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From Set-theoretic Coinduction to Coalgebraic Coinduction: some results, some problems

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Abstract

We investigate the relation between the *set-theoretical* description of coinduction based on Tarski Fixpoint Theorem, and the *categorical* description of coinduction based on *coalgebras*. In particular, we introduce set-theoretic generalizations of the *coinduction proof principle*, in the spirit of Milner's *bisimulation "up-to"*, and we discuss categorical counterparts for these. Moreover, we investigate the connection between these and the equivalences induced by T*-coiterative functions*. These are morphisms into *final coalgebras*, satisfying the T*-coiteration scheme*, which is a generalization of the *corecursion scheme*. We show how to describe coalgebraic F*-bisimulations* as set-theoretical ones. A list of examples of set-theoretic coinductions which appear not to be easily amenable to coalgebraic terms are discussed.

Introduction

Coinductive definitions and *coinduction proof principles* are a natural tool for defining and reasoning on *infinite* and *circular* objects, such as streams, exact reals, processes. See e.g. [Mil83,Coq94,HL95,BM96,Fio96,Len96,Pit96,Rut96] [HJ98,HLMP98,Len98] for various approaches to infinite objects based on coinduction. Many of such objects and concepts arise in connection with a *maximal fixed point* construction of some kind. One of the advantages offered by the coinductive approach with respect to others based on domain theory or metric semantics, is that it allows for a *simple, operationally-based, implementation-independent* treatment of infinite objects, without requiring any heavy mathematical overhead. A purely set-theoretical approach, however, often appears quite ad-hoc, just think of how one would prove set-theoretically the existence of a *coiterative* function into streams.

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In recent years, a *categorical* explanation of coinduction has appeared, based on the notion of *coalgebra*, which we will call *coalgebraic coinduction* ([Acz88,AM89,Acz93,RT93,RT94,Tur96,TP97,Len98]). Coalgebraic coinduction has proved to be extremely fertile ([HL95,Jac96,Len96,Rut96,Jac97,RV97] [HLMP98,Mos?,Len99]). It has spurred the development of *Final Semantics*, a methodology for understanding the correspondence between syntax and operational semantics of programming languages. Whenever coalgebraic coinduction has been successful, it has overcome many of the defects of set-theoretic coinduction. It explains coinductive proof techniques uniformly and suggestively, it allows to treat simultaneously definitions by corecursion and to phrase proofs by coinduction in a more principled and uniform way.

We feel, however, that there is still a wide range of contexts where set-theoretic coinductive tools have not yet been explained coalgebraically (see Section 4.2 below). Moreover, very few attempts have been made to formulate precisely the correspondence between set-theoretic and coalgebraic coinduction, and the scope of the latter.

In this paper, which expands ideas in [Len98], we offer some contributions along these directions of research, so far little explored. First (possibly new), we introduce various generalizations of the classical set-theoretical coinduction principle based on Tarski Fixpoint Theorem, which go in the direction of Milner's bisimulation "up-to" principle ([Mil83]). We call these proof principles *coinduction principles "up-to"*. Then we try to develop categorical counterparts of these set-theoretic coinduction principles "up-to". In particular, we present a categorical version of the set-theoretic principle "up-to-T", for T suitable operator on relations. We call the categorical version Coalgebraic Coinduction "up-to-T", for T suitable monad. This is based on the new notion of F-bisimulation "up-to-T. The Coalgebraic Coinduction "up-to-T" is put to use in order to get a *proof principle* for reasoning on equivalences induced by *T-coiterative morphisms*. I.e. morphisms into final coalgebras defined by the *T-coiteration scheme*. This generalizes the *corecursion scheme* (see [Geu92]), which is dual to the (primitive) recursion scheme. The T-coiteration scheme allows to capture many interesting functions into final coalgebras, which escape the pure coiteration scheme (see Section 3 for an example). This illustrates the advantages that a coalgebraic description offers w.r.t. a set-theoretical one, as far as uniformity and generality.

The correspondence between Tarski's coinduction principle and the categorical principle based on F-bisimulations can be formalized precisely. Namely, for all functors which preserve weak pullbacks, one can derive set-theoretic coinduction principles from categorical coinduction principles ([Rut98]). We show, moreover, that the translation from coalgebraic to set-theoretic coinduction is *compositional*. Going the other way round, i.e. providing categorical principles from set-theoretic ones, seems to be vey problematic. We provide some critical situations which seem to indicate limitations of the coalgebraic approach.

The paper is organized as follows. In Section 1, we recall the classical coinduction principle deriving from Tarski's characterization of greatest fixed points of monotone operators, and we introduce a number of (possibly new) set-theoretic coinduction principles "up-to". In Section 2, we present the categorical framework based on coalgebras for describing coinduction and coiteration. In Section 3, we introduce the T*-coiteration scheme*, and we present a categorical counterpart for the set-theoretic coinduction principle "up-to-T" introduced in Section 1. Moreover, we derive a *sound* proof principle for establishing equivalences induced by T-coiterative morphisms. In Section 4, we study the relations between the set-theoretic coinduction principles of Section 1, and the categorical coinduction principles based on F-bisimulations of Sections 2 and 3.

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1 Set-theoretic Coinduction

In this section, we list a number of set-theoretic coinduction principles, ranging from the basic coinduction principle, originally used by Milner in [Mil83] for reasoning on CCS processes, to the principles "up-to" ([San95,Len98]), which generalize the idea behind the notion of Milner's *bisimulation "up-to"*. All these coinduction principles arise naturally from suitable characterizations of maximal fixed points of monotone operators. For simplicitly, we discuss operators on binary relations, but many results apply more generally to operators on complete lattices.

Throughout this section, $\Phi : \mathcal{P}(X \times Y) \to \mathcal{P}(X \times Y)$ will denote a *monotone* operator over the complete lattice of set-theoretic binary relations on the cartesian product of two sets X and Y, and $\approx_\Phi$ will denote the greatest fixed point of Φ.

Theorem 1.1 (Coinduction Principle (Tarski)) *The following principle is* sound*:*

$$\frac{x \,\mathcal{R}\, y \quad \mathcal{R} \subseteq \Phi(\mathcal{R})}{x \approx_\Phi y} \ .$$

The Coinduction Principle is also complete in the sense that

$$x \approx_\Phi y \implies \exists\, \mathcal{R} \,.\, (x \,\mathcal{R}\, y \wedge \ \mathcal{R} \subseteq \Phi(\mathcal{R}))\ .$$

As usual, we call Φ-*bisimulation* a relation $\mathcal{R}$ which satisfies the second premise of the principle of Coinduction.

Coinduction principles are the more useful, the easier is to show the inclusion in the premise. It is therefore natural to look for alternative characterizations of maximal fixed points, possibly exploiting special properties of the operator Φ, which allow to relax the condition $\mathcal{R} \subseteq \Phi(\mathcal{R})$. A simple and natural example is given by the following theorem:

Theorem 1.2 (Coinduction "up-to-∪") *Let* $\overline{\mathcal{R}} \in \mathcal{P}(X \times Y)$ *be such that* $\overline{\mathcal{R}} \subseteq \approx_\Phi$*. Then*
i)

$$\approx_\Phi = \bigcup \{\mathcal{R} \mid \mathcal{R} \subseteq \Phi(\mathcal{R} \cup \overline{\mathcal{R}})\} \ .$$

ii) The following principle is sound and complete:

$$\frac{x \, \mathcal{R} \, y \qquad \mathcal{R} \subseteq \Phi(\mathcal{R} \cup \overline{\mathcal{R}})}{x \approx_\Phi y} \ .$$

Interesting instances of the above principle arise when $\overline{\mathcal{R}}$ is taken to be $\approx_\Phi$, or the identity relation, if $\approx_\Phi$ is reflexive.

Now we give two possible generalizations of the coinduction principle, in the spirit of Milner's bisimulation "up-to", which we call principle of *Coinduction "up-to-T"* and principle of *Coinduction "up-to-*$(\approx, \bullet)$*"*. The first generalizes also the principle of Coinduction "up-to-∪".

