbf{E}[\langle M_n - M_m\rangle_T] \\ &\le 4\frac{\mathrm{e}^{2\alpha T}}{\epsilon^2}\int_0^T \int_{E\setminus\{0\}} \mathbf{E}[\|f_n(t,u) - f_m(t,u)\|^2]\mathrm{d}t\beta(\mathrm{d}u), \end{aligned} \tag{22}$$

where the second inequality follows from [45]. By the Borel–Cantelli Lemma and $f_n \to f$ in

$M_\beta^{T,2}(E/F)$ there is a subsequence $\{Y_t^{n_k}(\omega)\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \sup_{0\le t\le T} \|Y_t^{n_k}(\omega) - Y_t^{n_{k+1}}(\omega)\| = 0 \quad P\text{-a.s.} \tag{23}$$

It follows

$$Y_t(\omega) = \lim_{k\to\infty} Y_t^{n_k}(\omega) \quad \text{uniformly in } [0,T], \qquad P\text{-a.s.} \tag{24}$$

We get that Y_t is càd-làg, since $Y_t^{n_k}$ is càd-làg. ∎

Remark 2.6. The result in Proposition 2.5 allows us to set up a contraction in the set of càd-làg processes, thus giving the solution of (1) to be càd-làg. In [26] the càd-làg property of the solution has been proven using Aldous criteria for compactness in the Skorohod space. We used instead Ichikawa's inequality in Lemma 5 of [28], for the stochastic integrals (6), which is more straightforward. Reference [26] considers the case where the state space is an M-type-p Banach space. In an article of Kotelenez [34] it was proven that the stochastic integrals $\int_0^t S_{t-s}\mathrm{d}M_s(\omega)$ have a càd-làg version, in the general case of $(M_t)_{t\in[0,T]}$ being càd-làg martingales. A precise statement and proof can also be found in Chapter 9.4.2 of the book by Peszat, Zabczyk [43].

3. Existence and uniqueness of solutions under non-Markovian Lipschitz conditions

In this Section we assume that $q(\mathrm{d}s\mathrm{d}x) := N(\mathrm{d}s\mathrm{d}x)(\omega) - \beta(\mathrm{d}s\mathrm{d}x)$ is the cPrm associated to a canonical Lévy process $(X_t)_{t\in\mathbb{R}_+}$. See e.g. [2] Section 2, or [46], for the definition of "canonical Lévy process" and [1] Definition 2.10 for the definition of "cPrm associated to a Lévy process" (or e.g. [5,4,25,29,35,44,46]). The canonical Lévy process is defined on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ introduced in Section 1. I.e. $\Omega := D(\mathbb{R}_+, H)$ and the filtration is given in (2).

In this Section we solve (3) for the case where the coefficients are mappings

$$A : \mathbb{R}_+ \times D(\mathbb{R}_+, H) \to H, \tag{25}$$
$$F : \mathbb{R}_+ \times H \setminus \{0\} \times D(\mathbb{R}_+, H) \to H. \tag{26}$$

The coefficients A and F are in general nonlinear (see the conditions (a), (b), (29) and (38) below).

In the whole Section we assume:

(a) $F(t,u,Z)$ is jointly measurable, and for all $u \in H$ and $t \in \mathbb{R}_+$ fixed, $F(t,u,\cdot)$ is $\mathcal{F}_t$-adapted.

(b) $A(t,Z)$ is jointly measurable, and for all $t \in \mathbb{R}_+$ fixed, $A(t,\cdot)$ is $\mathcal{F}_t$-adapted.

Moreover, for each $t \in \mathbb{R}_+$, we consider the function

$$\begin{aligned} &\theta_t : D(\mathbb{R}_+; H) \to D(\mathbb{R}_+; H) \\ &Z \to \theta_t(Z) \end{aligned} \tag{27}$$

defined by the following formula

$$\begin{aligned} \theta_t(Z)(s) &:= Z_s, \quad \text{if } 0 \le s < t \\ &:= Z_t, \quad \text{if } t \le s. \end{aligned} \tag{28}$$

Let us assume that $F(t,u,Z) = F(t,u,\theta_t(Z))$ and $A(t,Z) = A(t,\theta_t(Z))$.

We assume that

(c) there is $l > 0$, so that for any $t_1, t_2 \in \mathbb{R}_+$

$$\int_{t_1}^{t_2} \|F(t,u,Z)\|_H^2 \beta(\mathrm{d}u)\mathrm{d}t + \int_{t_1}^{t_2} \|A(t,Z)\|_H^2 \mathrm{d}t \le l \int_{t_1}^{t_2} (1 + \|\theta_t(Z)\|_\infty^2)\mathrm{d}t. \tag{29}$$

Let, for $Z \in D(\mathbb{R}_+; H)$,

$$I(t,Z) := \int_0^t S_{t-s} A(s,Z)\mathrm{d}s + \int_0^t \int_{H\setminus\{0\}} S_{t-s} F(s,u,Z) q(\mathrm{d}s\mathrm{d}u), \quad t \in [0,T]. \tag{30}$$

Theorem 3.1. *There exists a constant $C_{l,T,\alpha}$ such that for any $(\mathcal{F}_t)$-stopping time τ*

$$\mathbf{E}[\sup_{0\le s\le t\wedge\tau} \|I(s,Z)\|_H^2] \le C_{l,T,\alpha}\left(t + \int_0^t \mathbf{E}[\sup_{0\le v\le s\wedge\tau} \|Z_v\|^2]\mathrm{d}s\right), \quad t \in [0,T]. \tag{31}$$

Proof.

$$\begin{aligned}\sup_{0\le s\le t\wedge\tau} \|I(s,Z)\|_H^2 \le\ & 2 \sup_{0\le s\le t\wedge\tau} \left\| \int_0^s S_{s-v} A(v,Z)\mathrm{d}v \right\|_H^2 \\ & + 2 \sup_{0\le s\le t\wedge\tau} \left\| \int_0^s \int_{H\setminus\{0\}} S_{t-v} F(v,u,Z) q(\mathrm{d}v\mathrm{d}u) \right\|_H^2,\end{aligned} \tag{32}$$

(where we used the inequality $\|x+y\|^2 \le 2\|x\|^2 + 2\|y\|^2$, valid for any $x, y \in H$). Using (7) and (29) we obtain

$$\begin{aligned}\mathbf{E}\left[\sup_{0\le s\le t\wedge\tau} \left\| \int_0^s S_{s-v} A(v,Z)\mathrm{d}v \right\|_H^2\right] &\le \mathbf{E}\left[\sup_{0\le s\le t} \left(l\mathrm{e}^{\alpha t} \int_0^s (1 + \|\theta_v(Z)\|_\infty)\mathrm{d}v \right)^2\right] \\ &\le 2\mathrm{e}^{2\alpha T} l^2 \left\{ t^2 + t\mathbf{E}\left[\int_0^{s\wedge\tau} \|\theta_v(Z)\|_\infty^2 \mathrm{d}v\right]\right\}.\end{aligned} \tag{33}$$

Moreover using also Theorem 3 of [28], as well as (16), (17) and (29) we get

$$\begin{aligned}&\mathbf{E}\left[\sup_{0\le s\le t\wedge\tau} \left\| \int_0^s \int_{H\setminus\{0\}} S_{t-v} F(v,u,Z) q(\mathrm{d}v\mathrm{d}u) \right\|_H^2\right] \\ &\le 2\mathrm{e}^{2\alpha T} l^2 (3+\sqrt{10})^2 \left\{ t^2 + t\mathbf{E}\left[\int_0^{s\wedge\tau} \|\theta_v(Z)\|_\infty^2 \mathrm{d}v\right]\right\}\end{aligned} \tag{34}$$

$$\begin{aligned}\mathbf{E}[\sup_{0\le s\le t\wedge\tau} \|I(s,Z)\|_H^2] &\le 4\mathrm{e}^{2\alpha T} l^2 (1 + (3+\sqrt{10})^2) \left\{ t^2 + t\mathbf{E}\left[\int_0^{s\wedge\tau} \|\theta_v(Z)\|_\infty^2 \mathrm{d}v\right]\right\} \\ &\le C_{l,T,\alpha}\left(t + \int_0^t \mathbf{E}[\sup_{0\le v\le s\wedge\tau} \|Z_v\|^2]\mathrm{d}s\right)\end{aligned} \tag{35}$$

with $C_{l,T,\alpha} := 4T\mathrm{e}^{2\alpha T} l^2 (1 + (3+\sqrt{10})^2)$. ■

Let $T > 0$ and

$$\mathcal{H}_2^T := \{\xi := (\xi_s)_{s\in[0,T]} : \xi_s(\omega) \text{is jointly measurable, } \mathcal{F}_t\text{-adapted;}$$
$$\mathbf{E}[\sup_{0\le s\le T} \|\xi_s\|_H^2] < \infty\}. \tag{36}$$

Let us observe that it follows from Theorem 3.1 that the map

$$\begin{aligned} I : \mathcal{H}_2^T &\to \mathcal{H}_2^T \\ \xi &\to I(\cdot, \xi). \end{aligned} \tag{37}$$

is well defined.

Starting from here, we assume in this Section that the following additional condition holds: (d) there is a constant $K > 0$ such that for any $t_1, t_2 \in \mathbb{R}$ fixed and $Z, Y \in D(\mathbb{R}_+; H)$ fixed

$$\int_{t_1}^{t_2} \|F(t,u,Z) - F(t,u,Y)\|_H^2 \beta(\mathrm{d}u)\mathrm{d}t + \int_{t_1}^{t_2} \|A(t,Z) - A(t,Y)\|_H^2 \mathrm{d}t$$
$$\le K \int_{t_1}^{t_2} \|\theta_v(Z) - \theta_v(Y)\|_\infty^2 \mathrm{d}t \quad P\text{-a.s.} \tag{38}$$

Lemma 3.2. *The map $I : \mathcal{H}_2^T \to \mathcal{H}_2^T$ is continuous. There is a constant $C_{\alpha,K,T}$, depending on α, K and T, such that*

$$\mathbf{E}[\sup_{0\le s\le T} \|I(s,Z^1) - I(s,Z^2)\|_H^2] \le C_{\alpha,K,T} \int_0^T \mathbf{E}[\sup_{0\le s\le T} \|Z_s^2 - Z_s^1\|_H^2]\mathrm{d}s. \tag{39}$$

The proof of Lemma 3.2 is omitted, as the arguments used are similar to the proof of Theorem 3.1.

