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Sound modes in holographic hydrodynamics for charged AdS black hole

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Abstract

In the previous paper we studied the transport coefficients of quark–gluon plasma in finite temperature and finite density in vector and tensor modes. In this paper, we extend it to the scalar modes. We work out the decoupling problem and hydrodynamic analysis for the sound mode in charged AdS black hole and calculate the sound velocity, the charge susceptibility and the electrical conductivity. We find that Einstein relation among the conductivity, the diffusion constant and the susceptibility holds exactly.

1. Introduction

The discovery of low viscosity in the theory with gravity dual [1] and its possible relation to the RHIC (Relativistic Heavy Ion Collider) experiment induced a great deal of efforts to establish the relevant calculational scheme that may be provided by AdS/CFT correspondence [2–4]. An attempt has been made to map the entire process of RHIC experiment in terms of the gravity dual [5]. The way to include a chemical potential in the theory was figured out in the context of probe brane embedding [6–13]. Phases of these theories were discussed and new phases were reported where instability due to the strong attraction is a feature [8–10].

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In spite of the difference between QCD and $\mathcal{N}=4$ SYM, it is expected that some of the properties are shared by the two theories based on the universality of low energy physics. In this respect, the hydrodynamic limit is particularly interesting since such limit can be shared by many theories. The calculation scheme for transport coefficients is to use Kubo formula, which gives a relation to the low energy limit of Wightman Green functions. In AdS/CFT correspondence, one calculate the retarded Green function which is related to the Wightman function by fluctuation–dissipation theorem. Such scheme has been developed in a series of papers [14–18].

For the hydrodynamic analysis, one may need to have master equations for the decoupled modes in vector and scalar at hands. Although the analysis for the decoupling problem were analyzed in [19], it was based on $SO(3)$ decomposition while more useful work for hydrodynamics should be based on $SO(2)$ decomposition, where longitudinal direction of the spatial direction is distinguished. For this purpose, Kovtun and Starinets worked out the decoupling problem based on $SO(2)$ for the AdS black hole case [18] before doing the hydrodynamic analysis. For the charged cases, there are extra difficulties: vector modes of gravity and those of the gauge fields couple. Furthermore there are extra couplings in scalar modes which are not present in the chargeless cases.

In the previous paper [20], some of us extended this work to charged case using the Reissner–Nordström–Anti-de Sitter (RN–AdS) black hole, which corresponds to the diagonal $(1,1,1)$ R-charged STU black hole.[1] However, analysis for the scalar mode of charged case was not done due to difficulties caused by extra mixing between various scalar modes in charged AdS black hole. In this paper, we work out the decoupling problem and hydrodynamics for the sound (scalar) mode in such case. Green functions are explicitly obtained. Our results show that the behavior of the transport coefficients in RN-black hole are very different from those in the $(1,0,0)$ charged black hole: the formers are much more smoother than the latters. We find that Einstein relation among the conductivity, the diffusion constant and the susceptibility holds exactly.

This paper is organized as follows: In Section 2, we introduce RN–AdS black hole and review correlation function calculation at finite temperature in AdS/CFT correspondence. In Section 3, a formulation on the metric and the gauge perturbations in RN–AdS background is considered. We then solve linearized perturbative equations of motion in hydrodynamic regime and obtain retarded Green functions in Section 4. We also observe the transport coefficients including the speed of sound, the diffusion constant for $U(1)$ charge and the electrical conductivity in this section. Conclusions and discussions are given in the final section. Three appendices are provided. In Appendix A, we summarize the results in the vector and the tensor type perturbations in our previous work [20]. The details of calculations to solve equations of motion are given in Appendices B and C.

2. Basic setup

2.1. Minkowskian correlators in AdS/CFT correspondence

Before introducing RN–AdS black hole, we briefly summarize to obtain Minkowskian correlators in AdS/CFT correspondence. We follow the prescription proposed in [14]. We work on the

[1] In fact much works had been done for charged case by various groups [21–25]. In [21,22], thermodynamics for STU black hole [26,27] and the hydrodynamic calculations for the $(1,0,0)$ charge were performed. In [23,24], charged AdS_5 and AdS_4 black hole backgrounds were considered, respectively, and it was shown numerically that the ratio (η/s) was $1/(4\pi)$ with very good accuracy. Later, it was also proven that the ratio might be universal in more general setup [25].

five-dimensional background,

$$ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu + g_{uu}(du)^2, \tag{2.1}$$

where x^μ and u are the four-dimensional and the radial coordinates, respectively. We refer the boundary as $u = 0$ and the horizon as $u = 1$. A solution of the equation of motion may be given,

$$\phi(u, x) = \int \frac{d^4k}{(2\pi)^4}\, e^{ikx} f_k(u)\phi^0(k), \tag{2.2}$$

where $f_k(u)$ is normalized such that $f_k(0) = 1$ at the boundary. An on-shell action might be reduced to surface terms by using the equation of motion,

$$S[\phi^0] = \int \frac{d^4k}{(2\pi)^4}\phi^0(-k)\mathcal{G}(k, u)\phi^0(k)\Big|_{u=0}^{u=1}. \tag{2.3}$$

Here, the function $\mathcal{G}(k, u)$ can be written in terms of $f_{\pm k}(u)$ and $\partial_u f_{\pm k}(u)$. Accommodating Gubser–Klebanov–Polyakov/Witten relation [3,4] to Minkowski spacetime, Son and Starinets proposed the formula to get retarded Green functions,

$$G^{R}(k) = 2\mathcal{G}(k, u)|_{u=0}, \tag{2.4}$$

where the incoming boundary condition at the horizon is imposed. In general, there are several fields in the model. We write the Green function as $G_{ij}(k)$, where indices i and j distinguish corresponding fields.

In this paper, we work in RN–AdS background and consider its perturbations so that essential ingredients are perturbed metric field and $U(1)$ gauge field. Here we define the precise form of the retarded Green functions which we discuss later:

$$\begin{aligned}
G_{\mu\nu\,\rho\sigma}(k) &= -i\int d^4x\, e^{-ikx}\theta(t)\big\langle\big[T_{\mu\nu}(x), T_{\rho\sigma}(0)\big]\big\rangle,\\
G_{\mu\nu\,\rho}(k) &= -i\int d^4x\, e^{-ikx}\theta(t)\big\langle\big[T_{\mu\nu}(x), J_{\rho}(0)\big]\big\rangle,\\
G_{\mu\,\nu}(k) &= -i\int d^4x\, e^{-ikx}\theta(t)\big\langle\big[J_{\mu}(x),\ J_{\nu}(0)\big]\big\rangle,
\end{aligned} \tag{2.5}$$

where the operators $T_{\mu\nu}(x)$ and $J_\mu(x)$ are energy–momentum tensor and $U(1)$ current which couple to the metric and the gauge field, respectively.

2.2. *Reissner–Nordström–AdS background*

The charge in RN–AdS black hole is usually regarded as R-charge of SUSY [28]. We here consider an another interpretation in the following way: One can introduce quarks and mesons by considering the bulk-filling branes in AdS_5 space. The overall $U(1)$ of the flavor branes is identified as the baryon charge. The $U(1)$ charge in this model [29] minimally couples to the bulk gravity since the bulk and the world volume of brane are identified. Then, the baryon charge and the R-charge have the same description in terms of the $U(1)$ gauge field living in the AdS_5 space. A charged black hole (RN–AdS black hole) is then induced by its back reaction. Therefore the $U(1)$ charge in RN–AdS can be identical to the baryon charge. As a result, we can interpret our result as a calculation of the transport coefficients in the presence of the baryon density.

The effective action of this gauge field is given the quadratic piece of Dirac–Born–Infeld action[2]

$$S_{\mathrm{DBI}} = -\frac{1}{4e^2}\int \mathrm{d}^5x\sqrt{-g}\,\mathrm{Tr}\big(\mathcal{F}_{mn}\mathcal{F}^{mn}\big), \tag{2.6}$$

where the gauge coupling constant e is given by [29]

$$\frac{l}{e^2} = \frac{N_cN_f}{4\pi^2}, \tag{2.7}$$

with l the radius of the AdS space. We pick up an overall $U(1)$ part of this gauge field in order to consider the baryon current at the boundary. Together with the gravitation part, we arrive at the following action which is our starting point:

$$S_0 = \frac{1}{2\kappa^2}\int \mathrm{d}^5x\,\sqrt{-g}(R-2\Lambda) - \frac{1}{4e^2}\int \mathrm{d}^5x\,\sqrt{-g}\mathcal{F}_{mn}\mathcal{F}^{mn}, \tag{2.8}$$

where we denote the gravitation constant and the cosmological constant as $\kappa^2 = 8\pi G_5$ and Λ, respectively. The $U(1)$ gauge field strength is given by $\mathcal{F}_{mn}(x) = \partial_m\mathcal{A}_n(x) - \partial_n\mathcal{A}_m(x)$. The gravitation constant is related to the gauge theory quantities by

$$\frac{l^3}{\kappa^2} = \frac{N_c^2}{4\pi^2}. \tag{2.9}$$

Suppose we have baryon charge Q. This should be identified to the source of $U(1)$ charge on the brane hence on the bulk. Then we can relate it to the parameter in RN black hole solution by considering the full solution to the equation of motion,

$$R_{mn} - \frac{1}{2}g_{mn}R + g_{mn}\Lambda = \kappa^2T_{mn}, \tag{2.10}$$

where energy–momentum tensor $T_{mn}(x)$ is given by

$$T_{mn} = \frac{1}{e^2}\left(\mathcal{F}_{mk}\mathcal{F}_{nl}g^{kl} - \frac{1}{4}g_{mn}\mathcal{F}_{kl}\mathcal{F}^{kl}\right). \tag{2.11}$$

An equation of motion for the gauge field $\mathcal{A}_m(x)$ gives Maxwell equation,

$$\nabla_m\mathcal{F}^{mn} = \frac{1}{\sqrt{-g}}\partial_m\big(\sqrt{-g}g^{mk}g^{nl}(\partial_k\mathcal{A}_l - \partial_l\mathcal{A}_k)\big) = 0. \tag{2.12}$$