Theorem 1.3 (Coinduction "up-to-T") *Let* $T : \mathcal{P}(X \times Y) \to \mathcal{P}(X \times Y)$*. If* T *satisfies the following properties*

1) T *is a monotone operator on the complete lattice* $(\mathcal{P}(X \times Y), \subseteq)$*,*

2) *for all* $\mathcal{R} \in \mathcal{P}(X \times Y)$*,* $\mathcal{R} \subseteq T(\mathcal{R})$*,*

3) *for all* $\mathcal{R} \in \mathcal{P}(X \times Y)$*,* $(T \circ \Phi)(\mathcal{R}) \subseteq (\Phi \circ T)(\mathcal{R})$*,*

then
i)

$$\approx_\Phi = \bigcup \{\mathcal{R} \mid \mathcal{R} \subseteq (\Phi \circ T)(\mathcal{R})\} \ .$$

ii) The following principle is sound and complete:

$$\frac{x \, \mathcal{R} \, y \qquad \mathcal{R} \subseteq (\Phi \circ T)(\mathcal{R})}{x \approx_\Phi y} \ .$$

Proof. i) If $\mathcal{R} \subseteq \Phi(\mathcal{R})$, then, since T is a closure operator, $\mathcal{R} \subseteq T(\mathcal{R})$. Using the monotonicity of Φ, $\mathcal{R} \subseteq \Phi(\mathcal{R}) \subseteq \Phi(T(\mathcal{R}))$.
Vice versa, if $\mathcal{R} \subseteq \Phi \circ T(\mathcal{R})$, we have to show that $\exists \, \mathcal{S}$ such that $\mathcal{R} \subseteq \mathcal{S} \ \wedge \ \mathcal{S} \subseteq \Phi(\mathcal{S})$. Consider the following inductively defined sequence $\{\mathcal{R}_n\}_{n \geq 0}$:

$$\mathcal{R}_0 = \mathcal{R}$$
$$\mathcal{R}_{n+1} = T(\mathcal{R}_n) \ .$$

We prove by induction on n that $\mathcal{R}_n \subseteq \Phi(\mathcal{R}_{n+1})$:
For $n = 0$ the thesis is immediate, since $\mathcal{R} \subseteq \Phi \circ T(\mathcal{R})$ by hypothesis.
Let $n > 0$:
$\mathcal{R}_n = T(\mathcal{R}_{n-1})$
$\subseteq T \circ \Phi(\mathcal{R}_n)$, by induction hypothesis and monotonicity of T,
$= \Phi \circ T(\mathcal{R}_n)$, by hypothesis 3,
$= \Phi(\mathcal{R}_{n+1})$, by definition of the sequence.
Hence, taking $\mathcal{S} = \bigcup_n \mathcal{R}_n$, we have $\mathcal{R} \subseteq \mathcal{S}$ and $\mathcal{S} \subseteq \Phi(\mathcal{S})$.
ii) Immediate consequence of item i) of this theorem. □

If we take both X, Y in the above principle to be the domain of CCS processes, Φ to be the operator corresponding to strong bisimulation, and we take T to be defined by $T(\mathcal{R}) = \approx_\Phi \circ \mathcal{R} \circ \approx_\Phi$, we have that a relation $\mathcal{R}$ such that $\mathcal{R} \subseteq (\Phi \circ T)(\mathcal{R})$ is a bisimulation "up-to" in the sense of Milner.

If we drop hypothesis 2 in Theorem 1.3, then we can prove only soundness, but not completeness of the Coinduction Principle "up-to-T". A simple counterexample is the following. If the operator T is the constant operator equal to the least fixed point of Φ, $\sim_\Phi$, and moreover $\sim_\Phi \neq \approx_\Phi$, then $\approx_\Phi \neq \bigcup\{\mathcal{R} \mid \mathcal{R} \subseteq (\Phi \circ T)(\mathcal{R})\}$.

The Coinduction Principle "up-to-T" can be viewed as a variant of the principle introduced by Sangiorgi in [San95] for labelled transition systems. In particular, the principle of [San95] is obtained by replacing hypotheses 1,2,3 in Theorem 1.3 above by the hypothesis of *respectfulness* of T[2]. Sangiorgi's principle is complete when considered over all respectful operators, but, for each fixed T, it is only sound. In particular, the respectfulness condition is already implied by the sole hypotheses 1 and 3 of Theorem 1.3.

The second generalization of the of the Coinduction Principle is based on the following theorem.

Theorem 1.4 (Coinduction "up-to-$(\approx, \bullet)$") *Let $\Phi : \mathcal{P}(X \times X) \rightarrow \mathcal{P}(X \times X)$ be a monotone operator, and let $\bullet : \mathcal{P}(X \times X) \times \mathcal{P}(X \times X) \rightarrow \mathcal{P}(X \times X)$ be an associative operation. If*

1) *for all $\mathcal{R}, \mathcal{R}', \mathcal{R}_1, \mathcal{R}'_1$,*

$$\mathcal{R} \subseteq \Phi(\mathcal{R}_1) \wedge \mathcal{R}' \subseteq \Phi(\mathcal{R}'_1) \implies \mathcal{R} \bullet \mathcal{R}' \subseteq \Phi(\mathcal{R}_1 \bullet \mathcal{R}'_1),$$

2) $\approx_\Phi \subseteq \approx_\Phi \bullet \approx_\Phi$,

then

i)

$$\approx_\Phi = \bigcup\{\mathcal{R} | \mathcal{R} \subseteq \Phi(\approx_\Phi \bullet \mathcal{R} \bullet \approx_\Phi)\} \ .$$

ii) The following principle is sound and complete:

$$\frac{x \, \mathcal{R} \, y \quad \mathcal{R} \subseteq \Phi(\approx_\Phi \bullet \mathcal{R} \bullet \approx_\Phi)}{x \approx_\Phi y} \ .$$

Proof. i) It is sufficient to prove that

a) $\mathcal{R} \subseteq \Phi(\approx_\Phi \bullet \mathcal{R} \bullet \approx_\Phi) \implies \exists \, \mathcal{S} \, . \ \mathcal{R} \subseteq \mathcal{S} \subseteq \Phi(\mathcal{S})$.
b) $\mathcal{R} \subseteq \Phi(\mathcal{R}) \implies \exists \, \mathcal{S} \, . \ \mathcal{R} \subseteq \mathcal{S} \subseteq \Phi(\approx_\Phi \bullet \mathcal{R} \bullet \approx_\Phi)$.

Proof of item a): We prove that $\Phi(\approx_\Phi \bullet \mathcal{R} \bullet \approx_\Phi) \subseteq \Phi(\Phi(\approx_\Phi \bullet \mathcal{R} \bullet \approx_\Phi))$. Then we can take $\mathcal{S} = \Phi(\approx_\Phi \bullet \mathcal{R} \bullet \approx_\Phi)$. ¿From $\mathcal{R} \subseteq \Phi(\approx_\Phi \bullet \mathcal{R} \bullet \approx_\Phi)$ and $\approx_\Phi \subseteq \Phi(\approx_\Phi)$, by item 1, $\approx_\Phi \bullet \mathcal{R} \bullet \approx_\Phi \subseteq \Phi(\approx_\Phi \bullet \approx_\Phi \bullet \mathcal{R} \bullet \approx_\Phi \bullet \approx_\Phi)$. From $\approx_\Phi \subseteq \Phi(\approx_\Phi)$, using item 1, $\approx_\Phi \bullet \approx_\Phi \subseteq \approx_\Phi$. Hence, by item 2, $\approx_\Phi \bullet \approx_\Phi = \approx_\Phi$

[2] $T : X \rightarrow X$ is *respectful* if, for all $x, y \in X$, $(x \leq y \ \wedge \ x \leq \Phi(y)) \ \Rightarrow \ (T(x) \leq T(y) \ \wedge \ T(x) \leq \Phi(T(y)))$.

and $\approx_\Phi \bullet \mathcal{R} \bullet \approx_\Phi \subseteq \Phi(\approx_\Phi \bullet \mathcal{R} \bullet \approx_\Phi)$. Finally, by monotonicity of Φ, $\Phi(\approx_\Phi \bullet \mathcal{R} \bullet \approx_\Phi) \subseteq \Phi(\Phi(\approx_\Phi \bullet \mathcal{R} \bullet \approx_\Phi))$.
Proof of item b): Since, by the proof of item a), $\approx_\Phi \bullet \approx_\Phi = \approx_\Phi$, then $\approx_\Phi \subseteq \Phi(\approx_\Phi) \implies \approx_\Phi \subseteq \Phi(\approx_\Phi \bullet \approx_\Phi \bullet \approx_\Phi)$. Hence take $\mathcal{S} = \approx_\Phi$.
ii) Immediate consequence of item i of this theorem.

□

Milner's bisimulation "up-to" principle is recovered by simply taking as X the domain of CCS processes, as Φ the operator corresponding to strong bisimulation, and as $\bullet$ relational composition.

Hypothesis 1 in Theorem 1.4 can be viewed as a *generalized transitivity*. In fact, if $\bullet$ is relational composition, then hypothesis 1 implies transitivity of the relation $\approx_\Phi$.

If $\bullet$ is relational composition, and $\approx_\Phi$ is reflexive, then hypothesis 2 of Theorem 1.4 is satisfied.

Dropping hypothesis 2 in Theorem 1.4, and assuming the monotonicity of $\bullet$ w.r.t. $\subseteq$, i.e., for all $\mathcal{R}, \mathcal{R}_1, \mathcal{R}', \mathcal{R}'_1$, $\mathcal{R} \subseteq \mathcal{R}_1 \ \wedge \ \mathcal{R}' \subseteq \mathcal{R}'_1 \implies \mathcal{R} \bullet \mathcal{R}' \subseteq \mathcal{R}_1 \bullet \mathcal{R}'_1$, we get soundness of the principle of Coinduction "up-to-($\approx, \bullet$)", but we loose completeness.