Theorem 3.3. *Let $T > 0$, $x \in H$. There is a unique solution $Z := (Z_s)_{s\in[0,T]}$ in $\mathcal{H}_2^T$ which satisfies*

$$Z_t = S_t x + \int_0^t S_{t-s} A(s,Z)\mathrm{d}s + \int_0^t \int_{H\setminus\{0\}} S_{t-s} F(s,u,Z) q(\mathrm{d}s\mathrm{d}u). \tag{40}$$

Proof. We shall prove that the solution can be approximated in $\mathcal{H}_2^T$ by $Z^n := (Z_s^n)_{s\in[0,T]}$, for $n \to \infty$, $n \in \mathbb{N}$, where

$$Z_s^0(\omega) := S_s x \quad P\text{-a.s.} \tag{41}$$

$$Z_s^{n+1}(\omega) := I(s, Z^n(\omega)). \tag{42}$$

Remark that $(Z_t^n)_{t\in[0,T]}$ is $\mathcal{F}_t$-adapted. Let

$$v_t^n := \mathbf{E}[\sup_{0\le s\le t} \|Z_s^{n+1} - Z_s^n\|_H^2]. \tag{43}$$

Then from Theorem 3.1 it follows that there is a constant $V_{\alpha,l,T}(x)$, depending on α, l and T and the initial data x, such that

$$v_t^0 \le \mathbf{E}[\sup_{0\le s\le T} \|Z_s^1 - Z_s^0\|_H^2] \le V_{\alpha,l,T}(x). \tag{44}$$

Similarly as in the proof of Theorem 3.1, it can be proven that there is a constant $C_{\alpha,K,T}$ depending on α, K and T, such that

$$v_t^1 \le C_{\alpha,K,T} \int_0^t \mathbf{E}[\sup_{0\le s\le t} \|Z_s^1 - Z_s^0\|_H^2]\mathrm{d}s \le \frac{T^2(C_{\alpha,K,T})^2}{2} V_{\alpha,l,T}(x). \tag{45}$$

In a similar way we get by induction

$$v_t^n \le C_{\alpha,K,T} \int_0^t v_s^{n-1}\mathrm{d}s \le \frac{(TC_{\alpha,K,T})^{n+1}}{(n+1)!} V_{\alpha,l,T}(x). \tag{46}$$

Let $\epsilon_n := \left(\frac{(TC_{\alpha,K,T})^{n+1}}{(n+1)!}\right)^{\frac{1}{3}}$. Then:

$$P(\sup_{0\le t\le T} \|Z_t^{n+1} - Z_t^n\|^2 \ge \epsilon_n) \le \frac{\frac{(TC_{\alpha,K,T})^{n+1}}{(n+1)!} V_{\alpha,l,T}(x)}{(\frac{(TC_{\alpha,K,T})^{n+1}}{(n+1)!})^{\frac{1}{3}}} = \epsilon_n^2 V_{\alpha,l,T}(x). \tag{47}$$

As $\sum_n \epsilon_n^2$ is convergent, we get that $\sum_{n=1}^\infty \sup_{0\le t\le T} \|Z_t^{n+1} - Z_t^n\|^2$ converges P-a.s. It follows that there is a process $Z := (Z_t)_{t\in[0,T]}$, $Z \in D([0,T]; H)$, such that Z^n converges, when n goes to infinity, to Z in the space $D([0,T]; H)$ (with the supremum norm), P-a.s. Moreover

$$\begin{aligned}
\mathbf{E}[\sup_{0\le t\le T} \|Z_t - Z_t^n\|^2] &= \mathbf{E}\left[\lim_{m\to\infty} \sup_{0\le t\le T} \left\|\sum_{k=n}^{n+m-1} (Z_t^{k+1} - Z_t^k)\right\|^2\right] \\
&\le \mathbf{E}\left[\lim_{m\to\infty} \left(\sum_{k=n}^{n+m-1} \sup_{0\le t\le T} \|Z_t^{k+1} - Z_t^k\| k \frac{1}{k}\right)^2\right] \\
&\le \sum_{k=n}^{\infty} \mathbf{E}[\sup_{0\le t\le T} \|Z_t^{k+1} - Z_t^k\|^2 k^2] \sum_{k=n}^{\infty} \frac{1}{k^2} \\
&\le V_{\alpha,l,T}(x) \left(\sum_{k=n}^{\infty} \frac{(TC_{\alpha,K,T})^{k+1} k^2}{(k+1)!}\right) \left(\sum_{k=n}^{\infty} \frac{1}{k^2}\right) \to 0 \quad \text{when } n \to \infty,
\end{aligned} \tag{48}$$

where we used the Schwarz inequality. It follows that Z^n converges, when n goes to infinity, to Z in the space $\mathcal{H}_2^T$, too. From Lemma 3.2 it follows that $(Z_t)_{0\le t\le T}$ solves (40). We shall prove that the solution is unique. Suppose that $(Z_t)_{0\le t\le T}$ and $(Y_t)_{0\le t\le T}$ are two solutions of (40). Let

$$\mathcal{V}_t := \mathbf{E}[\sup_{0\le s\le t} \|Z_s - Y_s\|_H^2]. \tag{49}$$

Then similarly as for (46) we get

$$\mathcal{V}_t \le C_{\alpha,K,T} \int_0^t \mathcal{V}_s \tag{50}$$

and by induction

$$\mathcal{V}_t \le \frac{(C_{\alpha,K,T}t)^n}{n!} \mathbf{E}[\sup_{0\le s\le T} \|Z_s - Y_s\|_H^2] \to 0 \quad \text{when } n \to \infty \tag{51}$$

i.e. $\mathcal{V}_t = 0\, \forall t \in [0,T]$. ■

4. Existence and uniqueness of solutions under Markovian Lipschitz conditions

Let us assume that we are given

$$a : \mathbb{R}_+ \times H \to H, \tag{52}$$

$$f : \mathbb{R}_+ \times H \setminus \{0\} \times H \to H. \tag{53}$$

Assume
(A) $f(t, u, z)$ is jointly measurable,
(B) $a(t, z)$ is jointly measurable,
and for $T > 0$ fixed
(C) there is a constant $L > 0$, s.th.

$$T\|a(t,z) - a(t,z')\|^2 + \int \|f(t,u,z) - f(t,u,z')\|^2 \beta(\mathrm{d}u) \le L\|z - z'\|^2$$
$$\text{for all } t \in [0, T], z, z' \in F. \tag{54}$$

(D) There is a constant $K > 0$ such that

$$T\|a(t,z)\|^2 + \int \|f(t,u,z)\|^2 \beta(\mathrm{d}u) \le K(\|z\|^2 + 1) \quad \text{for all } t \in [0,T], z \in F. \tag{55}$$

Let $A(t, Z) := a(t, Z_t)$ and $F(t, u, Z) := f(t, u, Z_t)$. Then $A(t, Z)$ and $F(t, u, Z)$ satisfy the conditions in Theorem 3.3 so that there is a unique solution $Z := (Z_t)_{t\in[0,T]}$ in $\mathcal{H}_2^T$ of

$$Z_t = S_t Z_0 + \int_0^t S_{t-s} a(s, Z_s)\mathrm{d}s + \int_0^t \int S_{t-s} f(s, u, Z_s) q(\mathrm{d}s\mathrm{d}u) \quad P\text{-a.s. } \forall t \in [0, T]. \tag{56}$$

Starting from here we assume, like in Section 2, that in the rest of the present article the cPrm $q(\mathrm{d}s\mathrm{d}x) := N(\mathrm{d}s\mathrm{d}x)(\omega) - \mathrm{d}s\beta(\mathrm{d}x)$ is a random measure on $(E, \mathcal{B}(E))$, where E is a separable Banach space, and is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le+\infty}, P)$, satisfying the "usual hypothesis" (i)–(iv), given in Section 2.

We shall for this case prove the following result: let the coefficient a be like in (52) and

$$f : \mathbb{R}_+ \times E \setminus \{0\} \times H \to H. \tag{57}$$

Theorem 4.1. *Let* $0 < T < \infty$ *and suppose that* (A), (B), (C), (D) *are satisfied. Suppose also that*

$$\mathbf{E}[\|Z_0\|^2] < \infty. \tag{58}$$

Then there exists a unique càd-làg process $(Z_t)_{t\in[0,T]}$ *s.th.* P*-a.s.* $Z_t(\omega)$ *solves* (56) *and such that*
(i) Z_t *is* $\mathcal{F}_t$*-measurable,*
(ii) $\int_0^T \mathbf{E}[\|Z_s\|^2]\mathrm{d}s < \infty$.

Remark 4.2. From Theorem 15 in Chapter 89-IV [18] it follows that $(Z_t)_{t\in[0,T]}$ is progressively measurable.