Here we assumed that there is no electromagnetic source outside the black hole. One can confirm that the following metric and gauge potential satisfy the equations of motion (2.10) and (2.12),

$$\mathrm{d}s^2 = \frac{r^2}{l^2}\left(-f(r)(\mathrm{d}t)^2 + \sum_{i=1}^{3}\big(\mathrm{d}x^i\big)^2\right) + \frac{l^2}{r^2f(r)}(\mathrm{d}r)^2, \tag{2.13a}$$

$$\mathcal{A}_t = -\frac{Q}{r^2} + \mu, \tag{2.13b}$$

with

$$f(r) = 1 - \frac{ml^2}{r^4} + \frac{q^2l^2}{r^6}, \qquad \Lambda = -\frac{6}{l^2},$$

[2] The indices m and n run through five-dimensional spacetime while μ and ν would be reserved for four-dimensional Minkowski spacetime. Their spatial coordinates are labeled by i and j.

if and only if q is related to the Q by

$$e^2 = \frac{2Q^2}{3q^2}\kappa^2. \tag{2.14}$$

It should be noted that a ratio of the gauge coupling constant e^2 to the gravitation constant κ^2 is

$$\frac{e^2}{\kappa^2} = \frac{N_c}{N_f} l^{-2}. \tag{2.15}$$

Since the gauge potential $\mathcal{A}_t(x)$ must vanish at the horizon, the charge Q and the chemical potential μ are related.[3] The parameters m and q are the mass and charge of AdS space, respectively. This is nothing but Reissner–Nordström–Anti-de Sitter (RN–AdS) background in which we are interested throughout this paper.

The horizons of RN–AdS black hole are located at the zero for $f(r)$,[4]

$$f(r) = 1 - \frac{ml^2}{r^4} + \frac{q^2l^2}{r^6} = \frac{1}{r^6}\left(r^2 - r_+^2\right)\left(r^2 - r_-^2\right)\left(r^2 - r_0^2\right), \tag{2.16}$$

where their explicit forms of the horizon radiuses are given by

$$r_+^2 = \left(\frac{m}{3q^2}\left(1 + 2\cos\left(\frac{\theta}{3} + \frac{4}{3}\pi\right)\right)\right)^{-1}, \tag{2.17a}$$

$$r_-^2 = \left(\frac{m}{3q^2}\left(1 + 2\cos\left(\frac{\theta}{3}\right)\right)\right)^{-1}, \tag{2.17b}$$

$$r_0^2 = \left(\frac{m}{3q^2}\left(1 + 2\cos\left(\frac{\theta}{3} + \frac{2}{3}\pi\right)\right)\right)^{-1}, \tag{2.17c}$$

with

$$\theta = \arctan\left(\frac{3\sqrt{3}q^2\sqrt{4m^3l^2 - 27q^4}}{2m^3l^2 - 27q^4}\right),$$

and satisfy a relation $r_+^2 + r_-^2 = -r_0^2$. The positions expressed by r_+ and r_- correspond to the outer and the inner horizon, respectively. It is useful to notice that the charge q can be expressed in terms of θ and m by

$$q^4 = \frac{4m^3l^2}{27}\sin^2\left(\frac{\theta}{2}\right).$$

[3] The chemical potential μ can be expressed by using gauge invariant quantity as

$$\mu = \int_{r_+}^{\infty} \mathrm{d}r\, \mathcal{F}_{rt} = \mathcal{A}_t(\infty),$$

where r_+ and ∞ represent the horizon and the boundary, respectively. This definition gives thermodynamic relations consistently.

[4] In order to define the horizon, the charge q must satisfy a relation $q^4 \leqslant 4m^3l^2/27$.

The outer horizon takes a value in

$$\sqrt{\frac{m}{3}}l \leqslant r_+^2 \leqslant \sqrt{m}l,$$

where the upper bound and the lower bound correspond to the case for $q = 0$ and the extremal case, respectively.

We shall give various thermodynamic quantities of RN–AdS black hole [28,29]. The temperature is defined from the conical singularity free condition around the horizon r_+,

$$T = \frac{r_+^2 f'(r_+)}{4\pi l^2} = \frac{r_+}{\pi l^2}\left(1 - \frac{1}{2}\frac{q^2 l^2}{r_+^6}\right) \equiv \frac{1}{2\pi b}\left(1 - \frac{a}{2}\right) \quad (> 0), \tag{2.18}$$

where we defined the parameters a and b by

$$a \equiv \frac{q^2 l^2}{r_+^6}, \qquad b \equiv \frac{l^2}{2r_+}. \tag{2.19}$$

In the limit $q \to 0$, these parameters go to

$$a \to 0, \qquad b \to \frac{l^{3/2}}{2m^{1/4}},$$

and the temperature becomes

$$T \to T_0 = \frac{m^{1/4}}{\pi l^{3/2}}.$$

The entropy density s, the energy density ϵ, the pressure p, the chemical potential μ and the density of physical charge ρ can be also computed as

$$s = \frac{2\pi r_+^3}{\kappa^2 l^3} = \frac{\pi l^3}{4b^3\kappa^2}, \tag{2.20}$$

$$\epsilon = \frac{3m}{2\kappa^2 l^3} = \frac{3l^3}{32b^4\kappa^2}(1 + a), \tag{2.21}$$

$$p = \frac{\epsilon}{3}, \tag{2.22}$$

$$\mu = \frac{Q}{r_+^2} = \frac{4b^2 Q}{l^4}, \tag{2.23}$$

$$\rho = \frac{2Q}{e^2 l^3} = \frac{l}{e^2}\frac{\mu}{2b^2}. \tag{2.24}$$

In order to obtain a well-defined boundary term from the gravitational part, we have to add the Gibbons–Hawking term into the action, which is given by

$$S_{\mathrm{GH}} = \frac{1}{\kappa^2}\int \mathrm{d}^4x\,\sqrt{-g^{(4)}}K, \tag{2.25}$$

where integration is taken on the boundary of the AdS space. The four-dimensional metric $g_{\mu\nu}^{(4)}(x)$ is the induced metric on the boundary and $K(x)$ is the extrinsic curvature. We also need to add counter terms to regularize the action [30],

$$S_{\mathrm{ct}} = \frac{1}{\kappa^2}\int \mathrm{d}^4x\,\sqrt{-g^{(4)}}\left(\frac{3}{l} - \frac{l}{4}R^{(4)}\right). \tag{2.26}$$

3. Perturbations in RN–AdS background

In RN–AdS background, we study small perturbations of the metric $g_{mn}(x)$ and the gauge field $\mathcal{A}_m(x)$,

$$g_{mn} \equiv g^{(0)}_{mn} + h_{mn},$$
$$\mathcal{A}_m \equiv A^{(0)}_m + A_m, \tag{3.1}$$

where the background metric $g^{(0)}_{mn}(x)$ and the background gauge field $A^{(0)}_m(x)$ are given in (2.13a) and (2.13b), respectively. In the metric perturbation, one can define an inverse metric as

$$g^{mn} = g^{(0)mn} - h^{mn} + h^{ml}h_l{}^n + \mathcal{O}(h^3),$$

and raise and lower indices by using the background metric $g^{(0)}_{mn}(x)$ and $g^{(0)mn}(x)$.

Let us now consider a linearized theory of the symmetric tensor field $h_{mn}(x)$ and the vector field $A_m(x)$ propagating in RN–AdS background. We shall work in the $h_{rm}(x) = 0$ and $A_r(x) = 0$ gauges and use the Fourier decomposition

$$h_{\mu\nu}(t,z,r) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \mathrm{e}^{-i\omega t + ikz} h_{\mu\nu}(k,r),$$
$$A_\mu(t,z,r) = \int \frac{\mathrm{d}^4k}{(2\pi)^4} \mathrm{e}^{-i\omega t + ikz} A_\mu(k,r),$$

where we choose the momenta which are along the z-direction. In this case, one can categorize the metric perturbations to the following three types by using the spin under the $SO(2)$ rotation in (x, y)-plane [15]:

- vector type: $h_{tx} \neq 0$, $h_{zx} \neq 0$, (others) $= 0$ (equivalently, $h_{ty} \neq 0$, $h_{zy} \neq 0$, (others) $= 0$);
- tensor type: $h_{xy} \neq 0$, (others) $= 0$ (equivalently, $h_{xx} = -h_{yy} \neq 0$, (others) $= 0$);
- scalar type: $h_{tt} \neq 0$, $h_{tz} \neq 0$, $h_{xx} = h_{yy} \neq 0$, and $h_{zz} \neq 0$, (others) $= 0$.

First two types of the perturbations were studied in [20]. We list the result in Appendix A. In this paper we consider the scalar type perturbation.