1.1 Coiterative and Corecursive Functions

We could give a purely set-theoretic treatment of *coiterative* and *corecursive* functions ([Geu92,Gim94]). But we feel that in this respect the coalgebraic setting is the most natural. We only point out that indeed it would be possible to justify definitions by coiteration and corecursion solely by means of maxiaml fixed points. In fact, the graphs of coiterative functions can be naturally defined as maximal fixed points, since they are bisimulations after all. This approach would also highlight the connections between corecursive morphisms and the Coinduction "up-to-$\cup \approx$" (see Section 4.2 for more details).

2 Coalgebraic Description of Coinduction and Coiterative Morphisms

In this section, we present the categorical description of coinduction based on the notion of *coalgebra* ([Acz88,AM89,RT93,RT94,Rut96,Tur96,TP97,Len98]). In this setting, the categorical counterparts of set-theoretic bisimulations are *F-bisimulations*, i.e. *spans of coalgebra morphisms* ([TP97]). One of the advantages of a categorical description is that we can deal uniformly with coinductively defined objects and *coiterative* morphisms. In fact, the latter arise naturally in a categorical context.

As we will see formally in Section 4, the coinduction principle based on F-bisimulations presented in this section is the categorical counterpart of the Coinduction Principle 1.1.

We start by introducing the category of F-coalgebras, for F endofunctor on a category $\mathcal{C}$. F-coalgebras, i.e. pairs (X, α_X), where $\alpha_X : X \to FX$ is an arrow in $\mathcal{C}$, can be endowed with the structure of a category by introducing the notion of F-coalgebra morphism as follows:

Definition 2.1 Let $F : \mathcal{C} \to \mathcal{C}$. Then $f : (X, \alpha_X) \to (Y, \alpha_Y)$ is an *F-coalgebra morphism* if $f : X \to Y$ is an arrow of the category $\mathcal{C}$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \alpha_Y} \\ F(X) & \xrightarrow[F(f)]{} & F(Y) \end{array}$$

Notice that, in set-theoretic categories, *graphs* of F-coalgebra morphisms are F-bisimulations.

Unique morphisms into final coalgebras are called *coiterative morphisms*:

Definition 2.2 [*Coiteration Scheme*] Let $F : \mathcal{C} \to \mathcal{C}$, let (X, α_X) be an F-coalgebra, and let $(\nu F, \alpha_{\nu F})$ be a final F-coalgebra. The *coiterative morphism* is the unique F-coalgebra morphism $f : (X, \alpha_X) \to (\nu F, \alpha_{\nu F})$.

Before introducing the notion of *F-bisimulation*, we need to introduce the notion of span.

Definition 2.3 Let F be an endofunctor on a category $\mathcal{C}$. A *span* $(\mathcal{R}, r_1, r_2)$ on objects X, Y consists of an object $\mathcal{R}$ in $\mathcal{C}$, and two ordered arrows, $r_1 : \mathcal{R} \to X$ and $r_2 : \mathcal{R} \to Y$.
Spans on objects X and Y can be ordered as follows:

$$(\mathcal{R}, r_1, r_2) \leq (\mathcal{R}', r_1', r_2') \iff \exists f : \mathcal{R} \to \mathcal{R}'.\ \forall i = 1, 2.\ r_i = r_i' \circ f\ .$$

The notion of binary relation is expressed, in a general categorical setting, as an equivalence class of monic spans (see [FS90] for more details). As pointed out in [TP97], F-bisimulations on F-coalgebras can be simply taken to be spans with a suitable structure of F-coalgebra:

Definition 2.4 [*F-bisimulation,* [TP97]] Let F be an endofunctor on the category $\mathcal{C}$. A span $(\mathcal{R}, r_1, r_2)$ on objects X, Y is an *F-bisimulation* on the F-coalgebras (X, α_X) and (Y, α_Y), if there exists an arrow of $\mathcal{C}$, $\gamma : \mathcal{R} \to F(\mathcal{R})$, such that $((\mathcal{R}, \gamma), r_1, r_2)$ is a coalgebra span, i.e.

$$\begin{array}{ccccc} X & \xleftarrow{\ r_1\ } & \mathcal{R} & \xrightarrow{\ r_2\ } & Y \\ {\scriptstyle \alpha_X}\downarrow & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \alpha_Y} \\ F(X) & \xleftarrow[F(r_1)]{} & F(\mathcal{R}) & \xrightarrow[F(r_2)]{} & F(Y) \end{array}$$

When the two F-coalgebras (X, α_X) and (Y, α_Y) in the definition above coincide, we will simply say that the span is an F-bisimulation on the F-coalgebra (X, α_X).

The following theorem generalizes the fact that, in set-theoretic categories, equivalences induced by coiterative morphisms can be characterized coinductively as the *greatest* F-bisimulation.

Theorem 2.5 ([TP97]) *Suppose that $F : \mathcal{C} \to \mathcal{C}$ has a final F-coalgebra $(\nu F, \alpha_{\nu F})$. Let (X, α_X) be an F-coalgebra, and let $\mathcal{M} : (X, \alpha_X) \to (\nu F, \alpha_{\nu F})$ be the coiterative morphism. If F preserves weak pullbacks, then*
i) for all F-bisimulations $(\mathcal{R}, r_1, r_2)$ on (X, α_X), $\mathcal{M}; r_1 = \mathcal{M}; r_2$;
ii) the kernel pair of $\mathcal{M}$ is an F-bisimulation on (X, α_X).

2.1 Coalgebraic Coinduction in Set-theoretic Categories

Since in this paper we are interested in giving explicit coinduction principles, and in formalizing the connections between set-theoretic and coalgebraic coinduction, we will focus in particular on the "concrete" setting of *set-theoretic categories*. These are categories whose objects are sets or classes of a possibly non-wellfounded universe of sets. In particular, the set-theoretic categories which we will consider are the following:

Definition 2.6
- Let $Set(U)$ $(Set^*(U))$ be the category whose objects are the (non-)wellfounded sets belonging to a Universe of $ZF_0^-(FCU)$.
- Let $Class(U)$ $(Class^*(U))$ be the category whose objects are the classes of (non-)wellfounded sets belonging to a Universe of $ZF_0^-(FCU)$.
- Let $\mathcal{H}C_\kappa(U)$ $((\mathcal{H}C_\kappa)^*(U))$ be the category whose objects are the wellfounded (non wellfounded) sets whose hereditary cardinal is less than κ.
- Let $Card$ $(CARD)$ be the category whose objects are the cardinals (including Ord).

In all the categories above arrows between objects A and B are functions with domain A and codomain B, tagged with A and B.

Throughout this paper, we will denote with $\mathcal{C}^{\mathcal{S}}$ one of the set-theoretic categories defined above.

One can easily check that, in set-theoretic categories, the general notion of F-bisimulation of Definition 2.4 above reduces to the classical Definition 2.7 below. These two definitions are equivalent, in set-theoretic categories, in the sense that they characterize the same equivalence (see Theorem 2.8 below), although in effect they give rise to different coinduction principles.

Definition 2.7 [AM89] Let $F : \mathcal{C}^{\mathcal{S}} \to \mathcal{C}^{\mathcal{S}}$. *An F-bisimulation on the F-coalgebras (X, α_X) and (Y, α_Y) is a set-theoretic relation $R \subseteq X \times Y$ such that there exists an arrow of $\mathcal{C}$, $\gamma : \mathcal{R} \to F(\mathcal{R})$, making the following diagram commute:*

$$\begin{array}{ccccc} X & \xleftarrow{\pi_1} & \mathcal{R} & \xrightarrow{\pi_2} & Y \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \alpha_Y} \\ F(X) & \xleftarrow[F(\pi_1)]{} & F(\mathcal{R}) & \xrightarrow[F(\pi_2)]{} & F(Y) \end{array}$$

Notice that the following theorem, which specializes Theorem 2.5 to set-theoretic categories, holds for both notions of F-bisimulation (either that of Definition 2.4 or that of Definition 2.7). The notation "$x\mathcal{R}y$", for a bisimulation $(\mathcal{R}, r_1, r_2)$ on (X, α_X) and (Y, α_Y) as in Definition 2.4, stands for $\exists u \in \mathcal{R}.\ < r_1, r_2 > (u) = (x, y)$.