Proof of Theorem 4.1. *Proof of the Uniqueness:*

Let $(Z_t^1)_{t\in[0,T]}$ and $(Z_t^2)_{t\in[0,T]}$ be two solutions with initial conditions $Z_0^1(\omega)$ and $Z_0^2(\omega)$ respectively which satisfy the hypothesis in Theorem 4.1. Let us define

$$\begin{aligned}
&C(s,u,\omega) := f(s,u,Z_s^1(\omega)) - f(s,u,Z_s^2(\omega))\\
&g(s,\omega) := a(s,Z_s^1(\omega)) - a(s,Z_s^2(\omega))\\
&v(t) := \mathbf{E}[\|Z_t^1 - Z_t^2\|^2]. \qquad (59)
\end{aligned}$$

Then

$$\begin{aligned}
v(t) \le 4\mathrm{e}^{2\alpha t}\Bigg\{&\mathbf{E}[\|Z_0^1 - Z_0^2\|^2] + t\int_0^t \mathbf{E}[\|g(s)\|^2]\mathrm{d}s\\
&+ \int_0^t\int_{E\setminus\{0\}} \mathbf{E}[\|C(s,u)\|^2]\mathrm{d}s\beta(\mathrm{d}u)\Bigg\}, \qquad (60)
\end{aligned}$$

where we used the contraction property of S_t and Lemma 2.4. Using property (C) we get

$$v(t) \le 4\mathrm{e}^{2\alpha t}\left\{\mathbf{E}[\|Z_0^1 - Z_0^2\|^2] + L\int_0^t \mathbf{E}[\|Z_s^1 - Z_s^2\|^2]\mathrm{d}s\right\}. \qquad (61)$$

By Gronwall's inequality we then obtain

$$v(t) \le 4\mathrm{e}^{2\alpha t}\mathbf{E}[\|Z_0^1 - Z_0^2\|^2]\mathrm{e}^{L\rho_t t} \qquad (62)$$

where $\rho_t := 4\mathrm{e}^{2\alpha t}$. This implies that if

$$Z_0^1(\omega) = Z_0^2(\omega) \quad P\text{-a.s.}, \qquad (63)$$

then (denoting as usual by $\mathbb{Q}$ the field of rational numbers)

$$P(Z_t^1 = Z_t^2 \text{ for } t \in [0,T]\cap\mathbb{Q}) = 1, \qquad (64)$$

which implies by the càd-làg property

$$P(Z_t^1 = Z_t^2 \text{ for } t \in [0,T]) = 1. \qquad (65)$$

Proof of the Existence:
Define, for $t \in [0,T]$:

$$\begin{aligned}
&Z_t^0(\omega) := S_t Z_0(\omega) \qquad (66)\\
&Z_t^k := S_t Z_0 + \int_0^t S_{t-s} a(s, Z_s^{k-1})\mathrm{d}s + \int_0^t\int_{E\setminus\{0\}} S_{t-s} f(s,u,Z_s^{k-1}) q(\mathrm{d}s\mathrm{d}u). \qquad (67)
\end{aligned}$$

Then by similar arguments as in the proof of the uniqueness we get

$$\begin{aligned}
&\mathbf{E}[\|Z_t^{k+1} - Z_t^k\|^2] \le 2L\mathrm{e}^{2\alpha T}\int_0^t \mathbf{E}[\|Z_s^k - Z_s^{k-1}\|^2]\mathrm{d}s \qquad (68)\\
&\mathbf{E}[\|Z_t^1 - Z_t^0\|^2] \le 2\mathrm{e}^{2\alpha T} K(1 + \mathbf{E}[\|Z_0\|^2]). \qquad (69)
\end{aligned}$$

Defining

$$C_{K,T,\alpha} := 2L\mathrm{e}^{2\alpha T} \tag{70}$$

we obtain from (68) and (69)

$$\mathbf{E}[\|Z_t^{k+1} - Z_t^k\|^2] \leq (C_{K,T,\alpha})^{k+1} \frac{t^{k+1}}{(k+1)!} 2\mathrm{e}^{2\alpha T} K(1 + \mathbf{E}[\|Z_0\|^2]). \tag{71}$$

Moreover, for $k \geq 1$.

$$P(\sup_{0\leq t\leq T} \|Z_t^{k+1} - Z_t^k\| > 2^{-k}) \leq P\left(\int_0^T \|S_{t-s}a(s, Z_s^k) - S_{t-s}a(s, Z_s^{k-1})\|\mathrm{d}s > 2^{-k-1}\right)$$
$$+ P\left(\sup_{0\leq t\leq T} \left\|\int_0^t \int_{E\setminus\{0\}} [S_{t-s}f(s,u,Z_s^k) - S_{t-s}f(s,u,Z_s^{k-1})]q(\mathrm{d}s\mathrm{d}u)\right\| > 2^{-k-1}\right).$$

Using Chebychev's inequality, (18), the Lipschitz condition (C) and the growth condition (D) we obtain

$$P(\sup_{0\leq t\leq T} \|Z_t^{k+1} - Z_t^k\| > 2^{-k}) \leq 2^{2k+2} T\mathbf{E}\left[\int_0^T \|S_{t-s}a(s, Z_s^k) - S_{t-s}a(s, Z_s^{k-1})\|^2 \mathrm{d}s\right]$$
$$+ 42^{2k+2}\mathrm{e}^{2\alpha T} \left|\int_0^T \int_{E\setminus\{0\}} \mathbf{E}[\|f(s,u,Z_s^k) - f(s,u,Z_s^{k-1})\|]\mathrm{d}s\beta(\mathrm{d}u)\right|$$
$$\leq 42^{2k+2}\mathrm{e}^{2\alpha T} L\mathbf{E}\left[\int_0^T \|Z_s^k - Z_s^{k-1}\|^2 \mathrm{d}s\right]$$
$$\leq 42^{2k+2}\mathrm{e}^{2\alpha T} L2\mathrm{e}^{2\alpha T} K(1 + \mathbf{E}[\|Z_0\|^2]) \int_0^T (C_{K,T,\alpha})^k \frac{t^k}{(k)!}\mathrm{d}t$$
$$\leq (C_{K,T,\alpha})^k \frac{(4T)^{k+1}}{(k+1)!} \mathrm{e}^{2\alpha T} L8\mathrm{e}^{2\alpha T} K(1 + \mathbf{E}[\|Z_0\|^2]). \tag{72}$$

From the Borel–Cantelli Lemma it follows that P-a.s. there is $k_0(\omega) \in \mathbb{N}$ s.th. for all $k \geq k_0(\omega)$, $k \in \mathbb{N}$,

$$\sup_{0\leq t\leq T} \|Z_t^{k+1} - Z_t^k\| \leq 2^{-k}. \tag{73}$$

We have

$$Z_t^n(\omega) = Z_t^0(\omega) + \sum_{k=0}^{n-1} [Z_t^{k+1}(\omega) - Z_t^k(\omega)]. \tag{74}$$

It follows that $Z_t^n(\omega)$ converges uniformly on $[0, T]$ P-a.s. We define

$$Z_t(\omega) := \lim_{n\to\infty} Z_t^n(\omega) \quad P\text{-a.s.} \tag{75}$$

$(Z_t(\omega))_{t\in[0,T]}$ is càd-làg and adapted, as each $(Z_t^n(\omega))_{t\in[0,T]}$ is càd-làg and adapted by induction.

We now prove that for all $t \in [0, T]$ convergence of $Z_t^n(\omega)$ to $Z_t(\omega)$ as $n \to \infty$ holds also in the L_2-norm, i.e. $\lim_{n\to\infty}(\mathbf{E}[\|Z_t^n - Z_t\|^2])^{1/2} = 0$
and that

$$\int_0^T \mathbf{E}[\|Z_s\|^2]\mathrm{d}s < \infty. \tag{76}$$

Let $n \geq m, n, m \in \mathbb{N}$, then from (71) it follows

$$\mathbf{E}[\|Z_t^n - Z_t^m\|^2] \leq \sum_m^{n-1} \mathbf{E}[\|Z_t^{k+1} - Z_t^k\|^2]$$
$$\leq 2\mathrm{e}^{2\alpha T} K(1 + \mathbf{E}[\|Z_0\|^2]) \sum_m^{\infty} (C_{K,T,\alpha})^{k+1} \frac{t^{k+1}}{(k+1)!} \quad \forall t \in [0, T]. \tag{77}$$

We get

$$\mathbf{E}[\|Z_t^n - Z_t^m\|^2] \to 0 \quad \text{as } n, m \to \infty. \tag{78}$$

Hence the limit in L_2 of Z_t^n for $n \to \infty$ exists, we call it

$$\xi_t := \lim_{n \to \infty} Z_t^n. \tag{79}$$

Then

$$\xi_t(\omega) = Z_t(\omega) \quad P\text{-a.s.} \tag{80}$$

From (66), (67), (73), (74) and (75) we have

$$\sup_{t \in [0,T]} E[\|Z_t\|^2] < \infty \tag{81}$$

hence

$$\int_0^T \mathbf{E}[\|Z_t\|^2] \mathrm{d}t < \infty. \tag{82}$$

Moreover

$$\lim_{n \to \infty} \int_0^T \mathbf{E}[\|Z_t^n - Z_t\|^2] \mathrm{d}t = 0. \tag{83}$$

We now prove that $(Z_t)_{t \in [0,T]}$ solves Eq. (3) P-a.s..

$$Z_t^{n+1} := S_t Z_0 + \int_0^t S_{t-s} a(s, Z_s^n) \mathrm{d}s + \int_0^t \int_{E \setminus \{0\}} S_{t-s} f(s, u, Z_s^n) q(\mathrm{d}s \mathrm{d}u). \tag{84}$$

We define

$$C_n(s, u, \omega) := f(s, u, Z_s(\omega)) - f(s, u, Z_s^n(\omega))$$
$$g_n(s, \omega) := a(s, Z_s(\omega)) - a(s, Z_s^n(\omega)).$$

Similarly as for (60) and (61) we obtain

$$\left\| \int_0^t \mathbf{E}[g_n(s)] \mathrm{d}s \right\|^2 \leq t \int_0^t \mathbf{E}[\|g_n(s)\|^2] \mathrm{d}s \leq \mathrm{e}^{2\alpha t} L \int_0^t \mathbf{E}[\|Z_s - Z_s^n\|^2] \mathrm{d}s \tag{85}$$

$$\left\| \int_0^t \int_{E \setminus \{0\}} \{\mathbf{E}[C_n(s, u)] \mathrm{d}s \beta(\mathrm{d}u)\} \right\|^2 \leq \mathrm{e}^{2\alpha t} L \int_0^t \mathbf{E}[\|Z_s - Z_s^n\|^2] \mathrm{d}s. \tag{86}$$

From (83), (85), (86) it follows that $(Z_t)_{t \in [0,T]}$ solves Eq. (3) P-a.s. ■

Corollary 4.3. *Let* $0 < T < \infty$ *and suppose that* (A), (B), (C), (D) *are satisfied. Let* $(Z_t^\xi)_{t\in[0,T]}$, *resp.* $(Z_t^\eta)_{t\in[0,T]}$ *be the solution of* (3) *with initial condition* ξ, *resp.* η, *then*

$$\mathbf{E}[\|Z_t^\xi - Z_t^\eta\|^2] \leq 4\mathrm{e}^{2\alpha t}\|\xi - \eta\|^2 \mathrm{e}^{L\rho_t t}. \tag{87}$$

Proof. This follows from (59) and (62). ∎

5. Yosida approximation

In this Section we assume again the hypothesis of the previous Section: $\mathcal{A}$ is the infinitesimal generator of a pseudo-contraction semigroup $(S_t)_{t\in[0,T]}$ and conditions (A), (B), (C), (D) hold.