3.1. Linearized equations of motion

From explicit calculation, one can show that t and z-components of the gauge field $A_\mu(x)$ could participate in the linearized perturbative equations of motion. Thus independent variables are

$$h_{tt}(x), \qquad h_{tz}(x), \qquad h_{xx}(x) = h_{yy}(x), \qquad h_{zz}(x),$$
$$A_t(x), \qquad A_z(x).$$

In the hydrodynamic regime, it is standard to introduce new dimensionless coordinate $u = r_+^2/r^2$ which is normalized by the outer horizon. In this coordinate system, the horizon and the boundary are located at $u = 1$ and $u = 0$, respectively. We also define new field variables

$$h^t_t = g^{(0)tt} h_{tt} = -\frac{l^2 u}{r_+^2 f} h_{tt},$$

$$h_t^z = g^{(0)zz} h_{zt} = \frac{l^2 u}{r_+^2} h_{zt},$$

$$h_x^x = g^{(0)xx} h_{xx} = \frac{l^2 u}{r_+^2} h_{xx},$$

$$h_z^z = g^{(0)zz} h_{zz} = \frac{l^2 u}{r_+^2} h_{zz},$$

$$B_\mu \equiv \frac{A_\mu}{\mu} = \frac{l^4}{4Qb^2} A_\mu,$$

where μ is the chemical potential given by (2.23). Nontrivial equations in the Einstein equation (2.8) appear from (t,t), (t,u), (t,z), (u,u), (u,z), (x,x) and (z,z) components, respectively:

$$\begin{aligned} 0 = {} & h_t^{t\,\prime\prime} + \frac{(u^{-1}f)'}{u^{-1}f}\left(\frac{3}{2}h_t^{t\,\prime} + h_x^{x\,\prime} + \frac{1}{2}h_z^{z\,\prime}\right) - \frac{b^2k^2}{uf}h_t^t \\ & + \frac{2b^2}{uf^2}\left(\omega^2 h_x^x + \frac{1}{2}\omega^2 h_z^z + \omega k h_t^z\right) + 2a\frac{u}{f}h_t^t + 4a\frac{u}{f}B_t', \end{aligned} \tag{3.2a}$$

$$0 = \omega\left(2h_x^{x\,\prime} + h_z^{z\,\prime} - \frac{f'}{f}\left(h_x^x + \frac{1}{2}h_z^z\right)\right) + k\left(h_t^{z\,\prime} - \frac{f'}{f}h_t^z\right), \tag{3.2b}$$

$$0 = h_t^{z\,\prime\prime} - \frac{1}{u}h_t^{z\,\prime} + \frac{2b^2\omega k}{uf}h_x^x - 3auB_z', \tag{3.2c}$$

$$0 = h_t^{t\,\prime\prime} + 2h_x^{x\,\prime\prime} + h_z^{z\,\prime\prime} + \frac{f'}{f}\left(\frac{3}{2}h_t^{t\,\prime} + h_x^{x\,\prime} + \frac{1}{2}h_z^{z\,\prime}\right) + 2a\frac{u}{f}h_t^t + 4a\frac{u}{f}B_t', \tag{3.2d}$$

$$0 = kh_t^{t\,\prime} + 2kh_x^{x\,\prime} - \frac{\omega}{f}h_t^{z\,\prime} + \frac{kf'}{2f}h_t^t + 3a\frac{u}{f}(kB_t + \omega B_z), \tag{3.2e}$$

$$\begin{aligned} 0 = {} & h_x^{x\,\prime\prime} + \frac{(u^{-2}f)'}{u^{-2}f}h_x^{x\,\prime} - \frac{1}{2u}\left(h_t^{t\,\prime} + h_z^{z\,\prime}\right) + \frac{b^2}{uf^2}\left(\omega^2 - k^2 f\right)h_x^x \\ & - a\frac{u}{f}h_t^t - 2a\frac{u}{f}B_t', \end{aligned} \tag{3.2f}$$

$$\begin{aligned} 0 = {} & h_z^{z\,\prime\prime} + \frac{(u^{-\frac{3}{2}}f)'}{u^{-\frac{3}{2}}f}h_z^{z\,\prime} - \frac{1}{u}\left(\frac{1}{2}h_t^{t\,\prime} + h_x^{x\,\prime}\right) + \frac{b^2}{uf^2}\left(\omega^2 h_z^z + 2\omega k h_t^z - k^2 f h_t^t - 2k^2 f h_x^x\right) \\ & - a\frac{u}{f}h_t^t - 2a\frac{u}{f}B_t', \end{aligned} \tag{3.2g}$$

with

$$f(u) = (1-u)\left(1 + u - au^2\right),$$

where the prime implies the derivative with respect to u. On the other hand, in the Maxwell equation (2.12), t, u and z-components give nontrivial contributions

$$0 = B_t'' - \frac{b^2}{uf}\left(k^2 B_t + k\omega B_z\right) + \frac{1}{2}\left(h_t^{t\,\prime} - 2h_x^{x\,\prime} - h_z^{z\,\prime}\right), \tag{3.3a}$$

$$0 = \omega B_t' + kfB_z' + \frac{\omega}{2}\left(h_t^t - 2h_x^x - h_z^z\right) - kh_t^z, \tag{3.3b}$$

$$0 = B_z'' + \frac{f'}{f} B_z' + \frac{b^2}{uf^2}\left(\omega^2 B_z + \omega k B_t\right) - \frac{1}{f} h_t^{z\,\prime}. \tag{3.3c}$$

Eqs. (3.3a) and (3.3b) imply (3.3c). In the set of equations for the metric perturbation (3.2a)–(3.2g), together with (3.3a) and (3.3b), the following four independent relations are obtained:

$$\begin{aligned}
h_x^{x\,\prime} &= \frac{3(\omega^2 - k^2 f) + k^2 u f'}{2k^2(3f - uf')} h_t^{t\,\prime} + \frac{2b^2\omega^2}{f(3f - uf')} h_x^x - \frac{f'(3f - uf') - 4b^2\omega^2}{4f(3f - uf')} h_t^t \\
&\quad + \frac{\omega(f'(3f - uf') - 4b^2\omega^2)}{2k^2 f^2(3f - uf')}\left(\omega h_x^x + \frac{\omega}{2} h_z^z + k h_t^z\right) \\
&\quad + \frac{3a\omega^2 u^2}{2k^2 f(3f - uf')}\left(h_t^t + 2B_t'\right) - \frac{3au}{2kf}(kB_t + \omega B_z),
\end{aligned} \tag{3.4a}$$

$$\begin{aligned}
h_z^{z\,\prime} &= -\frac{3\omega^2 + k^2 u f'}{k^2(3f - uf')} h_t^{t\,\prime} - \frac{2b^2k^2}{3f - uf'}\left(h_t^t + 2h_x^x\right) \\
&\quad + \frac{2b^2}{f(3f - uf')}\left(\omega^2\left(2h_x^x + h_z^z\right) + 2\omega k h_t^z\right) \\
&\quad + \frac{1}{2f(3f - uf')}\left(\left(f'(3f - uf') - 4b^2\omega^2\right)h_t^t - 8b^2\omega^2 h_x^x\right) \\
&\quad - \frac{\omega(f'(3f - uf') - 4b^2\omega^2)}{k^2 f^2(3f - uf')}\left(\omega h_x^x + \frac{\omega}{2} h_z^z + k h_t^z\right) \\
&\quad - \frac{3au^2(\omega^2 + k^2 f)}{k^2 f(3f - uf')}\left(h_t^t + 2B_t'\right) + \frac{3au}{kf}(kB_t + \omega B_z),
\end{aligned} \tag{3.4b}$$

$$\begin{aligned}
h_t^{z\,\prime} &= \frac{3\omega f}{k(3f - uf')} h_t^{t\,\prime} + \frac{2b^2\omega k}{3f - uf'}\left(h_t^t + 2h_x^x\right) \\
&\quad + \frac{f'(3f - uf') - 4b^2\omega^2}{kf(3f - uf')}\left(\omega h_x^x + \frac{\omega}{2} h_z^z + k h_t^z\right) + \frac{3a\omega u^2}{k(3f - uf')}\left(h_t^t + 2B_t'\right),
\end{aligned} \tag{3.4c}$$

$$\begin{aligned}
0 &= h_t^{t\,\prime\prime} + \frac{1}{2uf(3f - uf')}\Big\{-3(f - uf')(2f - uf')h_t^{t\,\prime} \\
&\quad + 4b^2\left(-k^2 f h_t^t + \left(2\omega^2 + (f - uf')k^2\right)h_x^x + \omega^2 h_z^z + 2\omega k h_t^z\right) \\
&\quad + au^2(15f - 7uf')\left(h_t^t + 2B_t'\right)\Big\}.
\end{aligned} \tag{3.4d}$$

The equations of motion (3.2a)–(3.2g) can be derived by using the above relations. Taking the limit $q \to 0$, the relations (3.4a)–(3.4d) coincide with the result in [18].

3.2. Surface terms

Before solving the equations of motion, we shall give a surface action in oder to obtain Green functions. By using the equations of motion, bilinear parts of on-shell action (2.8) reduce to surface terms:

$$\begin{aligned}
S_0 = \frac{l^3}{32\kappa^2 b^4}\int \frac{d^4k}{(2\pi)^4}\Big\{&+\frac{f}{u} h_t^t h_t^{t\,\prime} + \frac{f}{u} h_x^x h_x^{x\,\prime} + \frac{f}{u} h_z^z h_z^{z\,\prime} - \frac{3}{u} h_t^z h_t^{z\,\prime} \\
&- \frac{f}{u} h_x^x h_t^{t\,\prime} - \frac{f}{2u} h_z^z h_t^{t\,\prime} - \frac{f}{u} h_t^t h_x^{x\,\prime} - \frac{f}{u} h_z^z h_x^{x\,\prime} - \frac{f}{2u} h_t^t h_z^{z\,\prime} - \frac{f}{u} h_x^x h_z^{z\,\prime}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{uf'-f}{4u^2}\left(h_t^t\right)^2-\frac{f}{4u^2}\left(h_z^z\right)^2+\frac{uf'+f}{u^2 f}\left(h_t^z\right)^2 \\
&-\frac{uf'-2f}{2u^2}h_t^t h_x^x-\frac{uf'-2f}{4u^2}h_t^t h_z^z+\frac{f}{u^2}h_x^x h_z^z \\
&+3a\left(B_t B_t'-f B_z B_z'+\frac{1}{2}B_t h_t^t+B_z h_t^z-B_t h_x^x-\frac{1}{2}B_t h_z^z\right)\Bigg\}\Bigg|_{u=0}.
\end{aligned}
\tag{3.5}
$$

Relevant terms of the Gibbons–Hawking term (2.25) are explicitly given by

$$
\begin{aligned}
S_{\mathrm{GH}}=\frac{l^3}{32\kappa^2 b^4}\int\frac{\mathrm{d}^4k}{(2\pi)^4}&\left\{-\frac{f}{u}h_t^t h_t^{t\prime}-\frac{f}{u}h_z^z h_z^{z\prime}+\frac{4}{u}h_t^z h_t^{z\prime}\right.\\
&+\frac{2f}{u}h_x^x h_t^{t\prime}+\frac{f}{u}h_z^z h_t^{t\prime}+\frac{2f}{u}h_t^t h_x^{x\prime}+\frac{2f}{u}h_z^z h_x^{x\prime}+\frac{f}{u}h_t^t h_z^{z\prime}+\frac{2f}{u}h_x^x h_z^{z\prime}\\
&-\frac{uf'-4f}{4u^2}\left(h_t^t\right)^2-\frac{uf'-4f}{4u^2}\left(h_z^z\right)^2-\frac{uf'+4f}{u^2 f}\left(h_t^z\right)^2\\
&\left.+\frac{uf'-4f}{u^2}h_t^t h_x^x+\frac{uf'-4f}{2u^2}h_t^t h_z^z+\frac{uf'-4f}{u^2}h_x^x h_z^z\right\}.
\end{aligned}
\tag{3.6}
$$

The counter term (2.26) also can be evaluated as

$$
\begin{aligned}
S_{\mathrm{ct}}=\frac{3l^3}{32\kappa^2 b^4}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\sqrt{f}&\left\{-\frac{1}{4u^2}\left(h_t^t\right)^2-\frac{1}{4u^2}\left(h_z^z\right)^2+\frac{1}{u^2 f}\left(h_t^z\right)^2\right.\\
&\left.+\frac{1}{u^2}h_t^t h_x^x+\frac{1}{2u^2}h_t^t h_z^z+\frac{1}{u^2}h_x^x h_z^z\right\},
\end{aligned}
\tag{3.7}
$$

up to $\mathcal{O}(\omega^2, k^2, \omega k)$.