Theorem 2.8 (Coalgebraic Coinduction) *Suppose that $F : \mathcal{C}^{\mathcal{S}} \to \mathcal{C}^{\mathcal{S}}$ has final F-coalgebra $(\nu F, \alpha_{\nu F})$. Let (X, α_X) be an F-coalgebra. If F preserves weak pullbacks, then the following principle is sound and complete*

$$\frac{x\,\mathcal{R}\,y \quad \mathcal{R} \ \text{is an } F\text{-bisimulation on } (X, \alpha_X)}{x \sim^F_{(X,\alpha_X)} y} \ ,$$

where $\sim^F_{(X,\alpha_X)}$ is the equivalence induced by the coiterative morphism $\mathcal{M} : (X, \alpha_X) \to (\nu F, \alpha_{\nu F})$.

3 Corecursion and T-coiteration

The coiteration scheme introduced in Definition 2 is not powerful enough to capture many interesting functions into final coalgebras. E.g., let $h_0 : S_N \to S_N$ be the function which, given a stream of natural numbers s, yields the stream obtained by replacing the first element of s by the constant 0. One can easily check that the function h_0 cannot be defined using the pure coiteration scheme. More general forms of coiteration schemes are therefore required to overcome this limitation. A typical example is the *corecursion scheme* (see [Geu92]).

In this section, we study, from a categorical standpoint, a significant class of coiteration schemes. In particular, we introduce a suitable categorical generalization of the coiteration scheme, which we call *T-coiteration scheme*. In Section 3.1, we introduce the principle of Coalgebraic Coinduction "up-to-T". In Section 3.2, we will use it to derive a *sound* proof principle for establishing equivalences induced by T-coiterative morphisms. As we will see in Section 4, the principle of Coalgebraic Coinduction "up-to-T" can be viewed as the categorical counterpart of the set-theoretic Coinduction "up-to-T" of Section 1.

Definition 3.1 [*T-coiteration Scheme*] Let $F : \mathcal{C} \to \mathcal{C}$ be such that F has final coalgebra $(\nu F, \alpha_{\nu F})$, and let $< T, \eta, \mu >$ be a monad over $\mathcal{C}$. Then, for any F-coalgebra (TX, α), we can define the *T-coiterative* morphism $h : X \to \nu F$ as $f \circ \eta_X$, where f is the unique F-coalgebra morphism from (TX, α) to the final coalgebra $(\nu F, \alpha_{\nu F})$, i.e.

$$\begin{array}{ccccc} X & \xrightarrow{\eta_X} & TX & \xrightarrow{f} & \nu F \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha_{\nu F}} \\ & & FTX & \xrightarrow[Ff]{} & F(\nu F) \end{array}$$

The T-coiteration scheme subsumes the important case of the *corecursion scheme* ([Geu92]). This can be recovered in categories with coproducts, by considering the following functor (which induces a monad in a standard way):

Definition 3.2 [*Corecursion Monad*] Let $\mathcal{C}$ be a category with coproducts, and let $F : \mathcal{C} \to \mathcal{C}$ have final coalgebra $(\nu F, \alpha_{\nu F})$. Let $T_F^+ : \mathcal{C} \to \mathcal{C}$ be the functor defined by

$$T_F^+ X = X + \nu F \ .$$

The definition of T_F^+ on arrow is canonical.

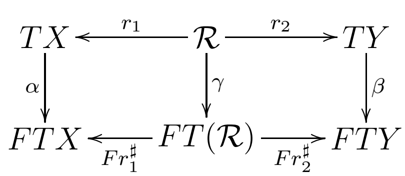
Then, by taking T in Definition 3.1 to be the monad induced by T_F^+, and by considering F-coalgebras of the shape $(X + \nu F, [\alpha_1, F(in_2)])$, where in_2 is the canonical sum injection, we recover exactly the corecursion scheme. The essence of the corecursion scheme is that we can make a choice between the two possibilities offered by the two branches of the disjoint sum in the F-coalgebra. For instance, the function $h_0 : S_N \to S_N$ mentioned at the beginning of this section is corecursive.

3.1 Coalgebraic Coinduction "up-to-T"

We present now the categorical version of the set-theoretical principle of Coinduction "up-to-T". We call this principle *Coalgebraic Coinduction "up-to-T"*. We will show that the Coalgebraic Coinduction "up-to-T" is related to the *T-coiteration scheme* introduced above, in the sense that it can be used to derive a proof principle for establishing equivalences induced by T-coiterative morphisms.

We start by introducing the notion of F-bisimulation "up-to-T":

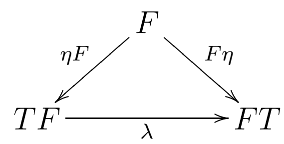
Definition 3.3 [*F-bisimulation "up-to-T"*] Let $F : \mathcal{C} \to \mathcal{C}$, let $< T, \eta, \mu >$ be a monad on $\mathcal{C}$, and let (TX, α) and (TY, β) be F-coalgebras. An F-bisimulation "up-to-T" on the the F-coalgebras (TX, α) and (TY, β) is a span $(\mathcal{R}, r_1, r_2)$ on TX and TY, such that there exists an arrow of $\mathcal{C}$, $\gamma : \mathcal{R} \to FT(\mathcal{R})$, making the following diagram commute:



where $r_1^\sharp, r_2^\sharp$ are the unique extensions of r_1, r_2 given by the universality property of η in the adjunction between the Eilenberg-Moore category of T-algebras and the category $\mathcal{C}$, i.e. $\eta_{\mathcal{R}}; r_i^\sharp = r_i$.

The following definition will be useful in Theorem 3.5 below.

Definition 3.4 Let $F : \mathcal{C} \to \mathcal{C}$ be a functor, and let $< T, \eta, \mu >$ be a monad over $\mathcal{C}$. We say that T *distributes over F w.r.t. η* if there exists a natural transformation $\lambda : TF \stackrel{\cdot}{\to} FT$ for which the following diagram commutes



The relation between F-bisimulations "up-to-T" and F-bisimulations is illustrated by the following theorem:

Theorem 3.5 *Let $\mathcal{C}$ be a category closed under ω-colimits, let $F : \mathcal{C} \to \mathcal{C}$, let $< T, \eta, \mu >$ be a monad over $\mathcal{C}$. If*

1) *F, T preserve ω-colimits,*

2) *T distributes over F w.r.t. η,*

3) *for all $((\mathcal{R}, \gamma), r_1, r_2)$ F-bisimulation "up-to-T" on F-coalgebras (TX, α) and (TY, β), also $((T(\mathcal{R}), T\gamma; \lambda), r_1^\natural, r_2^\natural)$ is an F-bisimulation "up-to-T" on F-coalgebras (TX, α) and (TY, β),*

Then
i) If $((\mathcal{R}, \gamma), r_1, r_2)$ is an F-bisimulation on (TX, α) and (TY, β), then $((\mathcal{R}, \gamma; F(\eta_{\mathcal{R}})), r_1, r_2)$ is an F-bisimulation "up-to-T" on (TX, α) and (TY, β).
ii) For all $(\mathcal{R}, r_1, r_2)$ F-bisimulation "up-to-T" on (TX, α) and (TY, β), there exists $(\widetilde{\mathcal{R}}, \widetilde{r_1}, \widetilde{r_2})$ F-bisimulation on (TX, α) and (TY, β) such that $\mathcal{R} \leq \widetilde{\mathcal{R}}$.

Proof. The proof of item i) is immediate. In order to prove item ii), let $\widehat{\mathcal{R}}$ be the ω-colimit of the ω-diagram $\{T^n(\mathcal{R}) \stackrel{T^n \eta_{\mathcal{R}}}{\to} T^{n+1}(\mathcal{R})\}_{n \geq 0}$, then we take as $\widetilde{\mathcal{R}}$ the ω-colimit $T(\widehat{\mathcal{R}})$ of the diagram $\{T^n(\mathcal{R}) \stackrel{T^n \eta_{\mathcal{R}}}{\to} T^{n+1}(\mathcal{R})\}_{n \geq 1}$. Let $r_1^{\sharp n}$, for $n \geq 1$ be defined by $r_1^{\sharp 1} = r_1^\sharp$, $r_1^{\sharp n+1} = (r_1^{\sharp n})^\sharp$. Notice that TX with $\{r_1^{\sharp n} : T^n(\mathcal{R}) \to TX\}_{n \geq 1}$ is a cocone for the latter diagram, hence we take as $\widetilde{r_1}$ the unique morphism from the colimit $T(\widehat{\mathcal{R}})$ to the cocone TX. Moreover, since F preserves colimits, $FT(\widehat{\mathcal{R}})$ is the colimit of the diagram $\{FT^n(\mathcal{R}) \stackrel{T^n \eta_{\mathcal{R}}}{\to} FT^{n+1}(\mathcal{R})\}_{n \geq 1}$, FTX with $\{Fr_1^{\sharp n} : FT^n(\mathcal{R}) \to FTX\}_{n \geq 1}$ is a cocone for the same diagram, and $F\widetilde{r_1} : FT(\widehat{\mathcal{R}}) \to FTX$ is the morphism given by the universal property of $FT(\widehat{\mathcal{R}})$. Using the naturality of η and the distributivity law, one can check that FTX with $\{\overbrace{T(\ldots(T(T}^{n}\gamma); \overbrace{\lambda); \lambda \ldots); \lambda}^{n}; Fr_1^{\sharp n} : T^n(\mathcal{R}) \to FTX\}_{n \geq 1}$ is a cocone for $\{T^n(\mathcal{R}) \stackrel{T^n \eta_{\mathcal{R}}}{\to} T^{n+1}(\mathcal{R})\}_{n \geq 1}$. In a similar way, also $FT(\widehat{\mathcal{R}})$ can be endowed with a structure of cocone for the same diagram. Let $\widetilde{\gamma} : T(\widehat{\mathcal{R}}) \to FT(\widehat{\mathcal{R}})$ be the unique morphism given by the universal property of $T(\widehat{\mathcal{R}})$. Finally, in order to prove that the following diagram commutes