Let

$$Z_0(\omega) = \xi \quad P\text{-a.s.} \tag{88}$$

and let $(Z_t)_{t\in[0,T]}$ be the unique càd-làg process solving P-a. s. (3) for every $t \in [0, T]$ such that (i), (ii) in Theorem 4.1 are satisfied.

Let $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ be the Yosida approximation to $\mathcal{A}$ (see Appendix A.2 in [15] or [21]). For every $T > 0$ fixed, there exists a unique càd-làg process $(Z_t^n)_{t\in[0,T]}$, such that $\int_0^T \mathbf{E}[\|Z_s^n\|^2]\mathrm{d}s < \infty$ and such that $(Z_t^n)_{t\in[0,T]}$ is a strong solution of

$$\mathrm{d}Z_t^n = \mathcal{A}_n Z_t^n \mathrm{d}t + a(t, Z_t^n)\mathrm{d}t + \int_{E\setminus\{0\}} f(s,u,Z_s^n) q(\mathrm{d}s\mathrm{d}u) \tag{89}$$

with initial condition (88) [37]. Moreover $(Z_t^n)_{t\in[0,T]}$ is also a mild solution, i.e. P-a.s.

$$Z_t^n = S_t^n \xi + \int_0^t S_{t-s}^n a(s, Z_s^n)\mathrm{d}s + \int_0^t \int_{E\setminus\{0\}} S_{t-s}^n f(s,u,Z_s^n) q(\mathrm{d}s\mathrm{d}u) \tag{90}$$

for every $t \in [0, T]$ and such that (i), (ii) in Theorem 4.1 are satisfied. We shall prove the following result:

Theorem 5.1.

$$\lim_{n\to\infty} \mathbf{E}[\|Z_t - Z_t^n\|^2] = 0 \tag{91}$$

uniformly in $[0, T]$.

Proof. We have

$$\mathbf{E}[\|Z_t - Z_t^n\|^2] \leq 2^3 \|S_t^n\xi - S_t\xi\|^2 + 2^3\mathbf{E}\left[\left\|\int_0^t S_{t-s}a(s,Z_s) - S_{t-s}^n a(s,Z_s^n)\mathrm{d}s\right\|^2\right]$$
$$+ 2^3\mathbf{E}\left[\left\|\int_0^t\int_{E\setminus\{0\}} S_{t-s}f(s,u,Z_s) - S_{t-s}^n f(s,u,Z_s^n) q(\mathrm{d}s\mathrm{d}u)\right\|^2\right]. \tag{92}$$

We shall analyze separately the three terms on the right-hand side of inequality (92). As for the first term, we remark that

$$\lim_{n\to\infty} \|S_t^n\xi - S_t\xi\| = 0. \tag{93}$$

Moreover, from equation (A.13) in [15] it follows that for all $\xi \in E$ there is a constant C_T and $n_0 \in \mathbb{N}$, such that for any $n \geq n_0$

$$\|S_t^n \xi - S_t \xi\| \leq C_T \|\xi\| \tag{94}$$

so that the convergence in (93) is uniform in $[0, T]$.

Let us consider the second term on the right-hand side of (92). We have:

$$\begin{aligned} &\mathbf{E}\left[\left\|\int_0^t S_{t-s}a(s, Z_s) - S_{t-s}^n a(s, Z_s^n)\mathrm{d}s\right\|^2\right] \\ &\quad \leq 2T\int_0^t \mathbf{E}[\|S_{t-s}a(s, Z_s) - S_{t-s}^n a(s, Z_s)\|^2]\mathrm{d}s \\ &\qquad + 2T\int_0^t \mathbf{E}[\|S_{t-s}^n a(s, Z_s) - S_{t-s}^n a(s, Z_s^n)\|^2]\mathrm{d}s \end{aligned} \tag{95}$$

$$\lim_{n\to\infty} \|S_{t-s}a(s, Z_s(\omega)) - S_{t-s}^n a(s, Z_s(\omega))\| = 0 \quad P\text{-a.s.} \tag{96}$$

and

$$\|S_{t-s}a(s, Z_s(\omega)) - S_{t-s}^n a(s, Z_s(\omega))\|^2 \leq C_T \|a(s, Z_s(\omega))\|^2 \leq C_T K(\|Z_s(\omega)\|^2 + 1), \tag{97}$$

where (97) is a consequence of (94) and condition (D). By the Lebesgue dominated convergence theorem it follows that the first term on the r.h.s. of (95) converges to zero.

Let us consider the second term on the r.h.s. of (95). We observe that from (94) and the Lipschitz condition (C) it follows that

$$T\|S_{t-s}^n a(s, Z_s(\omega)) - S_{t-s}^n a(s, Z_s^n(\omega))\|^2 \leq C_T L\|Z_s(\omega) - Z_s^n(\omega)\|^2 \tag{98}$$

so that

$$2T\int_0^t \mathbf{E}[\|S_{t-s}^n a(s, Z_s) - S_{t-s}^n a(s, Z_s^n)\|]^2\mathrm{d}s \leq 2C_T L\int_0^t \mathbf{E}[\|Z_s - Z_s^n\|^2]\mathrm{d}s. \tag{99}$$

It follows that for all $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\mathbf{E}\left[\left\|\int_0^t S_{t-s}a(s, Z_s) - S_{t-s}^n a(s, Z_s^n)\mathrm{d}s\right\|^2\right] \leq \epsilon + 2C_T L\int_0^t \mathbf{E}[\|Z_s - Z_s^n\|^2]\mathrm{d}s. \tag{100}$$

Let us consider the third term in (92). By similar arguments as in (100), it can be proved that

$$\begin{aligned} &\mathbf{E}\left[\left\|\int_0^t \int_{E\setminus\{0\}} S_{t-s}f(s, u, Z_s) - S_{t-s}^n f(s, u, Z_s^n)q(\mathrm{d}s\mathrm{d}u)\right\|^2\right] \\ &\quad \leq \epsilon + 2C_T L\int_0^t \mathbf{E}[\|Z_s - Z_s^n\|^2]\mathrm{d}s. \end{aligned} \tag{101}$$

It follows

$$\mathbf{E}[\|Z_t - Z_t^n\|^2] \leq 2^3\|S_t^n \xi - S_t \xi\|^2 + 2^4\epsilon 2^4 C_T L\int_0^t \mathbf{E}[\|Z_s - Z_s^n\|^2]\mathrm{d}s. \tag{102}$$

Using Gronwall's Lemma we get

$$\mathbf{E}[\|Z_t - Z_t^n\|^2] \leq (2^3\|S_t^n\xi - S_t\xi\|^2 + 2^4\epsilon)\exp(2^4 TLC_T) \tag{103}$$

so that (92) gives the result. ■

6. Markov property

Let $B_b(H)$ denote the set of bounded real valued functions on H. We first prove that the Markov property holds for the semigroup associated to the mild solutions of (1):

Let $0 < v < T$ and $\xi \in H$. Let $(Z(t, v, \xi))_{t\in[v,T]}$ denote the solution of the following integral equation

$$Z_t = S_{t-v}\xi + \int_v^t S_{t-s}a(s, Z_s)\mathrm{d}s + \int_v^t \int_{E\setminus\{0\}} S_{t-s}f(s, u, Z_s)q(\mathrm{d}s\mathrm{d}u) \tag{104}$$

(in the sense of Theorem 4.1). Let $\mathcal{F}_t^Z$ denote the σ-algebra generated by $Z(\tau, v, \xi)$, with $\tau \leq t$, $\tau \geq v$. Let $v \leq s \leq t \leq T$ and $P_{s,t}$ be the linear operator on $B_b(H)$, such that

$$(P_{s,t})(\phi)(x) = \mathbf{E}[\phi(Z(t, s; x))] \quad \text{for } \phi \in B_b(H)\, x \in H. \tag{105}$$

Then the Markov property holds, that is:

Theorem 6.1. *Let* $0 \leq v \leq s \leq t \leq T$.

$$\mathbf{E}[\phi(Z(t, v; \xi))/\mathcal{F}_s^Z] = (P_{s,t})(\phi)(Z(s, v; \xi)) \quad \textit{for any } \phi \in B_b(H). \tag{106}$$

Proof. As $\mathcal{F}_s^Z \subset \mathcal{F}_s$, it is sufficient to prove that

$$\mathbf{E}[\phi(Z(t, v; \xi))/\mathcal{F}_s] = (P_{s,t})(\phi)(Z(s, v; \xi)). \tag{107}$$

From the uniqueness (Theorem 4.1) we get

$$Z(t, v; \xi)(\omega) = Z(t, s; Z(s, v; \xi)(\omega))(\omega) \quad P\text{-a.s.} \tag{108}$$

Let

$$\eta(\omega) := Z(s, v; \xi)(\omega). \tag{109}$$

Then from (108) it follows that (107) can be written as

$$\mathbf{E}[\phi(Z(t, s; \eta))/\mathcal{F}_s] = (P_{s,t})(\phi)(Z(s, v; \eta)). \tag{110}$$

It is enough to show that (110) holds for every $\phi \in C_b(H)$, with $C_b(H)$ denoting the set of continuous real valued bounded functions on H. We first assume that ϕ is linear and bounded.