4. Pole structures and transport coefficients from hydrodynamics

We now look for solutions of our set of equations (3.4a)–(3.4d), and (3.3a) and (3.3b). We will consider these equations of motion in low frequency limit so-called hydrodynamic regime. In this regime we could obtain the sound velocity, the diffusion constant for $U(1)$ charge and the electrical conductivity from retarded Green functions.

4.1. Master variables

By using master variables derived by Kodama and Ishibashi in [19], the following field $\Phi(u)$ is introduced:

$$
\Phi\equiv\frac{1}{4u^{3/4}(4b^2k^2-3f')}\left(\left(4b^2k^2-3f'\right)h_x^x+2f\left(2h_x^{x\prime}+h_z^{z\prime}\right)\right).
\tag{4.1}
$$

For the gauge field, the corresponding variable is given by

$$
\mathcal{A}\equiv 2a\left(-h_t^t+3h_x^x-2B_t{}'\right).
\tag{4.2}
$$

In terms of these new variables $\Phi(u)$ and $\mathcal{A}(u)$, Einstein equations (3.4a)–(3.4d) and Maxwell equations (3.3a) and (3.3b) can be combined as follows:

$$
0=\left(u^{1/2}f\Phi'\right)'-\frac{1}{16u^{3/2}f(4b^2k^2-3f')^2}\Big\{-16b^2\omega^2 u\left(4b^2k^2-3f'\right)^2
$$

$$+ f^2\big(16\big(-b^4k^4 + 108ab^2k^2u^2 + 162a^2u^4\big) + 27f'\big(8b^2k^2 + 16au^2 + 5f'\big)\big)$$
$$+ 4uf\big(4b^2k^2 - 3f'\big)\big(16b^2k^2\big(b^2k^2 + 3au^2\big) + f'\big(8b^2k^2 + 36au^2 + 9f'\big)\big)\Big\}\Phi$$
$$+ \frac{1}{8u^{1/4}(4b^2k^2 - 3f')^2}\big\{3f\big(4b^2k^2 + 3f' + 18au^2\big) + u\big(16b^4k^4 - 9(f')^2\big)\big\}\mathcal{A}, \tag{4.3a}$$

$$0 = (uf\mathcal{A}')' + \frac{1}{f(4b^2k^2 - 3f')}\big\{b^2\big(4b^2k^2 - 3f'\big)\big(\omega^2 - k^2f\big) - 18auf^2\big\}\mathcal{A} - 48au^{3/4}f\Phi' + \frac{4a}{u^{1/4}(4b^2k^2 - 3f')}\big\{32b^4k^4u + 12f\big(b^2k^2 + 9au^2\big) + 3f'\big(-8b^2k^2u + 9f\big)\big\}\Phi. \tag{4.3b}$$

Next we would like to try to obtain decoupled equations from Eqs. (4.3a) and (4.3b). This will be done by introducing the following linear combinations of the variables:

$$\Phi_\pm \equiv \alpha_\pm\Phi + \beta\mathcal{A}, \tag{4.4}$$

where the coefficients $\alpha_\pm$ and β are

$$\alpha_\pm = C_\pm - 3au,$$
$$\beta = \frac{u^{1/4}}{8},$$

with the constants

$$C_\pm = (1+a) \pm \sqrt{(1+a)^2 + 4ab^2k^2}.$$

As a result, we can obtain second order ordinary differential equations in term of these new variables,

$$0 = \Phi_\pm'' + \frac{(u^{1/2}f)'}{u^{1/2}f}\Phi_\pm' + V_\pm\Phi_\pm, \tag{4.5}$$

where potentials $V_\pm(u)$ are given by

$$V_\pm = \frac{1}{16u^2f^2(4b^2k^2 - 3f')^2}\Big\{16b^2\omega^2u\big(4b^2k^2 - 3f'\big)^2 - 4uf\big(4b^2k^2 - 3f'\big)\big(16b^2k^2\big(b^2k^2 - C_\pm u + 3au^2\big) - 4f'\big(2b^2k^2 + 3C_\pm u - 9au^2\big) - 3(f')^2\big) + f^2\big\{16\big(b^4k^4 + 12C_\pm b^2k^2u - 108ab^2k^2u^2 + 54C_\pm au^3 - 162a^2u^4\big) - 24f'\big(b^2k^2 - 6C_\pm u - 18au^2\big) + 9(f')^2\big\}\Big\}. \tag{4.6}$$

Considering the perturbative expansion with respect to small ω and k, it might be convenient to introduce new variables $\tilde{\Phi}_\pm(u)$,

$$\Phi_\pm = H_\pm\tilde{\Phi}_\pm, \tag{4.7}$$

where the factors $F_\pm(u)$ are

$$H_\pm = \begin{cases} u^{-3/4} & (\text{for } \Phi_+), \\ \frac{u^{1/4}}{(1+a)-\frac{3}{2}au} & (\text{for } \Phi_-), \end{cases}$$

so that the second order differential equations (4.5) are reduced to be much simpler forms to solve,

$$0 = \tilde{\Phi}''_\pm + \frac{(H_\pm^2 u^{1/2} f)'}{H_\pm^2 u^{1/2} f}\tilde{\Phi}' + \tilde{V}_\pm \tilde{\Phi}_\pm, \tag{4.8}$$

where the potential $\tilde{V}_\pm(u,\omega,k)$ is newly defined from the original potential $V_\pm(u,\omega,k)$,

$$\tilde{V}_\pm(u,\omega,k) \equiv V_\pm(u,\omega,k) - V_\pm(u,0,0). \tag{4.9}$$

4.2. Perturbative solutions

Let us proceed to solve the differential equations. First we consider the equation for $\tilde{\Phi}_+(u)$. Following the usual way to solve differential equations, we impose a solution as $\tilde{\Phi}_+(u) = (1-u)^\nu F_+(u)$, where $F_+(u)$ is a regular function at the horizon $u = 1$. Plugging this form into the equation of motion, one can fix the parameter ν as $\nu = \pm i\omega/(4\pi T)$ where T is the temperature defined by Eq. (2.18). We here choose

$$\nu = -i\frac{\omega}{4\pi T} \tag{4.10}$$

as the incoming wave condition. We are now in the position to solve the equation of motion in the hydrodynamic regime. We start by introducing the following series expansion with respect to small ω and k:

$$F_+(u) = F_{+0}(u) + \omega F_{+1}(u) + k^2 G_{+1}(u) + \omega^2 F_{+2}(u) + \mathcal{O}\big(\omega^3,\ \omega k^2\big), \tag{4.11}$$

where $F_{+0}(u)$, $F_{+1}(u)$ and $G_{+1}(u)$ are determined by imposing suitable boundary conditions. The solution can be obtained recursively.[5] The result is as follows:

$$F_{+0}(u) = C \quad (\text{const.}), \tag{4.12a}$$

$$F_{+1}(u) = \frac{iCb}{2(2-a)}\left\{\log\big(1+u-au^2\big) - \frac{6K_1(u)}{\sqrt{1+4a}}\right\}, \tag{4.12b}$$

$$G_{+1}(u) = \frac{2}{3}Cb^2\left\{\frac{K_1(u)}{\sqrt{1+4a}} - \frac{1}{(1+a)u}\right\}, \tag{4.12c}$$

$$\begin{aligned} F_{+2}(u) = \int^u \mathrm{d}u \frac{Cb^2}{(1-u)(1+u-au^2)} \\ \times \Bigg\{1 - u + \frac{(1-u)(1+au)\log(1+u-au^2)}{2(2-a)^2} - \frac{3(1-u)(1+au)K_2(0)}{2(2-a)^2\sqrt{1+4a}} \\ - \frac{(1+a)K_2(1)u}{\sqrt{1+4a}} + \frac{(3+(5+3a-6a^2+2a^3)u-3au^2)K_2(u)}{2(2-a)^2\sqrt{1+4a}}\Bigg\}, \end{aligned} \tag{4.12d}$$

---

5 The derivation of the solution is given in Appendix B.

where

$$K_1(u) = \frac{1}{2}\log\bigl(1+u-au^2\bigr) - \log\left(1 - \frac{2au}{1+\sqrt{1+4a}}\right),$$
$$K_2(u) = \log\left(\frac{1+\sqrt{1+4a}-2au}{-1+\sqrt{1+4a}+2au}\right).$$