$$\begin{array}{ccc} TX & \xleftarrow{\widetilde{r_1}} & T(\widehat{\mathcal{R}}) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \widetilde{\gamma}} \\ FTX & \xleftarrow[F\widetilde{r_1}^{\sharp}]{} & FT(\widehat{\mathcal{R}}) \end{array}$$

we use the fact that, by hypothesis 3, for all $n \geq 0$, the following diagram commutes

$$\begin{array}{ccc} TX & \xleftarrow{r_1^{\sharp n}} & T^n(\mathcal{R}) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \overbrace{T(\ldots(T(T}^{n}\gamma);\overbrace{\lambda);\lambda\ldots);}^{n}\lambda} \\ FTX & \xleftarrow[Fr_1^{\sharp n+1}]{} & FT^{n+1}(\mathcal{R}) \end{array}$$

This implies that the cocone $FTX, \{\overbrace{T(\ldots(T(T}^{n}\ \gamma);\overbrace{\lambda);\lambda\ldots);}^{n}\lambda; Fr_1^{\sharp n} : T^n(\mathcal{R}) \rightarrow FTX\}_{n\geq 1}$ coincides with the cocone FTX with $\{r_1^{\sharp n};\alpha : T^n(\mathcal{R}) \rightarrow FTX\}_{n\geq 1}$, and the thesis follows exploiting the universality of $T(\widehat{\mathcal{R}})$ with respect to this cocone.

□

Specializing Theorem 3.5 to endofunctors on a category $\mathcal{C}^{\mathcal{S}}$, we obtain an alternative characterization of the greatest F-bisimulation $\sim^F_{(TX,\alpha)}$ on the F-coalgebra (TX, α), which yields the following

Theorem 3.6 (Coalgebraic Coinduction "up-to-T") *Let $F : \mathcal{C}^{\mathcal{S}} \rightarrow \mathcal{C}^{\mathcal{S}}$ and let T be a monad over $\mathcal{C}^{\mathcal{S}}$ satisfying the hypothesis 1,2,3 of Theorem 3.5 above. If moreover F preserves weak pullbacks and has final coalgebra, then the following principle is sound and complete*

$$\frac{x\,\mathcal{R}\,y \quad \mathcal{R}\ \ F\text{-bisimulation "up-to-}T\text{" on } (TX,\alpha)}{x \sim^F_{(TX,\alpha)} y} \ .$$

This principle can be specialized to the corecursion monad $T^+_{F^{S_N}} : \mathcal{C}^{\mathcal{S}} \rightarrow \mathcal{C}^{\mathcal{S}}$, where $F^{S_N} : \mathcal{C}^{\mathcal{S}} \rightarrow \mathcal{C}^{\mathcal{S}}$ is the functor defined by $F^{S_N}(X) = N \times X$ (with canonical definition on arrows), and N denotes the set of natural numbers. The principle above provides a generalization of the set-theoretic Coinduction "up-to-T" of Theorem 1.3, in the sense made precise by Proposition 4.4 of Section 4.

3.2 Reasoning on Equivalences of T-coiterative Morphisms

In this section, the principle of Coalgebraic Coinduction "up-to-T" is put to use for reasoning on equivalences induced by T-coiterative morphisms.

First, we need to recall the notions of *image* and *inverse image* of spans. These are to be intended as the (inverse) image of the subobject of $X_1 \times X_2$

determined by the relation underlying a span on X_1, X_2. See [FS90] for more details.

Definition 3.7 [*(Inverse) Image of Spans*] Let $\mathcal{C}$ be a category with products and pullpacks.

- The *image* of a span $(\mathcal{R}, r_1, r_2)$ on X_1, X_2 by $(f_1 : X_1 \to Y_1, f_2 : X_2 \to Y_2)$, denoted by $(f_1, f_2)^+(\mathcal{R}, r_1, r_2)$, is the span $(\mathcal{R}, r_1; f_1, r_2; f_2)$ on Y_1, Y_2.
- The *inverse image* of a span $(\mathcal{R}, r_1, r_2)$ on X_1, X_2 by $(f_1 : X_1 \to Y_1, f_2 : X_2 \to Y_2)$, denoted by $(f_1, f_2)^{-1}(\mathcal{R}, r_1, r_2)$, is the span $(\mathcal{P}, p_2; \pi_1, p_2; \pi_2)$ on Y_1, Y_2, where $(\mathcal{P}, p_1, p_2)$ is the pullback of $< r_1, r_2 >: \mathcal{R} \to X_1 \times X_2$ and $< \pi_1; f_1, \pi_2; f_2 >: Y_1 \times Y_2 \to X_1 \times X_2$.

If $(\mathcal{R}, r_1, r_2)$ is a span on X and $f : X \to Y$, $g : Y \to X$, we simply denote by $f^+(\mathcal{R}, r_1, r_2)$ the image of $(\mathcal{R}, r_1, r_2)$ by (f, f), and we denote by $g^{-1}(\mathcal{R}, r_1, r_2)$ the inverse image of $(\mathcal{R}, r_1, r_2)$ by (g, g).

Using the principle of Coalgebraic Coinduction "up-to-T", we now prove the following theorem

Theorem 3.8 (Coalgebraic Coinduction for T-coiterative Functions) *Let $F : \mathcal{C}^\mathcal{S} \to \mathcal{C}^\mathcal{S}$ and let $< T, \eta, \mu >$ be a monad on $\mathcal{C}^\mathcal{S}$ satisfying all the hypotheses of Theorem 3.6. Let h be the T-coiterative morphism induced by the F-coalgebra (TX, α), i.e. $h = f \circ \eta_X$, where $f : TX \to \nu F$ is the coiterative morphism. Then the following principle is sound*

$$\frac{x\ \mathcal{R}\ y \quad \eta_X^+(\mathcal{R}, r_1, r_2)\ F\text{-bisimulation “up-to-}T\text{” on } (TX, \alpha)}{x \sim_h y} ,$$

where $\sim_h$ denotes the equivalence induced by the T-coiterative morphism h.

Proof. One can easily check, using Theorem 3.6, that

$$\eta_X^+(\mathcal{R}, r_1, r_2)\ F\text{-bisimulation “up-to-}T\text{” on } (TX, \alpha) \Longrightarrow R \leq \eta_X^{-1}(\sim^F_{(TX,\alpha)}, \pi_1, \pi_2)\ .$$

□

The completeness of the categorical coinduction principle for T-coiterative functions deserves further study.

4 From Coalgebras to Sets and back

In this section we study and discuss the relations between set-theoretic and coalgebraic accounts of coinduction. As we pointed out in the Introduction, this area is quite unexplored and problematic. Here we present some results and raise some problems.

As far as the direction "From Coalgebras to Sets", in the case of functors which preserve weak pullbacks, one can show how to generate, from the coalgebraic coinduction principle based on F-bisimulations of Theorem 2.8, the corresponding set-theoretic Coinduction Principle 1.1. For a special class of covariant functors, we show that this translation is *compositional*, in a sense to be made precise.

We work in set-theoretic categories. It would be interesting to extend these results to other possibly more general categorical settings, where also contravariant and mixed functors could be used. A similar result can be given also for the Coalgebraic Coinduction "up-to-T" 3.6, exploiting the correspondence between it and the set-theoretic Coinduction "up-to-T" 1.3.

It would be extremely interesting to be able to provide coalgebraic coinduction principles in all contexts where set-theoretic coinduction principles of some kind are at work, but it appears very difficult. On one hand F-bisimulations convey more information than set-theoretic bisimulations. On the other hand it might not be always the case that one *can* give categorical descriptions at all of set-theoretic coinduction, see the examples in Section 4.2.