Moreover, let us first consider the case where

$$\eta(\omega) = x \in H \quad P\text{-a.s.} \tag{111}$$

(instead of (109)). As x is constant and because of the independent increment property of the cPrm, $Z(t, s; \eta(\omega))$ is independent of $\mathcal{F}_s$. In fact $\mathcal{F}_s$ is the σ-algebra generated by the pure jump Lévy process with compensator $\mathrm{d}s\beta(\mathrm{d}x)$, (see e.g. [1], Section 2).

$$\mathbf{E}[\phi(Z(t, s; \eta))/\mathcal{F}_s] = \mathbf{E}[\phi(Z(t, s, x))] = P_{s,t}(\phi(x)) \tag{112}$$

so that (110) holds for this particular case.

Now we prove (110) for the case where

$$\eta(\omega) := \sum_1^n a_j \mathbf{1}_{A_j}(Z(s,v;\xi)) \tag{113}$$

with $\{A_j, j = 1, \ldots, n\}$ a partition of H and $a_1, \ldots, a_n \in H$. In this case

$$Z(t,s;\eta(\omega))(\omega) = \sum_1^n Z(t,s;a_j)\mathbf{1}_{A_j}(Z(s,v;\xi)) \quad P\text{-a.s.}, \tag{114}$$

$$\phi(Z(t,s;\eta(\omega))(\omega)) = \sum_1^n \phi(Z(t,s;a_j))\mathbf{1}_{A_j}(Z(s,v;\xi)) \quad P\text{-a.s.}, \tag{115}$$

and

$$\begin{aligned}\mathbf{E}[\phi(Z(t,s;\eta)/\mathcal{F}_s)] &= \mathbf{E}\left[\sum_1^n \phi(Z(t,s;a_j))\mathbf{1}_{A_j}(Z(s,v;\xi))/\mathcal{F}_s\right]\\ &= \sum_1^n P_{s,t}(\phi)(a_j)\mathbf{1}_{A_j}(Z(s,v,\xi)) = P_{s,t}(\phi)(\eta),\end{aligned} \tag{116}$$

where in (116) we used that $\phi(Z(t,s;a_j))$ are independent of $\mathcal{F}_s$ and $\mathbf{1}_{A_j}(Z(s,v;\xi))$ are $\mathcal{F}_s$-measurable.

Now we prove (110) for the case where $\eta(\omega)$ is given according to (109). (From the proof it follows in particular that the r.h.s of (107) is $\mathcal{F}_s^Z$-measurable.) There is a sequence of simple functions $\eta_n(\omega)$ of the form (113) such that, if for a given natural number M we denote $\eta_n^M := \eta_n \wedge M$, then

$$\lim_{M\to\infty}\lim_{n\to\infty} \mathbf{E}[\|\eta_n^M - \eta\|^2] = 0. \tag{117}$$

Similarly to the proof of Corollary 4.3 it follows that

$$\lim_{M\to\infty}\lim_{n\to\infty} \mathbf{E}[\|Z(t,s;\eta_n^M) - Z(t,s;\eta)\|^2] = 0. \tag{118}$$

There is a subsequence (by abuse of notation we denote it like the original sequence), for which

$$\lim_{M\to\infty}\lim_{n\to\infty} Z(t,s;\eta_n^M)(\omega) = Z(t,s;\eta)(\omega) \quad P\text{-a.s.} \tag{119}$$

As ϕ is continuous and bounded, it follows from (116) that

$$\begin{aligned}\mathbf{E}[\phi(Z(t,s;\eta)/\mathcal{F}_s)] &= \lim_{M\to\infty}\lim_{n\to\infty} \mathbf{E}[\phi(Z(t,s,\eta_n^M)/\mathcal{F}_s)]\\ &= \lim_{M\to\infty}\lim_{n\to\infty} P_{s,t}(\phi)(\eta_n^M) = P_{s,t}(\phi)(\eta).\end{aligned} \tag{120}$$

Given $\phi \in C_b(H)$ there exists a sequence of linear bounded functions ϕ_n converging up to a set of Borel measure zero to ϕ (see e.g. [48] Chapter V.5). It follows that $\phi_n(Z(t,s;\eta)) \to \phi(Z(t,s;\eta))$ P-a.s., when $n \to \infty$. ϕ_n can be chosen so as to be uniformly bounded, so that by Theorem 54.14 Chapter X [9]

$$\lim_{n\to\infty} \mathbf{E}[\phi_n(Z(t,s;\eta)/\mathcal{F}_s)] = \mathbf{E}[\phi(Z(t,s;\eta)/\mathcal{F}_s)]. \quad \blacksquare$$

Theorem 6.2. *Let $T > 0$, $f(s,u,z) = f(z)$, $a(s,z) = a(z)$ and $x \in H$, then $(Z(t,0;x)(\omega))_{t\in[0,T]}$ is an homogeneous Markov process.*

Proof. It is sufficient to prove that

$$P_{s,t} = P_{0,t-s} \quad \text{for all } 0 \le s \le t \le T. \tag{121}$$

(121) together with the Markov property (106) in Theorem 6.1 implies that the Chapman–Kolmogorov equation holds for the transition probabilities associated to $P_{s,t}$,$0 \le s \le t \le T$ and $(Z(t,0;x)(\omega))_{t\in[0,T]}$ is a Markov process.

Let us remark that the compensated Lévy random measure $q(\mathrm{d}s\mathrm{d}u)(\omega)$ is translation invariant in time. I.e. if $t > 0$ and $\tilde{q}(\mathrm{d}s\mathrm{d}u)(\omega)$ denotes the unique σ-finite measure on $\mathcal{B}(\mathbb{R}_+ \times E \setminus \{0\})$ which extends the pre-measure $\tilde{q}(\mathrm{d}s\mathrm{d}u)(\omega)$ on $S(\mathbb{R}_+) \times \mathcal{B}(E \setminus \{0\})$, such that $\tilde{q}((s,\tau],\Lambda) := q((s+t,\tau+t],\Lambda)$, for $(s,\tau] \times \Lambda \in S(\mathbb{R}_+) \times \mathcal{B}(E \setminus \{0\})$, then $\tilde{q}(B)$ and $q(B)$ are equally distributed for all $B \in \mathcal{B}(\mathbb{R}_+ \times E \setminus \{0\})$.

It follows that

$$\begin{aligned} &Z(t+h,t;x)\\ &= S_h x + \int_t^{t+h} S_{t+h-s} a(Z(s,t;x))\mathrm{d}s + \int_t^{t+h}\int_{E\setminus\{0\}} S_{t+h-s} f(Z(s,t;x)) q(\mathrm{d}s\mathrm{d}u)\\ &= S_h x + \int_0^{h} S_{h-s} a(Z(t+s,t;x))\mathrm{d}s + \int_0^{h}\int_{E\setminus\{0\}} S_{h-s} f(Z(t+s,t;x)) \tilde{q}(\mathrm{d}s\mathrm{d}u)\\ &= S_h x + \int_0^{h} S_{h-s} a(Z(t+s,t;x))\mathrm{d}s + \int_0^{h}\int_{E\setminus\{0\}} S_{h-s} f(Z(t+s,t;x)) q(\mathrm{d}s\mathrm{d}u). \end{aligned} \tag{122}$$

By uniqueness (Theorem 4.1) it follows that $Z(t+h,t;x)(\omega)$ and $Z(h,0;x)(\omega)$ have the same distribution, so that (121) holds. ■

7. Existence of solutions for random coefficients

Let $L_2^T := L_2^T([0,T] \times \Omega, (\mathcal{F}_t)_{t\in[0,T]})$ be the space of processes $(Z_t(\omega))_{t\in[0,T]}$ which are jointly measurable and satisfy (i) and (ii) in Theorem 4.1.

Definition 7.1. We say that two processes $Z_t^i(\omega) \in L_2^T$, $i = 1,2$, are $\mathrm{d}t \otimes P$-equivalent if they coincide for all $(t,\omega) \in \Gamma$, with $\Gamma \in \mathcal{B}([0,T]) \otimes \mathcal{F}_T$, and $\mathrm{d}t \otimes P(\Gamma^c) = 0$. We denote by $\mathcal{L}_2^T$ the set of $\mathrm{d}t \otimes P$-equivalence classes.

Remark 7.2. $\mathcal{L}_2^T$, with norm

$$\|Z_t\|_{\mathcal{L}_2^T} := \left(\int_0^T \mathbf{E}[\|Z_s\|^2]\mathrm{d}s\right)^{1/2}, \tag{123}$$

is a Hilbert space.