Next, we shall study the equation of motion for $\tilde{\Phi}_-(u)$. Assuming again $\tilde{\Phi}_-(u) = (1-u)^\nu F_-(u)$ where $F_-(u)$ is a regular function at $u=1$, the singularity might be extracted. We fix the constant as $\nu = -i\omega/(4\pi T)$ to use the incoming wave condition. We now impose a perturbative solution as

$$F_-(u) = F_{-0}(u) + \omega F_{-1}(u) + k^2 G_{-1}(u) + \omega^2 F_{-2}(u) + \mathcal{O}\bigl(\omega^3,\ \omega k^2\bigr), \tag{4.13}$$

and then we obtain the following result[6]:

$$F_{-0}(u) = \tilde{C} \quad \text{(const.)}, \tag{4.14a}$$

$$F_{-1}(u) = \frac{i\tilde{C}b}{2(2-a)^2}\bigl\{8(1+a)^2\log(u) - (2+a)(1+4a)\log\bigl(1+u-au^2\bigr) - 2\sqrt{1+4a}(2+5a)K_1(u)\bigr\}, \tag{4.14b}$$

$$G_{-1}(u) = \tilde{C}b^2\left\{-\frac{3a^2u}{2(1+a)^2(1+a-\frac{3}{2}au)} - \frac{2(1+a)(2+a)\log(u)}{(2-a)^2} + \frac{(1+a)(2+a)\log(1+u-au^2)}{(2-a)^2} + \frac{2(2+5a+6a^2)K_1(u)}{(2-a)^2\sqrt{1+4a}}\right\}, \tag{4.14c}$$

$$\begin{aligned}F_{-2}(u) = \int^u \mathrm{d}u\, &\frac{\tilde{C}b^2}{2(2-a)^4(1+4a)^{3/2}(1-u)u(1+u-au^2)} \\
&\times\Bigl\{8(2-a)(1+a)^2(1+4a)^{3/2}u\bigl(1+u-au^2\bigr)\log(u) \\
&- (2-a)(1+4a)^{3/2}\bigl(4(1+a)^2 + \bigl(2-3a-8a^2\bigr)u + \bigl(2+9a+13a^2\bigr)u^2 \\
&- a(2+a)(1+4a)u^3\bigr)\log\bigl(1+u-au^2\bigr) \\
&+ (1+4a)^2(2+5a)K_2(0)(1-u)\bigl(4(1+a)^2 + \bigl(2-3a-8a^2\bigr)u - (2-a)au^2\bigr) \\
&- 2(1+a)\bigl(2-2a+41a^2\bigr)K_2(1)\bigl(2(1+a)-3au\bigr)^2 \\
&- (2-a)\bigl(a(1+4a)^2(2+5a)u^3 - (2-a)\bigl(1+11a+46a^2+18a^3\bigr)u^2 \\
&- \bigl(2+9a+180a^2+224a^3+24a^4\bigr)u - 4(1+a)^2\bigl(1-10a-2a^2\bigr)\bigr)K_2(u)\Bigr\}.\end{aligned} \tag{4.14d}$$

Using these, we can get behaviors for the solutions of $\Phi_+(u)$ and $\Phi_-(u)$ around the boundary $u=0$,

$$\Phi_+ = \frac{C}{u^{3/4}}\left\{1 - \frac{2k^2b^2}{3(1+a)u} + \frac{1}{3}\bigl(k^2+3\omega^2\bigr)b^2u + \cdots\right\}, \tag{4.15a}$$

[6] The detail is given in Appendix C.

$$\Phi_- = \tilde{C} u^{1/4} \left\{ \frac{1}{1+a} + \left(\frac{4i(1+a)b\omega}{(2-a)^2} - \frac{2(2+a)b^2k^2}{(2-a)^2} + b^2 D_- \omega^2 \right) \log(u) + \cdots \right\}, \tag{4.15b}$$

where the constant D_- is

$$D_- = \frac{2}{(2-a)^4(1+4a)^{3/2}} \{ -27(2-a)a^2\sqrt{1+4a} + 4(1+4a)^{3/2}(1+a)^3 \log(2-a) + 4(2-2a+41a^2)(1+a)^2 K_1(1) \}.$$

Let us now consider the integration constants C and $\tilde{C}$. These could be estimated in terms of boundary values of the fields

$$\lim_{u\to 0} h_t^t(u) = (h_t^t)^0, \qquad \lim_{u\to 0} B_t(u) = (B_t)^0, \quad \text{etc.}$$

Using equations of motion, the integration constants C and $\tilde{C}$ are determined as

$$C = \frac{1}{2(k^2-3\omega^2)} \{ 3ak^2(B_t)^0 + 3ak\omega(B_z)^0 + (1+a)(-k^2(h_t^t)^0 + 2k\omega(h_t^z)^0 + (k^2-\omega^2)(h_x^x)^0 + \omega^2(h_z^z)^0) \}, \tag{4.16a}$$

$$\tilde{C} = \frac{(2-a)^2 abk(k(B_t)^0 + \omega(B_z)^0)}{2D_{\mathrm{p}}(\omega,k)}, \tag{4.16b}$$

where

$$D_{\mathrm{p}}(\omega,k) = 2(2+a)bk^2 - 4i(1+a)\omega - (2-a)^2 bD_-\omega^2.$$

In Eqs. (4.16a) and (4.16b), one can observe the existence of the sound and diffusion poles in the complex ω-plane.

4.3. Retarded Green functions

Let us evaluate the Minkowskian correlators. The relevant action is given by the sum of three parts (3.5), (3.6) and (3.7),

$$\begin{aligned} S &= S_0 + S_{\mathrm{GH}} + S_{\mathrm{ct}} \\ &= \frac{l^3}{32\kappa^2 b^4} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \left\{ \frac{1}{u} h_t^z h_t^{z\prime} + \frac{f}{u} h_x^x h_x^{x\prime} + \frac{f}{u} h_t^t h_x^{x\prime} + \frac{f}{2u} h_t^t h_z^{z\prime} \right. \\ &\quad + \frac{f}{u} h_x^x h_t^{t\prime} + \frac{f}{u} h_x^x h_z^{z\prime} + \frac{f}{2u} h_z^z h_t^{t\prime} + \frac{f}{u} h_z^z h_x^{x\prime} \\ &\quad + \frac{3}{4u^2}(f-\sqrt{f})(h_t^t)^2 + \frac{1}{4u^2}(3f - uf' - 3\sqrt{f})(h_z^z)^2 \\ &\quad - \frac{3}{u^2 f}(f-\sqrt{f})(h_t^z)^2 - \frac{1}{2u^2}(6f - uf' - 6\sqrt{f}) h_t^t h_x^x \\ &\quad - \frac{1}{4u^2}(6f - uf' - 6\sqrt{f}) h_t^t h_z^z - \frac{1}{u^2}(3f - uf' - 3\sqrt{f}) h_x^x h_z^z \\ &\quad \left. \left. + 3a\left(B_t B_t' - f B_z B_z' + \frac{1}{2} B_t h_t^t + B_z h_t^z - B_t h_x^x - \frac{1}{2} B_t h_z^z \right) \right\} \right|_{u=0}. \end{aligned} \tag{4.17}$$

Table 1
$G_{****}/(\frac{-(1+a)l^3}{128\kappa^2 b^4(k^2-3\omega^2)})$.

	tt	xx	zz	tz
tt	$3(5k^2-3\omega^2)$	$6(k^2+\omega^2)$	$3(k^2+\omega^2)$	$24(k^2+\omega^2)$
xx	–	$16\omega^2$	$2(k^2+\omega^2)$	$16k\omega$
zz	–	–	$-k^2+7\omega^2$	$8k\omega$
tz	–	–	–	$4(k^2+9\omega^2)$

Using equations of motion and solutions of $\Phi_+(u)$ and $\Phi_-(u)$, derivatives of h's and B's can be expressed in terms of their boundary values:

$$\frac{1}{u}h_t^{t\,\prime} \to \frac{3}{k^2-3\omega^2}\Big\{-3ak\big(k(B_t)^0+\omega(B_z)^0\big)$$
$$+(1+a)\Big(k^2\big(h_t^t\big)^0-2\omega k\big(h_t^z\big)^0-\omega^2\big(2\big(h_x^x\big)^0+\big(h_z^z\big)^0\big)\Big)\Big\}, \tag{4.18a}$$

$$\frac{1}{u}h_x^{x\,\prime} \to \frac{1}{k^2-3\omega^2}\Big\{3ak\big(k(B_t)^0+\omega(B_z)^0\big)$$
$$+(1+a)\Big(-k^2\big(h_t^t\big)^0+2k\omega\big(h_t^z\big)^0+\omega^2\big(2\big(h_x^x\big)^0+\big(h_z^z\big)^0\big)\Big)\Big\}, \tag{4.18b}$$

$$\frac{1}{u}h_z^{z\,\prime} \to \frac{1}{k^2-3\omega^2}\Big\{3ak\big(k(B_t)^0+\omega(B_z)^0\big)$$
$$+(1+a)\Big(-k^2\big(h_t^t\big)^0+2k\omega\big(h_t^z\big)^0+\omega^2\big(2\big(h_x^x\big)^0+\big(h_z^z\big)^0\big)\Big)\Big\}, \tag{4.18c}$$

$$\frac{1}{u}h_t^{z\,\prime} \to \frac{1}{k^2-3\omega^2}\Big\{-9a\omega\big(k(B_t)^0+\omega(B_z)^0\big)$$
$$-(1+a)k\Big(2k\big(h_t^z\big)^0+\omega\big(-3\big(h_t^t\big)^0+2\big(h_x^x\big)^0+\big(h_z^z\big)^0\big)\Big)\Big\}, \tag{4.18d}$$

$$B_t' \to \frac{1}{2(1+a)(k^2-3\omega^2)}\Big\{-9ak\big(k(B_t)^0+\omega(B_z)^0\big)$$
$$+(1+a)\Big(\big(2k^2+3\omega^2\big)\big(h_t^t\big)^0-6k\omega\big(h_t^z\big)^0-3\omega^2\big(2\big(h_x^x\big)^0+\big(h_z^z\big)^0\big)\Big)\Big\}$$
$$-\frac{(2-a)^2bk(k(B_t)^0+\omega(B_z)^0)}{(1+a)D_{\mathrm{p}}(\omega,k)}, \tag{4.18e}$$

$$B_z' \to \frac{1}{2(1+a)(k^2-3\omega^2)}\Big\{9a\omega\big(k(B_t)^0+\omega(B_z)^0\big)$$
$$+(1+a)\Big(-3k\omega\big(h_t^t\big)^0+2k^2\big(h_t^z\big)^0+k\omega\big(2\big(h_x^x\big)^0+\big(h_z^z\big)^0\big)\Big)\Big\}$$
$$+\frac{(2-a)^2b\omega(k(B_t)^0+\omega(B_z)^0)}{(1+a)D_{\mathrm{p}}(\omega,k)}. \tag{4.18f}$$

Substituting these expressions to the surface term (4.17), we can read off the Green functions defined by (2.5). Through the counter terms, the singularities around the boundary vanish completely. The results are listed in Tables 1, 2 and 3.