4.1 From Coalgebras to Sets

We start by introducing some notation. Let $(\mathcal{R}, r_1, r_2)$ be a span on X and Y, for X, Y objects of a set-theoretic category $\mathcal{C}^{\mathcal{S}}$. Let denote by $\mathcal{R}^{\mathcal{S}}_{r_1 r_2}$ the set-theoretic relation induced by $(\mathcal{R}, r_1, r_2)$, i.e.

$$\mathcal{R}^{\mathcal{S}}_{r_1 r_2} = \{(x,y) \in X \times Y \mid \exists u \in \mathcal{R} \, . \, < r_1, r_2 > (u) = (x,y)\} \ .$$

Rutten, in [Rut98], using the theory of *relators*, showed that, when $F : Set \to Set$ preserves weak pullbacks, $\mathcal{R}$ is an F-bisimulation if and only if $\mathcal{R}$ is an $\mathcal{F}$-coalgebra morphism, where $\mathcal{F} : Rel \to Rel$ is the relator extending F. In purely set-theoretical terms, Corollary 3.1 of [Rut98] can be spelled out as follows:

Proposition 4.1 *Let $F : \mathcal{C}^{\mathcal{S}} \to \mathcal{C}^{\mathcal{S}}$, let $(\mathcal{R}, r_1, r_2)$ be a span on X, Y, and let (X, α), (Y, β) be F-coalgebras. Then*
i) $(\mathcal{R}, r_1, r_2)$ is an F-bisimulation on (X, α) and $(Y, \beta) \Longleftrightarrow$

$$\mathcal{R}^{\mathcal{S}}_{r_1 r_2} \subseteq \Phi^F_{(X,\alpha),(Y,\beta)}(\mathcal{R}^{\mathcal{S}}_{r_1 r_2}) \ ,$$

where $\Phi^F_{(X,\alpha),(Y,\beta)} : \mathcal{P}(X \times Y) \to \mathcal{P}(X \times Y)$ is defined by

$$\Phi^F_{(X,\alpha),(Y,\beta)}(\mathcal{R}) = \{(x,y) \mid \exists u \in F(\mathcal{R}). \ (F(\pi^{\mathcal{R}}_1)(z) = \alpha(x) \ \wedge \ F(\pi^{\mathcal{R}}_2)(z) = \beta(y))\} \ .$$

ii) Moreover, if F preserves weak pullbacks, then $\Phi^F_{(X,\alpha),(Y,\beta)}$ is monotone.

Now we show that, for a special class $\mathcal{F}un$ of functors, the translation from the categorical coinduction to the set-theoretical coinduction of Proposition 4.1 is *compositional* on the structure of $F \in \mathcal{F}un$. I.e., given $F \in \mathcal{F}un$ and F-coalgebras $(X, \alpha), (Y, \beta)$, there is a natural way of inducing coalgebras of the "component" functors of F, in such a way that the operator $\Phi^F_{(X,\alpha),(Y,\beta)}$ is obtained by "composing" the operators induced by the component functors.

We start by specifying a class $\mathcal{F}un$ of *covariant* functors. The functors which we consider involve the constructors which are normally used for defining final semantics, i.e. identity, constants, cartesian and infinite cartesian products, disjoint sum, powerset constructors.

Definition 4.2 Let $\mathcal{F}un$ be the class of functors $F : \mathcal{C}^{\mathcal{S}} \rightarrow \mathcal{C}^{\mathcal{S}}$ defined as follows:

$$F(\cdot) ::= Id(\cdot) \mid F_C(\cdot) \mid F(\cdot) \times F(\cdot) \mid F(\cdot) + F(\cdot) \mid \mathcal{P}(F(\cdot)) \mid F_C(\cdot) \rightarrow F(\cdot) ,$$

where

- $Id(\cdot)$ is the identity functor, defined by
$$\begin{cases} Id(A) = A & \text{for } A \text{ object in } \mathcal{C}^{\mathcal{S}} \\ Id(f) = f & \text{for } f \text{ arrow in } \mathcal{C}^{\mathcal{S}} \ , \end{cases}$$
- $F_C(\cdot)$, for C object in $\mathcal{C}^{\mathcal{S}}$, is the constant functor, defined by
$$\begin{cases} F_C(A) = C & \text{for } A \text{ object in } \mathcal{C}^{\mathcal{S}} \\ F_C(f) = id_C & \text{for } f \text{ arrow in } \mathcal{C}^{\mathcal{S}} \ , \end{cases}$$

Compositionality of the translation given in Proposition 4.1 can be expressed as follows:

Theorem 4.3 *Let $F : \mathcal{C}^{\mathcal{S}} \rightarrow \mathcal{C}^{\mathcal{S}}$ be a functor in $\mathcal{F}un$, and let (X, α), (Y, β) be F-coalgebras.*

($\times$) *If $F(\cdot) = F_1(\cdot) \times F_2(\cdot)$, then, for all $\mathcal{R}$,*

$$\Phi^F_{(X,\alpha),(Y,\beta)}(\mathcal{R}) = \Phi^{F_1}_{(X,\pi_1\circ\alpha),(Y,\pi_1\circ\beta)}(\mathcal{R}) \ \cap \ \Phi^{F_2}_{(X,\pi_2\circ\alpha),(Y,\pi_2\circ\beta)}(\mathcal{R}) \ .$$

($+$) *If $F(\cdot) = F_1(\cdot) + F_2(\cdot)$, then, for all $\mathcal{R}$,*

$$\Phi^F_{(X,\alpha),(Y,\beta)}(\mathcal{R}) = \Phi^{F_1}_{(X_1,\alpha_1),(Y_1,\beta_1)}(\mathcal{R} \cap (X_1 \times Y_1)) \cup \Phi^{F_2}_{(X_2,\alpha_2),(Y_2,\beta_2)}(\mathcal{R} \cap (X_2 \times Y_2)) \ ,$$

where the F_i-coalgebras (X_i, α_i), (Y_i, β_i) are defined as follows. First of all, notice that the F-coalgebras (X, α) and (Y, β) are of the shape: $\alpha = [\alpha'_1, \alpha'_2] : X_1 + X_2 \rightarrow F_1(X) + F_2(X)$, with $\alpha'_i : X_i \rightarrow F_i(X)$, and $\beta = [\beta'_1, \beta'_2] : Y_1 + Y_2 \rightarrow F_1(Y) + F_2(Y)$, with $\beta'_i : Y_i \rightarrow F_i(Y)$. Then $\alpha_i : X \rightarrow F_i(X)$ is any F_i-coalgebra such that $(\alpha_i)_{|X_i} = \alpha'_i$, and $\beta_i : Y \rightarrow F_i(Y)$ is any F_i-coalgebra such that $(\beta_i)_{|Y_i} = \beta'_i$.

($\mathcal{P}$) *If $F(\cdot) = \mathcal{P}(F_1(\cdot))$, then, for all $\mathcal{R}$,*

$$\Phi^F_{(X,\alpha),(Y,\beta)}(\mathcal{R}) = \bigcap_i \bigcup_j \Phi^{F_1}_{(X,\alpha_i),(Y,\beta_j)}(\mathcal{R}) \ \cap \ \bigcap_j \bigcup_i \Phi^{F_1}_{(X,\alpha_i),(Y,\beta_j)}(\mathcal{R}) \ ,$$

where the F_1-coalgebras (X, α_i), (Y, β_j) are induced by the F-coalgebras (X, α) and (Y, β) as follows. $\alpha_i : X \rightarrow F_1(X)$ is such that $\forall x \in X.\ \alpha_i(x) \in \alpha(x)$, and $\beta_j : Y \rightarrow F_1(Y)$ is such that $\forall y \in Y.\ \beta_j(y) \in \beta(y)$.

($\rightarrow$) *If $F(\cdot) = C \rightarrow F_1(\cdot)$, then, for all $\mathcal{R}$,*

$$\Phi^F_{(X,\alpha),(Y,\beta)}(\mathcal{R}) = \bigcap_{c \in C} \Phi^{F_1}_{(X,\alpha_c),(Y,\beta_c)}(\mathcal{R}) \ ,$$

where $\alpha_c : X \rightarrow F_1(X)$, is $\lambda x.\alpha(x)(c)$, and $\beta_c : Y \rightarrow F_1(Y)$, is $\lambda y.\beta(y)(c)$.

We address now the problem of formalizing the correspondence between Coalgebraic Coinduction "up-to-T" and set-theoretic Coinduction "up-to-T".