In this section we assume that the coefficients are random and adapted to the filtration and prove the existence of a solution in $\mathcal{L}_2^T$. We assume here the growth and Lipschitz conditions of the coefficients independent of ω, but depending on the points in H. We assume in fact that we are given

$$a : \mathbb{R}_+ \times H \times \Omega \to H, \tag{124}$$

$$f : \mathbb{R}_+ \times E \setminus \{0\} \times H \times \Omega \to H, \tag{125}$$

such that
(A′) $f(t, u, z, \omega)$ is jointly measurable, s.th. for all $t \in [0, T]$, $u \in E$ and $z \in H$ fixed, $f(t, u, z, \cdot)$ is $\mathcal{F}_t$-adapted,
(B′) $a(t, z, \omega)$ is jointly measurable, s.th. for all $t \in [0, T]$, and $z \in H$ fixed, $a(t, z, \cdot)$ is $\mathcal{F}_t$-adapted,
and for $T > 0$ fixed;
(C′) there is a constant $L > 0$, s.th.

$$\begin{aligned} &T\|a(t, z, \omega) - a(t, z', \omega)\|^2 + \int_{E\setminus\{0\}} \|f(t, u, z, \omega) - f(t, u, z', \omega)\|^2 \beta(\mathrm{d}u) \\ &\quad \le L\|z - z'\|^2 \quad \text{for all } t \in [0, T], \quad z, z' \in H, \text{ and } P\text{-a.e. } \omega \in \Omega, \end{aligned} \tag{126}$$

(D′) there is a constant $K > 0$ such that

$$\begin{aligned} &T\|a(t, z, \omega)\|^2 + \int_{E\setminus\{0\}} \|f(t, u, z, \omega)\|^2 \beta(\mathrm{d}u) \le K(\|z\|^2 + 1) \\ &\quad \text{for all } t \in [0, T], z \in H, \text{ and } P\text{-a.e. } \omega \in \Omega. \end{aligned} \tag{127}$$

Theorem 7.3. *Let* $0 < T < \infty$ *and suppose that* (A′), (B′), (C′), (D′) *are satisfied. Let* $x \in H$. *Then there is a unique process* $(Z_t)_{0\le t\le T} \in \mathcal{L}_2^T$ *which satisfies*

$$\begin{aligned} Z_t(\omega) = S_t x &+ \int_0^t S_{t-s} a(s, Z_s(\omega), \omega)\mathrm{d}s \\ &+ \int_0^t \int_{E\setminus\{0\}} S_{t-s} f(s, u, Z_s(\omega), \omega) q(\mathrm{d}s\mathrm{d}u) \quad \forall t \in [0, T]. \end{aligned} \tag{128}$$

As a consequence of Theorem 7.3:

Corollary 7.4. *Let* $0 < T < \infty$ *and suppose that* (A′), (B′), (C′), (D′) *are satisfied. Then there is up to stochastic equivalence a unique process* $(Z_t)_{0\le t\le T} \in L_2^T$ *which satisfies* (3).

Remark 7.5. As a consequence of Proposition 2.5 we have that $(Z_t)_{0\le t\le T}$ is càd-làg.

Before proving Theorem 7.3 we show some property of the following function

$$\begin{aligned} K_t(x, \xi)(\omega) := S_t x &+ \int_0^t S_{t-s} a(s, \xi_s(\omega), \omega)\mathrm{d}s \\ &+ \int_0^t \int_{E\setminus\{0\}} S_{t-s} f(s, u, \xi_s(\omega), \omega) q(\mathrm{d}s\mathrm{d}u) \end{aligned} \tag{129}$$

with $x \in H$ and $\xi := (\xi_s)_{s\in[0,T]} \in \mathcal{L}_2^T$.

Lemma 7.6. *For any $T > 0$ there is a constant C_T^1 such that*

$$\int_0^T \mathbf{E}[\|K_t(x,\xi) - K_t(x,\eta)\|^2]\mathrm{d}t \leq C_T^1 \int_0^T \mathbf{E}[\|\xi_t - \eta_t\|^2]\mathrm{d}t. \tag{130}$$

Proof.

$$\begin{aligned}
&\int_0^T \mathbf{E}[\|K_t(x,\xi) - K_t(x,\xi)\|^2]\mathrm{d}t \\
&\leq 2\mathrm{e}^{2\alpha T} T \int_0^T \mathbf{E}\left[\left\|\int_0^t a(s,\xi_s) - a(s,\eta_s)\mathrm{d}s\right\|^2\right]\mathrm{d}t \\
&\quad + 2\mathrm{e}^{2\alpha T} \int_0^T \int_0^t \int_{E\setminus\{0\}} \mathbf{E}\left[\|f(s,u,\xi_s) - f(s,u,\eta_s)\|^2 \mathrm{d}s\beta(\mathrm{d}u)\right]\mathrm{d}t \\
&\leq 2LT\mathrm{e}^{2\alpha T} \int_0^T \mathbf{E}[\|\xi_s - \eta_s\|^2]\mathrm{d}t < \infty,
\end{aligned} \tag{131}$$

where we applied Lemma 2.4. This proves (130). ■

Let

$$K(x,\xi) : H \times \mathcal{L}_2^T \to \mathcal{L}_2^T \tag{132}$$

be such that its projection at time $t \in [0, T]$ is given by $K_t(x, \xi)$.

Lemma 7.7. *There exists a constant α_T, depending on T, such that $\alpha_T \in (0, 1)$ and*

$$\|K(x,\xi)(\omega) - K(x,\eta)(\omega)\|_{\mathcal{L}_2^T} \leq \alpha_T \|\xi - \eta\|_{\mathcal{L}_2^T}. \tag{133}$$

Proof. Let $\mathbf{S}\xi := K_t(x, \xi)$. We shall prove that $\mathbf{S}^n$ is a contraction operator on $\mathcal{L}_2^T$, for sufficiently large values of $n \in \mathbb{N}$. From (130) it follows by induction

$$\int_0^T \mathbf{E}[\|\mathbf{S}^n\xi_t - \mathbf{S}^n\eta_t\|^2]\mathrm{d}t \tag{134}$$

$$\leq {C_T^1}^n \int_0^T \mathrm{d}t \int_0^T \mathrm{d}s_1 \int_0^T \mathrm{d}s_2 \cdots \int_0^T \mathbf{E}[\|\xi_{s_n} - \eta_{s_n}\|^2]\mathrm{d}s_n \tag{135}$$

$$\leq {C_T^1}^n \frac{T^n}{n!} \int_0^T \mathbf{E}[\|\xi_s - \eta_s\|^2]\mathrm{d}s. \tag{136}$$

From this we get that, for sufficiently large values of $n \in \mathbb{N}$, the operator $\mathbf{S}^n$ is a contraction operator on $\mathcal{L}_2^T$ and has therefore a unique fixed point. Suppose that $\mathbf{S}^{n_0}$ is a contraction operator on $\mathcal{L}_2^T$. We get

$$\begin{aligned}
&\int_0^T \mathrm{d}t\mathbf{E}[\|\mathbf{S}\xi_t - \mathbf{S}\eta_t\|^2] = \int_0^T \mathrm{d}t\mathbf{E}[\|\mathbf{S}^{kn_0+1}\xi_t - \mathbf{S}^{kn_0+1}\eta_t\|^2] \\
&\leq \frac{{C_T^1}^{kn_0} T^{kn_0}}{kn_0!} \int_0^T \mathrm{d}t\mathbf{E}[\|\mathbf{S}\xi_t - \mathbf{S}\eta_t\|^2] \\
&\leq \frac{{C_T^1}^{kn_0+1} T^{kn_0}}{kn_0+1!} \int_0^T \mathrm{d}t\mathbf{E}[\|\xi_t - \eta_t\|^2] \to 0 \quad \text{when } k \to \infty. \quad ■
\end{aligned} \tag{137}$$

Proof of Theorem 7.3. From (133) it follows that $K(x,\xi)$ is a contraction on $\mathcal{L}_2^T$ for every $x \in H$ fixed. We get by the contraction principle that there exists $\phi \in C(H, \mathcal{L}_2^T)$ such that

$$K(x,\phi(x)) = \phi(x) \tag{138}$$

for every $x \in H$ fixed. $\phi(x) := (Z_t^x(\omega))_{t\in[0,T]}$ is the solution of (128). ■

8. Continuous dependence on initial data, drift and noise coefficients

Let $T > 0$. Let us assume that (A), (B), (C), (D) or (A′), (B′), (C′), (D′) are satisfied for $f_0(t,u,z,\omega) := f(t,u,z,\omega)$ and $a_0(t,z,\omega) := a(t,z,\omega)$. Moreover, we assume that this holds also for $f_n(t,u,z,\omega)$, and $a_n(t,z,\omega)$, for any $n \in \mathbb{N}$. Let $(Z_t)_{t\in[0,T]}$ be the solution of (128) (in the sense of Theorem 3.3, or Theorem 4.1 or Theorem 7.3, depending on the hypothesis). We denote by $(Z_t^n(\omega))_{[0,T]}$ the unique solution of

$$\begin{aligned} Z_t^n(\omega) = {} & S_t Z_0^n(\omega) + \int_0^t S_{t-s} a_n(s, Z_s^n(\omega), \omega)\mathrm{d}s \\ & + \int_0^t \int_{E\setminus\{0\}} S_{t-s} f_n(s,u,Z_s^n(\omega),\omega) q(\mathrm{d}s\mathrm{d}u) \end{aligned} \tag{139}$$

(in the sense of Theorem 3.3, resp. Theorem 4.1, resp. Theorem 7.3). We prove the following result.