In the final expression we rescaled the gauge field $(B_\mu)^0$ to the original one $(A_\mu)^0 = \frac{4Qb^2}{l^4}(B_\mu)^0$ and raised and lowered the indices by using the flat Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$ in four-dimensional boundary theory. Taking the limit which the charge goes to

Table 2
$G_{**\,*}/(\frac{-l\mu}{4e^2b^2(k^2-3\omega^2)})$.

	tt	xx	zz	tz
t	$3k^2$	$2k^2$	k^2	$6k\omega$
z	$3k\omega$	$2k\omega$	$k\omega$	$6\omega^2$

Table 3
$G_{**}/(\frac{-l}{4e^2(1+a)b^2}(\frac{9a}{k^2-3\omega^2}+\frac{2(2-a)^2b}{D_{\mathrm{p}}(\omega,k)}))$.

	t	z
t	k^2	$k\omega$
z	–	ω^2

zero, the correlators for energy–momentum tensors coincide with the known ones in [15]. In this limit, the correlators for the energy–momentum tensor and the $U(1)$ current vanish, while ones for the $U(1)$ currents have no sound poles, as we could see in the case of vector type perturbation for $(1,0,0)$ R-charged [22] and RN–AdS black holes [20].

Remember $l^3/\kappa^2 = N_c^2/(4\pi^2)$ and it should be noticed that the factor $l/e^2 = N_c^2/(16\pi^2)$ for the R-charge since $e = 2\kappa/l$ in that case, while $l/e^2 = N_c N_f/(4\pi^2)$ for the brane charge [29].

4.4. Transport coefficients

From the obtained Green functions, we can observe the value of the speed of sound without the medium effect,

$$v_{\mathrm{s}} = \frac{1}{\sqrt{3}}. \tag{4.19}$$

One should notice that there is no effect of the charge on the sound velocity.

We can also find the diffusion pole in the current–current correlators. The diffusion constant can be read off

$$D_A = \frac{(2+a)b}{2(1+a)}. \tag{4.20}$$

It should be compared with the diffusion constant for gravitation fields D_H obtained in the previous work [20],

$$D_H = \frac{b}{2(1+a)},$$

so that the relation between them is

$$D_A = (2+a)D_H. \tag{4.21}$$

In the chargeless case, we can reproduce the result in [15].

The electrical conductivity σ of the medium could be also determined by the current–current correlators via Kubo formula,

$$\sigma \equiv -\lim_{\omega\to 0}\frac{e_{\mathrm{E}}^2}{3\omega}\,\mathrm{Im}\big(\delta^{ij}G_{ij}(\omega,k=0)\big),$$

where e_{E} is a four-dimensional gauge coupling. Together with the result for vector type perturbation [20], we can obtain

$$\sigma = \frac{l e_{\mathrm{E}}^2 (2-a)^2}{24 e^2 (1+a)^2 b} \times 3 = \left(\frac{l}{e^2}\right) \frac{\pi {e_{\mathrm{E}}}^2 (2-a)}{2(1+a)^2} T. \tag{4.22}$$

We can also access to the charge susceptibility Ξ defined by

$$\Xi \equiv \frac{1}{T} \frac{\langle Q^2 \rangle}{(\text{volume})}. \tag{4.23}$$

Using Green function $G_{tt}(k)$ which might give an expectation value of Q^2, one can obtain the following relation in thermal equilibrium,

$$\frac{\langle Q^2 \rangle}{(\text{volume})} = \int \frac{\mathrm{d}\omega}{2\pi} \big(-\operatorname{Im}\big(G_{tt}(\omega, k \to 0)\big)\big) n_{\mathrm{b}}(\omega), \tag{4.24}$$

where $n_{\mathrm{b}}(\omega) = \frac{1}{\mathrm{e}^{\omega/T} - 1}$ is Bose–Einstein distribution function [31]. From Table 3, we can see

$$-\operatorname{Im}\big(G_{tt}(\omega, k)\big) = \frac{4\pi^2 l T^2}{e^2 (1+a)(2+a)} \left(\frac{\omega D_A k^2}{\omega^2 + (D_A k^2)^2}\right).$$

It should be noted that a quantity $D_A k^2/(\omega^2 + (D_A k^2)^2)$ approaches to $2\pi\delta(\omega)$ for $k \to 0$ limit. Taking the relation (4.23) into account, we can read off the charge susceptibility Ξ as

$$\Xi = \left(\frac{l}{e^2}\right) \frac{4\pi^2}{(1+a)(2+a)} T^2. \tag{4.25}$$

We can then observe that Einstein relation

$$\sigma / \big({e_{\mathrm{E}}}^2 \Xi\big) = D_A \tag{4.26}$$

holds exactly. (See also [32].)[7]

In R-charge case, taking the charge-free limit, the electric conductivity and the charge susceptibility coincide with the results in [31].

It is interesting to express physical constants in terms of the boundary variables: the temperature and the chemical potential. In fact, it is easy to verify that

$$a = 2 - \frac{4}{1 + \sqrt{1 + 4(\tilde{\mu}/T)^2}}, \qquad b = \left(\frac{1}{\pi T}\right) \frac{1}{1 + \sqrt{1 + 4(\tilde{\mu}/T)^2}}, \tag{4.27}$$

where $\tilde{\mu} \equiv \frac{\kappa}{\sqrt{3}\pi e l} \mu$. Notice that for the R-charge, $\tilde{\mu} = \frac{1}{2\sqrt{3}\pi} \mu$, while for the brane charge, $\tilde{\mu} = \sqrt{\frac{N_f}{3 N_c \pi^2}} \mu$. The behaviors of the diffusion constants D_A and D_H, the electrical conductivity σ and the charge susceptibility Ξ are drawn as functions of the $\tilde{\mu}/T$ in Figs. 1, 2, 3 and 4, respectively. Notice that for the fixed temperature, all of them are decreasing functions of the chemical potential. One should notice that there is no upper bound of μ/T for any of these quantities unlike $(1, 0, 0)$ charged black hole studied in [22].

It is particularly interesting to observe that the charge susceptibility modulo T^2 factor, which is an indicator of the degree of freedom, shows rapid change between low and high temperature

7 It is interesting to notice that $(\partial\rho/\partial\mu)_T$ gives a different value for Ξ given in Eq. (4.25). The authors thank J. Mas and J. Shock for pointing this out.

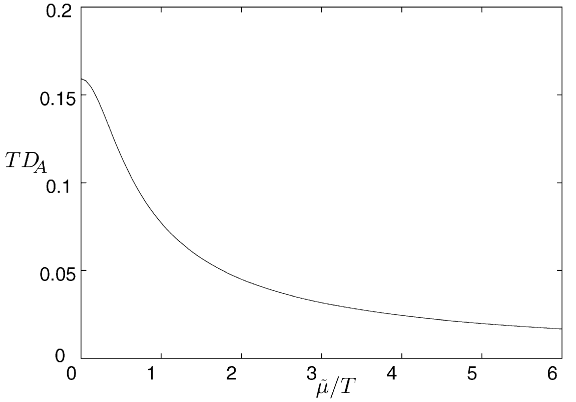

Fig. 1. TD_A vs. $\tilde{\mu}/T$.

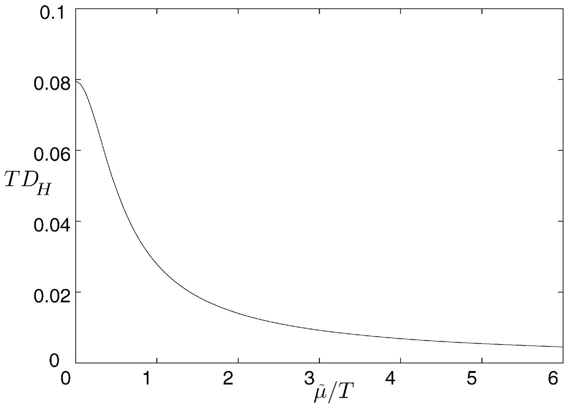

Fig. 2. TD_H vs. $\tilde{\mu}/T$.

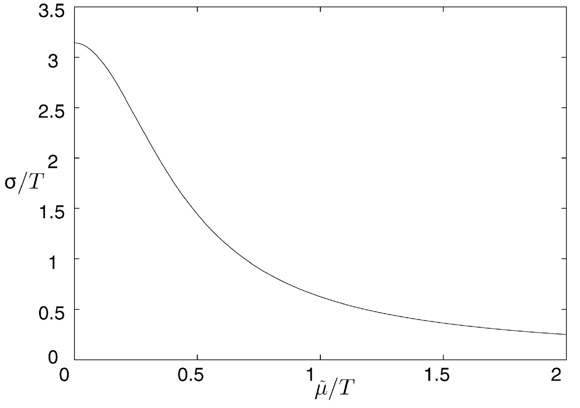

Fig. 3. $\sigma/(Tle_{\mathrm{E}}^2/e^2)$ vs. $\tilde{\mu}/T$.

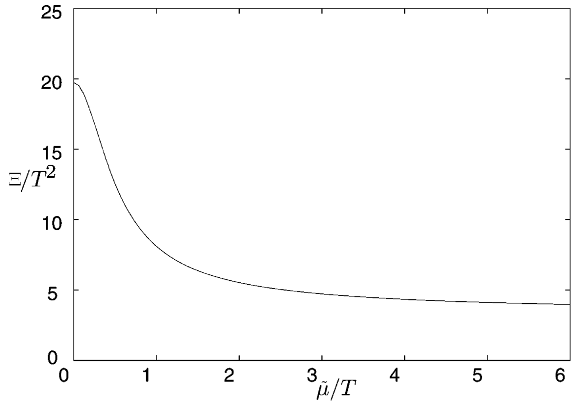

Fig. 4. $\Xi/(T^2 l/e^2)$ vs. $\tilde{\mu}/T$: Notice the rapid change between two finite values as T runs from 0 to ∞.