Proposition 4.4 *Let $F : \mathcal{C}^{\mathcal{S}} \to \mathcal{C}^{\mathcal{S}}$ be a functor, let $< T, \eta, \mu >$ be a monad on $\mathcal{C}^{\mathcal{S}}$, let $(\mathcal{R}, r_1, r_2)$ be a span on objects TX, TY, and let (TX, α), (TY, β) be F-coalgebras. Then*
i) $(\mathcal{R}, r_1, r_2)$ is an F-bisimulation on (TX, α) and $(TY, \beta) \iff$

$$\mathcal{R}^{\mathcal{S}}_{r_1 r_2} \subseteq \Phi^F_{(TX,\alpha),(TY,\beta)}(\Phi^T_{X,Y}(\mathcal{R}^{\mathcal{S}}_{r_1 r_2})) \ ,$$

where $\Phi^T_{X,Y} : \mathcal{P}(TX \times TY) \to \mathcal{P}(TX \times TY)$ is defined by

$$\Phi^T_{X,Y}(\mathcal{R}) = (TR)^{\mathcal{S}}_{r_1^\sharp r_2^\sharp} \ ,$$

and $\Phi^F_{(TX,\alpha),(TY,\beta)}$ is the operator defined in Proposition 4.1.
ii) Moreover, if F preserves weak pullbacks, T is monotone over relations[3]*, and T distributes over F w.r.t. η via a natural transformation λ which extends to a distributivity law on relations* [4]*, then the set-theoretic coinduction principle "up-to-Φ^T" is sound and complete.*

It would be interesting to find suitable counterparts to Theorem 4.3 for Coinduction "up-to-T".

Proposition 4.4 above applies immediately to the following example. Consider the monad for corecursion $T^+_{F^{S_N}}$, where F^{S_N} is the functor defined at the end of Subsection 3.1. Recall that a final F^{S_N}-coalgebra is the set S_N of all infinite streams on natural numbers. Let $(T^+_{F^{S_N}}(S_N), [\alpha_1, F^{S_N}(in_2)])$ be an F^{S_N}-coalgebra. Then, by Proposition 4.4, using strong extensionality of final coalgebras, one can easily check that a relation $(\mathcal{R}, r_1, r_2)$ is an F^{S_N}-bisimulation "up-to-$T^+_{F^{S_N}}$" on $(T^+_{F^{S_N}}(S_N), [\alpha_1, F^{S_N}(in_2)])$ if and only if $\mathcal{R}^{\mathcal{S}}_{r_1 r_2}$ is a Φ^+-bisimulation "up-to-$\cup\approx$" for the operator $\Phi^+ : \mathcal{P}(T^+_{F^{S_N}}(S_N) \times T^+_{F^{S_N}}(S_N)) \to \mathcal{P}(T^+_{F^{S_N}}(S_N) \times T^+_{F^{S_N}}(S_N))$ defined by

$$\Phi^+(\mathcal{R}) = \{(in_1(s), in_1(s')) \mid \pi_1(\alpha_1(s)) = \pi_1(\alpha_1(s')) \wedge \pi_2(\alpha_1(s)) \ \mathcal{R} \ \pi_2(\alpha_1(s'))\} \cup \{(in_2(s), in_2(s')) \mid s = s'\}.$$

The correspondence between Coalgebraic Coinduction "up-to-T^+_F" and set-theoretic Coinduction "up-to-$\cup\approx$" is quite intrinsic. Categorically, applying Theorem 3.8 to the corecursion monad T^+_F, we get a proof principle for reasoning on equivalences of corecursive morphisms. While, in a purely set-theoretic framework, one can give a coinductive characterization of the equivalence induced by corecursive functions, using the Coinduction "up-to-$\cup\approx$". For simplicity, we work out only the special case of the functor F^{S_N} introduced in Subsection 3.1.

Notice that in the set-theoretic case we derive a *complete* characterization. This immediately implies, by Proposition 4.4, also the completeness of the Coalgebraic Coinduction "up-to-$T^+_{F^{S^N}}$".

Theorem 4.5 (Set-theoretic Coinduction for Corecursive Functions) *Let $h : X \to S_N$ be the corecursive morphisms induced by the F^{S_N}-coalgebra*

[3] I.e.: $(\mathcal{R}, r_1, r_2) \leq (\mathcal{R}', r'_1, r'_2) \implies (T(\mathcal{R}), r_1^\sharp, r_2^\sharp) \leq (T(\mathcal{R}'), r_1'^\sharp, r_2'^\sharp)$.
[4] I.e.: for all $(\mathcal{R}, r_1, r_2)$, for all i, $\lambda; Fr_i^\sharp = (Fr_i)^\sharp$.

$(T^+_{F^{S_N}}X, [\alpha_1, F(in_2)])$, *i.e.* $h = f \circ in_1$, *where* $f : T^+_{F^{S_N}}X \to S_N$ *is the coiterative morphism. Then the following principle is sound and complete:*

$$\frac{x\,\mathcal{R}\,y \quad in_1^+(\mathcal{R}) \subseteq \Phi^+(in_1^+(\mathcal{R}) \cup \approx_{\Phi_f})}{x \sim_h y} \ ,$$

where $in_1^+(\mathcal{R})$ *denotes the set-theoretic image of* $\mathcal{R}$ *by* in_1, *and* Φ_f *is the monotone operator* $\Phi^{F^{S_N}}_{(T^+_{F^{S_N}}X,[\alpha_1,F(in_2)]),(T^+_{F^{S_N}}X,[\alpha_1,F(in_2)])}$ *given in Proposition 4.1.*

Proof. First of all notice that, using Theorem 2.8 of Section 2,
$\sim_h = in_1^{-1}(\sim_f) = \bigcup\{in_1^{-1}(\mathcal{R}) \mid \mathcal{R} \subseteq (X+S_N)^2 \ \wedge \ \mathcal{R} \subseteq \Phi_f(\mathcal{R})\}$.
We prove that
$\bigcup\{in_1^{-1}(\mathcal{R}) \mid \mathcal{R} \subseteq (X+S_N)^2 \ \wedge \ \mathcal{R} \subseteq \Phi_f(\mathcal{R})\} \ =$
$\bigcup\{\mathcal{R} \mid \mathcal{R} \subseteq X \times X \ \wedge \ in_1^+(\mathcal{R}) \subseteq \Phi_f(in_1^+(\mathcal{R}) \cup \approx_{\Phi_f})\}$.
($\subseteq$) Let $\mathcal{R} \subseteq \Phi_f(\mathcal{R})$. Then $in_1^+(in_1^{-1}(\mathcal{R})) \subseteq \mathcal{R} \subseteq \Phi_f(\mathcal{R}) \subseteq \Phi_f(\approx_{\Phi_f}) \subseteq \Phi_f(in_1^+(in_1^{-1}(\mathcal{R})) \cup \approx_{\Phi_f})$.
($\supseteq$) Let $\mathcal{R} \subseteq X \times X$ be such that $in_1^+(\mathcal{R}) \subseteq \Phi_f(in_1^+(\mathcal{R}) \cup \approx_{\Phi_f})$. Then, by the Principle "up-to-$\cup \approx_{\Phi_f}$", $in_1^+(\mathcal{R}) \subseteq \approx_{\Phi_f}$. Hence $\mathcal{R} \subseteq in_1^{-1}(\approx_{\Phi_f})$.

□

Corollary 4.6 *The Coalgebraic Coinduction "up-to-$T^+_{F^{S_N}}$" is complete.*

4.2 From Sets to Coalgebras?

In this subsection we list some critical situations where set-theoretic coinduction does not seem to be directly amenable to categorical terms. These examples possibly indicate some limitations of the coalgebraic approach.

4.2.1 Non-uniform Bisimulations.

Consider, for the sake of example, the following notion of bisimulation on CCS-like processes, obtained by sligthly modifying the definition of strong bisimulation:

$$p\,\mathcal{R}\,q \implies \exists a\ (\forall p_1(p \xrightarrow{a} p_1 \Rightarrow \exists q_1.\ q \xrightarrow{a} q_1 \wedge p_1\,\mathcal{R}\,q_1) \wedge \forall q_1(q \xrightarrow{a} q_1 \Rightarrow \exists p_1.\ p \xrightarrow{a} p_1 \wedge p_1\,\mathcal{R}\,q_1)).$$

It is not at all clear how to describe this notion of bisimulation coalgebraically. The problem is due to the presence of an $\exists$ quantifier, in place of a $\forall$. Intuitively, $\forall$ quantifiers guarantee a uniform property to hold over all objects. With $\exists$ quantifiers we loose this uniformity. But this uniformity seems necessary in providing a coalgebraic description. More in general, the problem with $\exists$ quantifiers can be rephrased as follows.

Let $\Phi_1, \ldots, \Phi_n : \mathcal{P}(X \times X) \to \mathcal{P}(X \times X)$ be monotone operators. Each of these operators generates a coinduction principle in the line of the Coinduction Principle 1.1. If we define $\Phi : \mathcal{P}(X \times X) \to \mathcal{P}(X \times X)$ by

$$\Phi(\mathcal{R}) = \bigcup_{1 \leq i \leq n} \Phi_i(\mathcal{R}) \ ,$$

we get a monotone operator which gives rise to a corresponding coinduction principle. Assuming we have coalgebraic descriptions of the set-theoretic coinduction principles induced by the operators Φ_i, it is not at all clear how to derive a coalgebraic description of the coinduction principle induced by Φ. A similar example occurs in [HL98] for the case of a generalized *applicative* coinduction principle for λ-calculus.