Theorem 8.1. *Assume that there is a constant $K > 0$ such that for all $n \in \mathbb{N}_0$, $t \in [0,T]$ and $z \in H$*

$$\|a_n(t,z,\omega)\|^2 + \int_{E\setminus\{0\}} \|f_n(t,u,z,\omega)\|^2 \beta(\mathrm{d}u) \le K(\|z\|^2+1) \quad P\text{-a.s.} \tag{140}$$

Assume that there is a constant L such that for all $n \in \mathbb{N}_0$, $t \in [0,T]$ and $z, z' \in H$:

$$\begin{aligned} & T\|a_n(t,z,\omega) - a_n(t,z',\omega)\|^2 + \int_{E\setminus\{0\}} \|f_n(t,u,z,\omega) - f_n(t,u,z',\omega)\|^2 \beta(\mathrm{d}u) \\ & \quad \le L\|z-z'\|^2 \quad P\text{-a.s.} \end{aligned} \tag{141}$$

Moreover, assume that

$$\sup_{n\in\mathbb{N}_0} \mathbf{E}[\|(Z_0^n)\|^2] < \infty. \tag{142}$$

$$\lim_{n\to\infty} \mathbf{E}[\|Z_0^n - Z_0\|^2] = 0, \tag{143}$$

(where $Z_0^0(\omega) := Z_0(\omega)$) and assume that for every $t \in [0,T]$ and $z \in H$ fixed

$$\begin{aligned} & \lim_{n\to\infty} \left\{ T\|a_n(t,z,\omega) - a(t,z,\omega)\|^2 + \int_{E\setminus\{0\}} \|f_n(t,u,z,\omega) - f(t,u,z,\omega)\|^2 \beta(\mathrm{d}u) \right\} \\ & \quad = 0 \quad P\text{-a.s.} \end{aligned} \tag{144}$$

Then

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \mathbf{E}[\|Z_t^n - Z_t\|^2] = 0. \tag{145}$$

Proof. Let $t \le T$, then:

$$\mathbf{E}[\|Z_t^n - Z_t\|^2] \le 2^5 \mathrm{e}^{2\alpha T} \Big\{ \mathbf{E}[\|Z_0^n - Z_0\|^2] + 2L \int_0^t \mathbf{E}[\|Z_t^n - Z_t\|^2]\mathrm{d}s + 2T \int_0^t \mathbf{E}[\|a_n(s, Z_s) - a(s, Z_s)\|^2]\mathrm{d}s \Big\} + 2^5 \mathrm{e}^{2\alpha T} \Big\{ 2 \int_0^t \int_{E\setminus\{0\}} \mathbf{E}[\|f_n(s, u, Z_s) - f(s, u, Z_s)\|^2]\beta(\mathrm{d}u)\mathrm{d}s \Big\}, \tag{146}$$

where the latter inequality is proven by using Lemma 2.4 and inequality (141).

Let

$$\gamma_t^n := T \int_0^t \mathbf{E}[\|a_n(s, Z_s) - a(s, Z_s)\|^2]\mathrm{d}s \tag{147}$$

$$\delta_t^n := \int_0^t \int_{E\setminus\{0\}} \mathbf{E}[\|f_n(s, u, Z_s) - f(s, u, Z_s)\|^2]\beta(\mathrm{d}u)\mathrm{d}s. \tag{148}$$

As

$$\lim_{n\to\infty} \|a_n(s, Z_s, \omega) - a(s, Z_s, \omega)\|^2 + \int_{E\setminus\{0\}} \|f_n(s, u, Z_s, \omega) - f(s, u, Z_s, \omega)\|^2 \beta(\mathrm{d}u) = 0, \quad P\text{-a.s.} \tag{149}$$

and (140) implies

$$\|a_n(t, Z_s(\omega), \omega)\|^2 + \int_{E\setminus\{0\}} \|f_n(t, u, Z_s(\omega), \omega)\|^2 \beta(\mathrm{d}u) \le K(\|Z_s(\omega)\|^2 + 1) \quad P\text{-a.s.}, \tag{150}$$

it follows

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \delta_t^n + \lim_{n\to\infty} \sup_{t\in[0,T]} \gamma_t^n = 0. \tag{151}$$

(145) follows then by using Gronwall's inequality. ■

9. Differential dependence of the solutions on the initial data

In this section we still assume that the coefficients a and f in (52) and resp. (57) satisfy the conditions (A), (B), (C) and (D). We shall prove the differential dependence of the solution of (56) with respect to the initial data.

Let

$$K_t(x, \xi) := S_t x + \int_0^t S_{t-s} a(s, \xi_s)\mathrm{d}s + \int_0^t \int_{E\setminus\{0\}} S_{t-s} f(s, u, \xi_s) q(\mathrm{d}s\mathrm{d}u) \tag{152}$$

with $x \in H$ and $\xi := (\xi_s)_{s\in[0,T]} \in \mathcal{L}_2^T$.

Lemma 9.1. *For any $T > 0$ there is a constant C_T^1, resp. C_T^2, such that*

$$\int_0^T \mathbf{E}[\|K_t(x, \xi) - K_t(x, \eta)\|^2]\mathrm{d}t \le C_T^1 \int_0^T \mathbf{E}[\|\xi_t - \eta_t\|^2]\mathrm{d}t \tag{153}$$

$$\int_0^T \mathbf{E}[\|K_t(x,\xi) - K_t(y,\xi)\|^2]\mathrm{d}t \le C_T^2\|x-y\|^2. \tag{154}$$

Proof. Eq. (153) is a special case of (130). (154) is proven similarly to Corollary 4.3. ■

Let

$$K(x,\xi) : H \times \mathcal{L}_2^T \to \mathcal{L}_2^T \tag{155}$$

be such that its projection at time $t \in [0,T]$ is given by $K_t(x,\xi)$.

Remark 9.2. From Theorem 4.1 we know that there is a unique solution $(Z_t^x(\omega))_{t\in[0,T]}$ of (56). From Theorem 7.3 we know that for every $x \in H$ fixed

$$K(x, Z_t^x(\omega)) = Z_t^x(\omega) \quad P\text{-a.s.} \tag{156}$$

We shall now prove some facts about the map K.

Theorem 9.3. *Let $\xi \in \mathcal{L}_2^T$ be fixed. The map*

$$K(\cdot,\xi) : H \to \mathcal{L}_2^T \tag{157}$$

is Fréchét differentiable and its derivative $\frac{\partial K}{\partial x}$ along the direction $h \in H$ is such that

$$\frac{\partial K_t(x,\xi)}{\partial x}(h) = S_t h. \tag{158}$$

The proof of Theorem 9.3 is easy and follows from the Frechét differentiability of S_t.

Remark 9.4. It follows in particular that $\frac{\partial K}{\partial x}$ is in $\mathcal{L}(H;\mathcal{L}_2^T)$.

Let us denote by $\frac{\partial}{\partial z}$ the Fréchét derivative in H. Starting from here we assume that the coefficients a and f in (52) and resp. (57) satisfy also the following conditions
(E) $\frac{\partial}{\partial z} f(t,u,z)$ exists for all $t \in (0,T]$ and $u \in E \setminus \{0\}$ fixed,
(F) $\frac{\partial}{\partial z} a(t,z)$ exists for all $t \in (0,T]$ fixed.
Moreover we assume that

$$\left|\left|\left| \frac{\partial}{\partial z} a(s,z) \right|\right|\right|^2 + \int_{E\setminus\{0\}} \left|\left|\left| \frac{\partial}{\partial z} f(s,z,u) \right|\right|\right|^2 \beta(\mathrm{d}u) < \infty \quad \text{uniformly in } z \in H, \text{ and } s \in [0,T], \tag{159}$$

where with $|||\cdot|||$ we denote the operator norm of the Fréchét derivative in H.

Theorem 9.5. *Let $x \in H$ be fixed.*

$$K(x,\cdot) : \mathcal{L}_2^T \to \mathcal{L}_2^T \tag{160}$$

is Gateaux differentiable and its derivative $\frac{\partial K}{\partial \xi}$ along the direction $\xi \in \mathcal{L}_2^T$ satisfies

$$\frac{\partial K_t(x,\xi)}{\partial \xi}(\eta_t) = \int_0^t S_{t-s}\frac{\partial}{\partial z} a(s,\xi_s)(\eta_s)\mathrm{d}s + \int_0^t \int_{E\setminus\{0\}} S_{t-s}\frac{\partial}{\partial z} f(s,u,\xi_s)(\eta_s) q(\mathrm{d}s\mathrm{d}u) \tag{161}$$

(with the notation $\frac{\partial}{\partial z}a(s,\xi_s(\omega))$ (resp. $\frac{\partial}{\partial z}f(s,u,\xi_s(\omega))$) for $\frac{\partial}{\partial z}a(s,z)$ (resp. $\frac{\partial}{\partial z}f(s,u,z)$), at $z=\xi_s(\omega)$).

Proof. For any fixed $x \in H$, and $\xi, \eta \in \mathcal{L}_2^T$ we consider the map $r \to K(x, \xi + r\eta)$ from $\mathbb{R}$ to $\mathcal{L}_2^T$. We have

$$K_t(x,\xi+r\eta) = S_t x + \int_0^t S_{t-s}a(s,\xi_s+r\eta_s)\mathrm{d}s + \int_0^t \int_{E\setminus\{0\}} S_{t-s}f(s,u,\xi_s+r\eta_s)q(\mathrm{d}s\mathrm{d}u). \tag{162}$$

It follows

$$\frac{1}{r}K_t(x,\xi+r\eta) - K(x,\xi) = \int_0^t S_{t-s}\frac{(a(s,\xi_s+r\eta_s)-a(s,\xi_s))}{r}\mathrm{d}s + \int_0^t \int_{E\setminus\{0\}} S_{t-s}\frac{(f(s,u,\xi_s+r\eta_s)-f(s,u,\xi_s))}{r}q(\mathrm{d}s\mathrm{d}u). \tag{163}$$

Let us fix $z \in H$ and define for any $r \neq 0$:

$$a_r(t,z,y) := \frac{a(t,z+ry)-a(t,z)}{r} \tag{164}$$

$$f_r(t,u,z,y) := \frac{f(t,u,z+ry)-f(t,u,z)}{r}, \tag{165}$$

where $t \in [0,T]$, $y \in H$. $a_r(t,y,\xi_s(\omega))$ and $f_r(t,u,y,\xi_s(\omega))$ satisfy the conditions (140), (141) with r instead of n (and y instead of z). Moreover, $\frac{\partial}{\partial z}a(s,\xi_s(\omega))y$ and $\frac{\partial}{\partial z}f(s,u,\xi_s(\omega))y$ satisfy also (140) and (141), due to condition (159).