(density) indicating a mild phase transition. Such behavior does not exist for chargeless case. See also [33].

5. Conclusions and discussions

In this paper, we worked out the decoupling of scalar modes of the charged AdS black hole background in $SO(2)$ basis for the mode classifications. We also perform the hydrodynamic analysis for the holographic quark–gluon plasma system. Master equations for the decoupled modes are worked out explicitly. The sound velocity is not modified by the presence of the charge. We calculated the diffusion constants, the charge susceptibility and the conductivity as a consequence and observed that Einstein relation holds between them. These transport coefficients are modified due to the charge effect. Interestingly, the susceptibility modulo T^2 factor, which is an indicator of the degree of freedom, shows rapid change between low and high temperature (density) indicating a mild phase transition. Such behavior does not exist for chargeless case.

One can give an explanation of hydrodynamic mode in meson physics. In our interpretation, the Maxwell fields are the fluctuations of bulk-filling branes, therefore they should be interpreted as master fields of the mesons. Then hydrodynamic modes are lowest lying massless meson spectrum. In terms of brane embedding picture, this masslessness is due to the touching of the brane on the black hole horizon. Near the horizon, the tension of the brane is zero due to the metric factor and it can lead to the massless fluctuation. Then the massless spectrum cannot go far from the horizon in radial direction. In this picture, hydrodynamic nature is closely related to the near horizon behavior of the branes. We will discuss the spectrum of meson mode by considering the quasinormal mode [34] of the vector modes in elsewhere.

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Table 4
$G_{**\,**}/(\frac{l^3}{16\kappa^2 b^3(i\omega - D_H k^2)})$.

	tx	zx
tx	k^2	$-\omega k$
zx	–	ω^2

Table 5
$G_{**\,*}/(\frac{l\mu}{4e^2 b^2(i\omega - D_H k^2)})$.

	tx	zx
x	$-2i\omega$	$\frac{b}{(1+a)}\omega k$

Table 6
$G_{**}/(\frac{l}{4e^2(1+a)b^2}(\frac{3a}{i\omega - D_H k^2} - \frac{(2-a)^2 b}{2(1+a)}))$.

	x
x	$i\omega$

Appendix A. Results for the vector and the tensor type perturbations

A.1. Vector type perturbation

In the vector type perturbation, independent variables are

$$h_{tx}(x), \qquad h_{zx}(x), \qquad A_x(x).$$

One can observe the diffusion constant for the metric perturbation D_H as

$$D_H = \frac{b}{2(1+a)}. \tag{A.1}$$

We list the retarded Green functions in Tables 4, 5 and 6. Notice that $l^3/\kappa^2 = N_c^2/(4\pi^2)$ and $l/e^2 = N_c^2/(16\pi^2)$ for the R-charge, $l/e^2 = N_c N_f/(4\pi^2)$ for the brane charge. From the Green function for $U(1)$ currents $G_{x\,x}(\omega, k)$, one can read off the thermal conductivity κ_T via Kubo formula,

$$\kappa_T \equiv -\frac{(\epsilon + p)^2}{\rho^2 T} \lim_{\omega\to 0} \frac{1}{\omega} \operatorname{Im}\bigl(G_{x\,x}(\omega, k=0)\bigr) = 2\pi^2 \frac{N_c}{N_f} \frac{\eta T}{\mu^2}. \tag{A.2}$$

A.2. Tensor type perturbation

In the tensor type perturbation, an independent variable is just

$$h_{xy}(x).$$

By using Kubo formula, one can obtain the shear viscosity η as

$$\eta \equiv -\lim_{\omega\to 0} \frac{1}{\omega} \operatorname{Im}\bigl(G_{xy\;xy}(\omega, k=0)\bigr) = \frac{l^3}{16\kappa^2 b^3}, \tag{A.3}$$

where the retarded Green function is given by Table 7.

Table 7
$G_{****}/(-\frac{l^3}{16\kappa^2 b^3})$.

	xy
xy	$i\omega + bk^2$

One can confirm the universal (within Einstein gravity[8]) ratio that is the ratio of the shear viscosity to the entropy density s,

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{A.4}$$

Appendix B. Perturbative solutions for $\tilde{\Phi}_+$

Substituting Eq. (4.11) into Eq. (4.8), one can read off one for $F_{+0}(u)$,

$$0 = \big(u^{-1}(1-u)\big(1+u-au^2\big)F'_{+0}\big)'. \tag{B.1}$$

A general solution is given by

$$F_{+0}(u) = C_0 + D_0\left\{2\log(1-u) - \log\big(1+u-au^2\big) - \frac{3K_2(u)}{\sqrt{1+4a}}\right\}. \tag{B.2}$$

Constants of integration C_0 and D_0 should be determined to be a regular function at the horizon. So we here choose $D_0 = 0$ and set

$$F_{+0}(u) = C_0 = C \quad \text{(const.)}. \tag{B.3}$$

By using this solution, one can get an equation for $F_{+1}(u)$ from Eq. (4.8),

$$0 = \big(u^{-1}(1-u)\big(1+u-au^2\big)F'_{+1}\big)' - i\frac{Cb(1+au^2)}{(2-a)u^2}. \tag{B.4}$$

A general solution is

$$F_{+1}(u) = C_1 + \frac{i(C(1-a)b-(2-a)D_1)\log(1-u)}{(2-a)^2} + \frac{i(Cb+(2-a)D_1)\log(1+u-au^2)}{2(2-a)^2} + \frac{3i(Cb+(2-a)D_1)K_2(u)}{2(2-a)^2\sqrt{1+4a}}. \tag{B.5}$$

Removing the singularity at the horizon, the constant D_1 should be

$$D_1 = Cb\frac{1-a}{2-a}.$$

We also impose a boundary condition $F_{+1}(u=0) = 0$ so as to fix the constant C_1. Therefore the final form is

$$F_{+1}(u) = \frac{iCb}{2(2-a)}\left\{\log\big(1+u-au^2\big) - \frac{6K_1(u)}{\sqrt{1+4a}}\right\}. \tag{B.6}$$

[8] If one consider higher derivative corrections to Einstein gravity, the viscosity bound could be modified [35–39].

A differential equation for $G_{+1}(u)$ is

$$0 = \big(u^{-1}(1-u)\big(1+u-au^2\big)G'_{+1}\big)' + \frac{Cb^2(6-(1+a)u^2)}{3(1+a)u^4}. \tag{B.7}$$

A general solution is

$$\begin{aligned} G_{+1}(u) = \tilde{C}_1 + \frac{1}{6(2-a)(1+a)}\bigg\{&-\frac{4C(2-a)b^2}{u} \\ &+ \frac{(C(1+5a-2a^2)b^2 - 9(1+a)\tilde{D}_1)K_2(u)}{\sqrt{1+4a}} \\ &+ \big(C(1-a)b^2 + 3(1+a)\tilde{D}_1\big)\big(2\log(1-u) - \log\big(1+u-au^2\big)\big)\bigg\}, \end{aligned} \tag{B.8}$$

and the constant $\tilde{D}_1$ can be fixed as

$$\tilde{D}_1 = -\frac{C(1-a)b^2}{3(1+a)}.$$

From the condition $(uG_{+1})'|_{u=0} = 0$, we can fix the constant $\tilde{C}_1$. The final form of the solution becomes

$$G_{+1}(u) = \frac{2}{3}Cb^2\left\{\frac{K_1(u)}{\sqrt{1+4a}} - \frac{1}{(1+a)u}\right\}. \tag{B.9}$$

A differential equation for $F_{+2}(u)$ is

$$\begin{aligned} 0 = \bigg[&u^{-1}(1-u)\big(1+u-au^2\big) \\ &\times\left(\frac{Cb^2\log(1-u)}{(2-a)^2(1-u)} + \frac{ibF_{+1}(u)}{(2-a)(1-u)} - \frac{ib\log(1-u)F'_{+1}(u)}{2-a} + F'_{+2}(u)\right)\bigg]' \\ &- \frac{Cb^2}{u^2(1-u)(1+u-au^2)}. \end{aligned} \tag{B.10}$$

Integrating over u, we have

$$\begin{aligned} F'_{+2}(u) = &\frac{1}{2(2-a)^2\sqrt{1+4a}(1-u)(1+u-au^2)} \\ &\times \big\{2\sqrt{1+4a}(2-a)^2\big(Cb^2 - D_2u\big) - 3Cb^2K_2(0)\big(1+u-au^2\big) \\ &+ Cb^2\big(3+\big(5+3a-6a^2+2a^3\big)u - 3au^2\big)K_2(u) \\ &+ C\sqrt{1+4a}b^2(1-u)(1+au)\log\big(1+u-au^2\big)\big\}, \end{aligned} \tag{B.11}$$

and the constant D_2 can be fixed as

$$D_2 = Cb^2\left\{1 - \frac{3K_2(0)}{2(2-a)\sqrt{1+4a}} + \frac{(1+a)K_2(1)}{\sqrt{1+4a}}\right\},$$

so that $((1-u)F'_{+2}(u))|_{u=1} = 0$. Then, we find

$$F'_{+2}(u) = Cb^2 + \mathcal{O}(u). \tag{B.12}$$

Hence, we can write the solution of $F_{+2}(u)$ as

$$\begin{aligned}F_{+2}(u) = \int^{u} \mathrm{d}u \frac{Cb^2}{(1-u)(1+u-au^2)} \\ \times \bigg\{ 1-u+\frac{(1-u)(1+au)\log(1+u-au^2)}{2(2-a)^2} \\ - \frac{3K_2(0)(1-u)(1+au)}{2(2-a)^2\sqrt{1+4a}} - \frac{(1+a)K_2(1)u}{\sqrt{1+4a}} \\ + \frac{(3+(5+3a-6a^2+2a^3)u-3au^2)K_2(u)}{2(2-a)^2\sqrt{1+4a}} \bigg\},\end{aligned} \tag{B.13}$$

which satisfies a boundary condition $F_{+2}(0) = 0$.