Other examples of bisimulations which have a problematic coalgebraic description are those where "side-conditions" depending on the structure of the objects to be related appear. Both *early* and *late* bisimulations in Milner's π-calculus ([MPW92]), are of this form. Also in this case, like in the previous example with quantifiers, we lack a uniform description. Luckily, in the π-calculus case, it is still possible to get rid of the local side-conditions in the definitions of bisimulations (see [HLMP98]), thereby making possible a coalgebraic description. This latter situation seems related to the difficulty of obtaining a "generalized minimal automata".

4.2.2 Coinduction "up-to".

In this paper, we have discussed at length coalgebraic counterparts to set-theoretic Coinduction "up-to-T". Not all operators T, however, seem to be easily treated coalgebraically. For example, consider the set-theoretic operator T defined by $\approx \circ \_ \circ \_ \circ \approx$, which captures Milner's bisimulation "up-to" principle. The theory of functors and relators could shed some light on this problem.

4.2.3 Binary Operators.

Let $\Phi : \mathcal{P}(X \times Y) \times \mathcal{P}(X \times Y) \to \mathcal{P}(X \times Y) \times \mathcal{P}(X \times Y)$ be a monotone operator over the complete lattice of relations ordered by $(\leq_1, \leq_2)$, where $\leq_i \in \{\subseteq, \supseteq\}$. A special case is that of the *mixed induction-coinduction* operators. These are monotone operators on the complete lattice $\mathcal{P}(X \times Y) \times \mathcal{P}(X \times Y)$ ordered by $(\subseteq, \supseteq)$, which have a unique fixed point $(\approx, \approx)$. These kind of operators induce an *induction-coinduction principle* of the following form (see e.g. [HL95,Pit96]):

$$\frac{\mathcal{R}^- \subseteq \pi_1(\Phi(\mathcal{R}^-, \mathcal{R}^+)) \quad \mathcal{R}^+ \supseteq \pi_2(\Phi(\mathcal{R}^-, \mathcal{R}^+))}{\mathcal{R}^- \subseteq \approx \subseteq \mathcal{R}^+} \ .$$

It is not at all clear how to describe coalgebraically coinduction principles induced by these binary operators. In particular, induction-coinduction principles seem to require an extension of the coalgebraic approach to contravariant (mixed) functors. This could lead us to Freyd's algebraically compact categories. However, also the "purely covariant" case, in which both components of the binary operator are ordered by $\subseteq$, seems to be problematic to deal with coalgebraically in full generality. Similar problems, of course, arise for n-ary operators.

References

[Acz88] P.Aczel. *Non-well-founded sets*, CSLI Lecture Notes **14**, Stanford 1988.

[Acz93] P.Aczel. *Final Universes of Processes*, *MFPS'93* Conference Proceedings, Brookes et al. eds., Springer LNCS **802**, New Orleans 1993.

[AM89] P.Aczel, N.Mendler. A Final Coalgebra Theorem, in *Category Theory and Computer Science*, D.H.Pitt et al. eds., Springer LNCS **389**, 1989, 357–365.

[BV96] J.de Bakker, E.de Vink. *Control Flow Semantics*, MIT Press, 1996.

[BM96] J.Barwise, L.Moss. *Vicious circles: On the mathematics of non-wellfounded phenomena*, CSLI Publications, Stanford, 1996.

[Coq94] T.Coquand. Infinite Objects in Type Theory, in *TYPES-93*, SLNCS **806**, 1994, 62–78.

[Fio96] M.Fiore. A Coinduction Principle for Recursive Data Types Based on Bisimulation, *Information and Computation* **127**, 1996, 186–198.

[FH83] M.Forti, F.Honsell. Set theory with free construction principles, *Ann. Scuola Norm. Sup. Pisa*, Cl. Sci. (4) **10**, 1983, 493– 522.

[FHL94] M.Forti, F.Honsell, M.Lenisa. Processes and Hyperuniverses, *MFCS'94* Conference Proceedings, I.Prívara et al. eds., Springer LNCS **841**, Berlin 1994, 352–361.

[FS90] P.Freyd, A.Scedrov. *Categories, Allegories*, North-Holland, Amsterdam, 1990.

[Geu92] H.Geuvers. Inductive and Coinductive types with Iteration and Recursion, Draft, 1992.

[Gim94] E.Giménez. Codifying guarded definitions with recursive schemes, *Workshop on Types for Proofs and Programs*, P.Dybjer et al. eds., Springer LNCS **996**, 1994, 39–59.

[HJ98] C.Hermida, B.Jacobs. Structural induction and coinduction in a fibrational setting, to appear in *Inf. and Comp.*.

[HL95] F.Honsell, M.Lenisa. Final Semantics for Untyped Lambda Calculus, *TLCA'95* Conf. Proc., M.Dezani, G.Plotkin eds., Springer LNCS **902**, Berlin 1995, 249–265.

[HL98] F.Honsell, M.Lenisa. Coinductive Characterizations on Applicative Structures, to appear in *MSCS*.

[HLMP98] F.Honsell, M.Lenisa, U.Montanari, M.Pistore. Final Semantics for the π-calculus, *PROCOMET'98* Conf. Proc., D. Gries et al. eds, Chapman & Hall, 1998.

[Jac96] B.Jacobs. Objects and Classes, co-algebraically, *Object-Orientation with Parallelism and Book Persistence*, B.Freitag et al. eds., Kluwer Academic Publishers, 1996.

[Jac97] B.Jacobs. Behaviour-refinement of object-oriented specifications with coinductive correctness proofs, *TAPSOFT'97*, M.Bidoit, et. al. eds., Springer LNCS **1214**, 1997, 787–802.

[Len96] M.Lenisa. Final Semantics for a Higher Order Concurrent Language, *CAAP'96*, H.Kirchner et. al. eds., Springer LNCS **1059**, 1996, 102–118.

[Len98] M.Lenisa. Themes in Final Semantics, Ph.D. Thesis **TD-6/98**, Dipartimento di Informatica, Università di Pisa, March 1998.

[Len99] M.Lenisa. A Complete Coinductive Logical System for Bisimulation Equivalence on Circular Objects, to appear in *FoSSaCS'99* Conf. Proc..

[Mil83] R.Milner. Calculi for synchrony and asynchrony, *TCS* **25**, 1983, 267–310.

[MPW92] R.Milner, J.Parrow, D.Walker. A calculus of mobile processes, *Inf. and Comp.* **100**, 1992, 1–77.

[Mos?] L.Moss. Coalgebraic Logic, to appear in the *Annals of Pure and Applied Logic.*

[Par81] D.M.R.Park. Concurrency and automata on infinite sequences, *MFPS'93*, P.Deussen ed., Springer LNCS **104**, 1981, 167–183.

[Pit96] A.M.Pitts. Relational Properties of Domains, *Inf. and Comp.* **127**, 1996, 66–90.

[Plo85] G.Plotkin. Types and Partial Functions, Post-Graduate Lecture Notes, Department of Computer Science, University of Edinburgh, 1985.

[Rut96] J.J.M.M.Rutten. Universal coalgebra: a theory of systems, **CS-R9652**, CWI, 1996.

[Rut98] J.J.M.M.Rutten. Relators and Metric Bisimulations, *CMCS'98* Conf. Proc., ENTCS **11**, 1998.

[RT93] J.J.M.M.Rutten, D.Turi. On the Foundations of Final Semantics: Non-Standard Sets, Metric Spaces, Partial Orders, *REX* Conference Proceedings, J.de Bakker et al. eds., Springer LNCS **666**, 1993, 477–530.

[RT94] J.J.M.M.Rutten, D.Turi. Initial algebra and final coalgebra semantics for concurrency: A Decade of Concurrency - Reflections and Perspectives, *REX* Conference Proceedings, J.de Bakker et al. eds., Springer LNCS **803**, 1994, 530–582.

[RV97] J.J.M.M.Rutten, E.de Vink, Bisimulation for probabilistic transition systems: a coalgebraic approach (extended abstract), *ICALP'97* Conference Proceedings, P.Degano et al, eds., Springer LNCS **1256**, 1997, 460–470.

[San95] D.Sangiorgi. On the proof method for bisimulation, *MFCS'95* Conference Proceedings, J.Wiedermann et al. eds., Springer LNCS **969**, 1995, 479–488.

[SP82] M.Smyth, G.Plotkin. The category-theoretic solution of recursive domain equations, *SIAM J. of Computing* **11**(4), 1982, 761–783.

[Tur96] D.Turi. *Functorial Operational Semantics and its Denotational Dual*, PhD thesis, 1996.

[TP97] D.Turi, G.Plotkin. Towards a mathematical operational semantics, Proc 12^{th} *LICS* Conf., IEEE, Computer Science Press, 1997, 280–291.