Analogous to (144), we have (by using also the Lipschitz conditions) that

$$\lim_{r\to 0}\left\{T\left\|a_r(t,y,\xi_t(\omega)) - \frac{\partial}{\partial z}a(t,\xi_t(\omega))y\right\|^2 + \int_{E\setminus\{0\}}\left\|f_r(t,u,y,\xi_t(\omega)) - \frac{\partial}{\partial z}f(t,u,\xi_t(\omega))y\right\|^2\beta(\mathrm{d}u)\right\} = 0 \quad P\text{-a.s.} \tag{166}$$

Defining similarly as for (147) and (148)

$$\gamma_t^r := T\int_0^t \mathbf{E}\left[\left\|a_r(s,\eta_s,\xi_s,) - \frac{\partial}{\partial z}a(s,\xi_s)\eta_s\right\|^2\right]\mathrm{d}s \tag{167}$$

$$\delta_t^r := \int_0^t \int_{E\setminus\{0\}} \mathbf{E}\left[\left\|f_r(s,u,\eta_s,\xi_s) - \frac{\partial}{\partial z}f(s,u,\xi_s)\eta_s\right\|^2\right]\beta(\mathrm{d}u)\mathrm{d}s, \tag{168}$$

and operating in a similar way as in the proof of Theorem 8.1 we obtain the desired result. ∎

We assume also:

(G) $\frac{\partial}{\partial z}a(s,z)$ is continuous in $z\mathrm{d}s$-a.s.

(H) $\frac{\partial}{\partial z}f(s,u,z)$ is continuous $\mathrm{d}s$-a.s. in the norm $\|\cdot\|_{\mathcal{L}^2(\mathrm{d}\beta)}$ of $\mathcal{L}^2(\mathrm{d}\beta)$.

Theorem 9.6. *For any fixed $\eta \in \mathcal{L}_2^T$ the function*

$$\frac{\delta}{\delta\xi} K(x,\xi)\eta : H \times \mathcal{L}_2^T \to \mathcal{L}_2^T \tag{169}$$

is continuous.

Proof of Theorem 9.6. Let (x^n, ξ^n) converge to (x, ξ) in $H \times \mathcal{L}_2^T$. For any $n \in \mathbb{N}$ we have that

$$\begin{aligned} \frac{\partial}{\partial\xi} K(x^n,\xi^n)\eta_t - \frac{\partial}{\partial\xi} K(x,\xi)\eta_t &= \int_0^t S_{t-s}\left(\frac{\partial}{\partial z} a(s,\xi_s^n)\eta_s - \frac{\partial}{\partial z} a(s,\xi_s)\eta_s\right) \\ &\quad + \int_0^t \int_{E\setminus\{0\}} S_{t-s}\left(\frac{\partial}{\partial z} f(s,u,\xi_s^n)\eta_s - \frac{\partial}{\partial z} f(s,u,\xi_s)\eta_s\right) q(\mathrm{d}s\mathrm{d}x). \end{aligned} \tag{170}$$

From Lemma 2.4 it follows that

$$\begin{aligned} &\int_0^T \mathbf{E}\left[\left\|\frac{\partial}{\partial\xi} K(x^n,\xi^n)\eta_t - \frac{\partial}{\partial\xi} K(x,\xi)\eta_t\right\|^2\right] \mathrm{d}t \\ &\le 2T\mathrm{e}^{2\alpha T} \int_0^T \mathbf{E}\left[\left\|\frac{\partial}{\partial z} a(s,\xi_s^n)\eta_s - \frac{\partial}{\partial z} a(s,\xi_s)\eta_s\right\|^2\right] \mathrm{d}s \\ &\quad + 2T\mathrm{e}^{2\alpha T} \int_0^T \int_{E\setminus\{0\}} \mathbf{E}\left[\left\|\frac{\partial}{\partial z} f(s,u,\xi_s^n)\eta_s - \frac{\partial}{\partial z} f(s,u,\xi_s)\eta_s\right\|^2\right] \mathrm{d}s\beta(\mathrm{d}u). \end{aligned} \tag{171}$$

$\xi^n \to \xi$ in $\mathcal{L}_2^T$ when $n \to \infty$ implies that there is a subsequence $\{n_k\}_{k\in\mathbb{N}}$ such that $\xi_s^{n_k} \to \xi_s \mathrm{d}s \otimes \mathrm{d}P$-a.s. in $[0,T] \times \Omega$, when $k \to \infty$. Hence we get

$$\begin{aligned} &\left\|\left\|\frac{\partial}{\partial z} a(s,\xi_s^{n_k}(\omega))\eta_s - \frac{\partial}{\partial z} a(s,\xi_s(\omega))\eta_s\right\|\right\| \to 0 \quad \mathrm{d}s \otimes \mathrm{d}P\text{-a.e.} \\ &\text{in } [0,T] \times \Omega \text{ when } k \to \infty \end{aligned} \tag{172}$$

and

$$\begin{aligned} &\int_{E\setminus\{0\}} \left\|\left\|\frac{\partial}{\partial z} f(s,u,\xi_s^n(\omega))\eta_s - \frac{\partial}{\partial z} f(s,u,\xi_s(\omega))\eta_s\right\|\right\|^2 \beta(\mathrm{d}u) \to 0 \quad \text{a.e. } \mathrm{d}s \otimes \mathrm{d}P \\ &\text{in } [0,T] \times \Omega. \end{aligned} \tag{173}$$

We get by the Lebesgue dominated convergence theorem that $\frac{\partial}{\partial\xi} K(x,\xi)\eta$ is continuous. ■

Corollary 9.7. *Let us assume that all the hypotheses of Theorem 9.6 hold. Let $(Z_t^x)_{t\in[0,T]}$ denote the solution of (3) with initial condition*

$$Z_0(\omega) = x \quad P\text{-}a.s.$$

Then $(\frac{\partial}{\partial x} Z_t^x)_{t\in[0,T]}$ is a solution of

$$\begin{aligned} \frac{\partial}{\partial x} Z_t^x &= \int_0^t \left(S_{t-s}\frac{\partial}{\partial z} a(s,Z_s^x)\frac{\partial}{\partial x} Z_s^x\right) \mathrm{d}s \\ &\quad + \int_0^t \int_{E\setminus\{0\}} \left(S_{t-s}\frac{\partial}{\partial z} f(s,u,Z_s^x)\frac{\partial}{\partial x} Z_s^x\right) q(\mathrm{d}s\mathrm{d}x). \end{aligned} \tag{174}$$

Proof. The statement of Corollary 9.7 is a consequence of the Theorems 9.3, 9.5 and 9.6, Remark 9.4 and Proposition C.0.3. in Appendix C of [12] (cf. also Appendix C of [16], where the Gaussian case is considered). ■

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Appendix

Let $g(s, u, \omega) \in M_\beta^{T,2}(E/H)$. From Theorem 3.13 and Proposition 3.15 in [37] it follows that $(M_t)_{0\le t\le T}$ defined in (4) is an $\mathcal{F}_t$-square integrable martingale and is càd-làg.

Theorem A.1. *Let the state space H be a separable Hilbert space. The corresponding Meyer process to $(M_t)_{0\le t\le T}$ is the process $(\langle M\rangle_t)_{0\le t\le T}$, with*

$$\langle M\rangle_t := \int_0^t \int_{E\setminus\{0\}} \|g\|^2 \mathrm{d}s\beta(\mathrm{d}u). \tag{175}$$

(See e.g. [40], or [41] page 121, for the definition of Meyer process.)

Proof. $(\langle M\rangle_t)_{0\le t\le T}$ is an increasing and continuous process and hence also a process with paths of bounded variation. Moreover $\langle M\rangle_0 = 0P$-a.s.. Hence it is sufficient to show that $(\|M_t\|^2 - \langle M\rangle_t)_{0\le t\le T}$ is a martingale.

This follows by proving that for all $0 < s < t \le T$ the following holds:

$$\mathbf{E}[(\|M_t\|^2 - \langle M\rangle_t)\mathbf{1}_{A_s}] = \mathbf{E}[(\|M_s\|^2 - \langle M\rangle_s)\mathbf{1}_{A_s}] \quad \forall A_s \in \mathcal{F}_s. \tag{176}$$

Proof of (176):

$$\begin{aligned}\mathbf{E}[\|M_t\|^2\mathbf{1}_{A_s}] &= \mathbf{E}[\|M_t - M_s + M_s\|^2\mathbf{1}_{A_s}] = \mathbf{E}[\|M_s\|^2\mathbf{1}_{A_s}] + \mathbf{E}[\|M_t - M_s\|^2\mathbf{1}_{A_s}] \\ &= \mathbf{E}[\|M_s\|^2\mathbf{1}_{A_s}] + \mathbf{E}\left[\left\|\int_s^t \int_{E\setminus\{0\}} g\mathbf{1}_{A_s} q(\mathrm{d}s'\mathrm{d}u)\right\|^2\right],\end{aligned} \tag{177}$$

where the second equality follows from the fact that $(M_t)_{t\in[0,T]}$ is a martingale, while the last equality follows from Proposition 3.3 of [38]. Due to the isometry (12) we get

$$\begin{aligned}\mathbf{E}[\|M_t\|^2\mathbf{1}_{A_s}] &= \mathbf{E}[\|M_s\|^2\mathbf{1}_{A_s}] + \int_s^t \int_{E\setminus\{0\}} \mathbf{E}[\|g\|^2\mathbf{1}_{A_s}\mathrm{d}s']\beta(\mathrm{d}u) \\ &= \mathbf{E}[\|M_s\|^2\mathbf{1}_{A_s}] + \mathbf{E}[(\langle M\rangle_t - \langle M\rangle_s)\mathbf{1}_{A_s}],\end{aligned} \tag{178}$$

which implies (176). ■

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