Appendix C. Perturbative solutions for $\tilde{\Phi}_-$

For $F_{-0}(u)$, one can get an equation

$$0 = \left(u(1-u)\big(1+u-au^2\big)\left(1+a-\frac{3}{2}au\right)^{-2} F'_{-0} \right)'. \tag{C.1}$$

A general solution is given by

$$\begin{aligned}F_{-0}(u) = C_0 + D_0 \bigg\{ (2-a)\log(1-u) - 4(1+a)^2 \log(u) \\ + \frac{1}{2}(2+a)(1+4a)\log\big(1+u-au^2\big) - \frac{\sqrt{1+4a}(2+5a)K_2(u)}{2} \bigg\}.\end{aligned} \tag{C.2}$$

Since the function $F_{-0}(u)$ should be regular at the horizon, we choose $D_0 = 0$ and get

$$F_{-0}(u) = C_0 = \tilde{C} \quad \text{(const.)}. \tag{C.3}$$

Substituting the solution to Eq. (4.8), we get an equation for $F_{-1}(u)$,

$$\begin{aligned}0 = \left(u(1-u)\big(1+u-au^2\big)\left(1+a-\frac{3}{2}au\right)^{-2} F'_{-1} \right)' \\ + \frac{i\tilde{C}b(2(1+a)+(4+7a)u-6a(1+a)u^2+3a^2u^3)}{2(2-a)(1+a-\frac{3}{2}au)^3}.\end{aligned} \tag{C.4}$$

A general solution is given as

$$\begin{aligned}F_{-1}(u) = C_1 - \frac{i}{54(2-a)a^2} \\ \times \Big\{ 8(1+a)^2\big(\tilde{C}(1+4a)b - 27D_1(2-a)a^2\big)\log(u) \\ - 2\big(\tilde{C}\big(2+7a+23a^2\big)b - 27D_1(2-a)^2a^2\big)\log(1-u) \\ - (2+a)(1+4a)\big(\tilde{C}(1+4a)b - 27D_1(2-a)a^2\big)\log\big(1+u-au^2\big) \\ + \sqrt{1+4a}(2+5a)\big(\tilde{C}(1+4a)b - 27D_1(2-a)a^2\big)K_2(u) \Big\}.\end{aligned} \tag{C.5}$$

The constant of integration D_1 should be

$$D_1 = \frac{\tilde{C}b(2+7a+23a^2)}{27(2-a)^2a^2},$$

so that the singularity at the horizon could be removed. In addition, we require the condition

$$\left[F_{-1}(u) - \log(u)\lim_{u\to 0}\left(\frac{F_{-1}(u)}{\log(u)}\right)\right]_{u=0} = 0$$

to fix the constant C_1. Then we get the final form of the solution

$$F_{-1}(u) = \frac{i\tilde{C}b}{2(2-a)^2}\bigl\{8(1+a)^2\log(u) - (2+a)(1+4a)\log\bigl(1+u-au^2\bigr) - 2\sqrt{1+4a}(2+5a)K_1(u)\bigr\}. \tag{C.6}$$

Similarly we have a differential equation for $G_{-1}(u)$,

$$0 = \left(u(1-u)\bigl(1+u-au^2\bigr)\left(1+a-\frac{3}{2}au\right)^{-2}G'_{-1}\right)' + \frac{\tilde{C}b^2}{8(1+a)(1+a-\frac{3}{2}au)^5}\bigl\{-4(1+a)\bigl(2+6a+3a^2+2a^3\bigr) + 2a\bigl(10+30a+57a^2+10a^3\bigr)u - 42a^2(1+a)^2u^2 + 21a^3(1+a)u^3\bigr\}. \tag{C.7}$$

A general solution of this equation can be obtained

$$\begin{aligned}G_{-1}(u) = {}& \tilde{C}_1 - \frac{\tilde{C}a^2b^2}{a(1+a)(1+a-\frac{3}{2}ua)} + \frac{1}{54a(1+a)(2-a)} \\ & \times \Bigl\{4(1+a)^2(2-a)\bigl(7\tilde{C}b^2 + 54\tilde{D}_1a(1+a)\bigr)\log(u) \\ & - \bigl(2\tilde{C}\bigl(14+13a+17a^2\bigr)b^2 + 54\tilde{D}_1(2-a)^2a(1+a)\bigr)\log(1-u) \\ & - (2+a)\bigl(\tilde{C}\bigl(7+11a-14a^2\bigr)b^2 \\ & + 27\tilde{D}_1a(2-a)(1+a)(1+4a)\bigr)\log\bigl(1+u-au^2\bigr) \\ & + \frac{1}{\sqrt{1+4a}}\bigl(\tilde{C}\bigl(14+57a+81a^2-16a^3\bigr)b^2 \\ & + 27\tilde{D}_1a(2-a)(1+a)(1+4a)(2+5a)\bigr)K_2(u)\Bigr\}.\end{aligned} \tag{C.8}$$

The constant of integration $\tilde{D}_1$ might be fixed to remove the singularity at $u=1$,

$$\tilde{D}_1 = -\frac{\tilde{C}b^2(14+13a+17a^2)}{27a(1+a)(2-a)^2}.$$

Another constant of integration $\tilde{C}_1$ is fixed to satisfy the condition

$$\left[G_{-1}(u) - \log(u)\lim_{u\to 0}\left(\frac{G_{-1}(u)}{\log(u)}\right)\right]_{u=0} = 0.$$

The final result of the solution is

$$G_{-1}(u)=\frac{1}{3}\tilde{C}b^2\left\{-\frac{9a^2u}{2(1+a)^2(1+a-\frac{3}{2}au)}-\frac{6(1+a)(2+a)}{(2-a)^2}\log(u)\right.$$
$$\left.+\frac{3(1+a)(2+a)}{(2-a)^2}\log\bigl(1+u-au^2\bigr)+\frac{6(2+5a+6a^2)}{(2-a)^2\sqrt{1+4a}}K_1(u)\right\}. \tag{C.9}$$

A differential equation for $F_{-2}(u)$ is

$$0=\left[\frac{u(1+u-au^2)}{(2-a)^2(1+a-\frac{3}{2}au)^2}\right.$$
$$\times\bigl((2-a)^2(1-u)F'_{-2}(u)+i(2-a)bF_{-1}(u)$$
$$\left.-i(2-a)b(1-u)\log(1-u)F'_{-1}(u)+\tilde{C}b^2\log(1-u)\bigr)\right]'$$
$$+\frac{\tilde{C}b^2}{(1+a-\frac{3}{2}au)^2(1-u)(1+u-au^2)}. \tag{C.10}$$

Integrating over u, we have

$$F'_{-2}(u)=\frac{(1+a-\frac{3}{2}au)^2}{(1-u)u(1+u-au^2)}$$
$$\times\left\{D_2+\frac{18\tilde{C}ab^2}{(2-a)^2(1+4a)(1+a-\frac{3}{2}au)}\right.$$
$$+\frac{4\tilde{C}(1+a)^2b^2u(1+u-au^2)\log(u)}{(2-a)^3(1+a-\frac{3}{2}au)^2}$$
$$-\frac{2\tilde{C}b^2}{(2-a)^3}\left(1+\frac{(2+a)(1+4a)u(1+u-au^2)}{4(1+a-\frac{3}{2}au)^2}\right)\log\bigl(1+u-au^2\bigr)$$
$$-\frac{\tilde{C}\sqrt{1+4a}(2+5a)b^2K_2(0)u(1+u-au^2)}{2(2-a)^3(1+a-\frac{3}{2}au)^2}$$
$$+\frac{2\tilde{C}b^2}{(2-a)^3(1+4a)^{3/2}}\left(1-2a(5+a)\right.$$
$$\left.\left.+\frac{(2+5a)(1+4a)^2u(1+u-au^2)}{4(1+a-\frac{3}{2}au)^2}\right)K_2(u)\right\}, \tag{C.11}$$

and the constant D_2 can be fixed as

$$D_2=\frac{4\tilde{C}b^2}{(2-a)^4(1+4a)^{3/2}}\left\{\sqrt{1+4a}\bigl(2(1+4a)(1+a)^2\log(2-a)-9a(2-a)\bigr)\right.$$
$$\left.+\frac{1}{2}(2+5a)(1+4a)^2K_2(0)-(1+a)\bigl(2-2a+41a^2\bigr)K_2(1)\right\},$$

so that $((1-u)F'_{-2}(u))|_{u=1}=0$. Then, we can write the solution as

$$F_{-2}(u)=\int^u du\,\frac{\tilde{C}b^2}{2(2-a)^4(1+4a)^{3/2}(1-u)u(1+u-au^2)}$$

$$
\begin{aligned}
&\times \Big\{8(2-a)(1+a)^2(1+4a)^{3/2}u\big(1+u-au^2\big)\log(u) \\
&+ (2-a)(1+4a)^{3/2}\big(a(2+a)(1+4a)u^3 - \big(2+9a+13a^2\big)u^2 \\
&- \big(2-3a-8a^2\big)u - 4(1+a)^2\big)\log\big(1+u-au^2\big) \\
&+ (1+4a)^2(2+5a)K_2(0)(1-u)\big(4(1+a)^2 \\
&+ \big(2-3a-8a^2\big)u - a(2-a)u^2\big) \\
&- 8(1+a)\big(2-2a+41a^2\big)K_2(1)\left(1+a-\frac{3}{2}au\right)^2 \\
&- (2-a)\big(a(1+4a)^2(2+5a)u^3 - (2-a)\big(1+11a+46a^2+18a^3\big)u^2 \\
&- a\big(1+11a+46a^2+18a^3\big)u - 4(1+a)^2\big(1-10a-2a^2\big)\big)K_2(u)\Big\}.
\end{aligned}
\tag{C.12}
$$

Expanding this expression around $u=0$, we have

$$F_{-2}(u) = \tilde{C}b^2 D_-(1+a)\log(u) + C_2 + \mathcal{O}(u), \tag{C.13}$$

where C_2 is an integration constant.

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