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ON THE PRODUCT OF HOMOGENEOUS SPACES*

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Within the class of Tychonoff spaces, and within the class of topological groups, most of the natural questions concerning 'productive closure' of the subclasses of countably compact and pseudocompact spaces are answered by the following three well-known results: (1) [ZFC] There is a countably compact Tychonoff space X such that $X\times X$ is not pseudocompact; (2) [ZFC] The product of any set of pseudocompact topological groups is pseudocompact; and (3) [ZFC+MA] There are countably compact topological groups G_0, G_1 such that $G_0\times G_1$ is not countably compact.

In this paper we consider the question of 'productive closure' in the intermediate class of homogeneous spaces. Our principal result, whose proof leans heavily on a simple, elegant result of V.V. Uspenskiĭ, is this: In ZFC there are pseudocompact, homogeneous spaces X_0, X_1 such that $X_0\times X_1$ is not pseudocompact; if in addition MA is assumed, the spaces X_i may be chosen countably compact.

Our construction yields an unexpected corollary in a different direction: Every compact space embeds as a retract in a countably compact, homogeneous space. Thus for every cardinal number α there is a countably compact, homogeneous space whose Souslin number exceeds α.

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homogeneous space	countably compact	Martin's Axiom
retract	universal topological property	pseudocompact
extension property	Souslin number	cellular number

0. Introduction. References to the literature

0.1. In this paper we consider only Tychonoff spaces; that is, completely regular, Hausdorff spaces. The word 'space' is to be interpreted in this way throughout.

We use the symbol ω to denote both the first infinite cardinal and the countably infinite discrete space.

The Čech–Stone compactification of a space X is denoted βX; and for a continuous function f from a space X to a space Y we denote by $\bar{f}$ the (unique) continuous extension of f mapping βX into βY.

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For $p \in \beta(\omega)\backslash\omega$ and $\mathcal{S}$ the set of permutations of ω, we write

$$T(p) = \{\bar{\pi}(p): \pi \in \mathcal{S}\}.$$

(Following W. Rudin [40], $T(p)$ is called the *type* of p.) It is clear for p, $q \in \beta(\omega)\backslash\omega$ that either $T(p) = T(q)$ or $T(p) \cap T(q) = \emptyset$. It is well known (see for example [7, Section 9] or [35, Section 3] that $|T(p)| = c$ for each $p \in \beta(\omega)\backslash\omega$ and $|\{T(p): p \in \beta(\omega)\backslash\omega\}| = 2^c$.

0.2. The symbol MA used in the abstract refers to Martin's Axiom. The reader unfamiliar with MA and its uses in topology might refer to M.E. Rudin [38, 39] or to Weiss [50].

0.3. Here are some bibliographical citations concerning the results (1), (2) and (3) in our abstract.

(1) An example, credited to Novák [36] and Terasaka [45], is described in detail by Gillman and Jerison [19, (9.15)].

(2) This is due to Comfort and Ross [10]. A simplified proof based on an argument of De Vries [48] appears in [5; (6.4), (2.6)].

(3) Such groups G_i are constructed by Van Douwen [11] inside the compact 'Boolean group' $\{-1, +1\}^c$.

0.4. The theorem of Uspenskiĭ [46] on which our work depends reads as follows:

Theorem ([46]). *Let Y be a space with $|Y| = \alpha \geq \omega$ and let*

$$H(Y) = \{f \in Y^\alpha: |f^{-1}(\{y\})| = \alpha \text{ for all } y \in Y\}.$$

Then:

(a) *$H(Y)$ is homogeneous*;

(b) *$H(Y)$ is homeomorphic to $Y \times H(Y)$; and*

(c) *$H(Y)$ admits a retraction onto a homeomorph of Y.*

(Uspenskiĭ's theorem is easily proved. For (a) let g, $h \in H(Y)$ and let π be a permutation of α such that $\pi \circ g^{-1}(\{y\}) = h^{-1}(\{y\})$ for each $y \in Y$; the map $f \to f \circ \pi$ is a homeomorphism of $H(Y)$ onto $H(Y)$ such that $g \to h$. For (b), from which (c) follows, it is enough to write as usual

$$\alpha + 1 = \alpha \cup \{\alpha\} = \{\xi: \xi \leq \alpha\}$$

and to consider the map

$$Y \times H(Y) \to \{g \in Y^{\alpha+1}: |g^{-1}(\{y\})| = \alpha \text{ for all } y \in Y\}$$

given by $\langle y, f\rangle \to g$ with $g(\xi) = f(\xi)$ for $\xi < \alpha$, $g(\alpha) = y$.)

Given an infinite space Y, we use throughout this paper the symbol $H(Y)$ to denote the homogeneous space defined in the statement of Theorem 0.4.

1. On the product of pseudocompact, homogeneous spaces

Let us begin with a warm-up exercise.

1.1. Theorem. [ZFC] *There are pseudocompact, homogeneous spaces X_0, X_1 such that $X_0 \times X_1$ is not pseudocompact.*

Proof. Choose $p_0, p_1 \in \beta(\omega)\backslash\omega$ such that $T(p_0) \cap T(p_1) = \emptyset$, and for $i = 0, 1$ define $Y_i = \omega \cup T(p_i)$ and $X_i = H(Y_i)$.

In $Y_0 \times Y_1$, the family $\{U_n : n < \omega\}$ defined by $U_n = \{\langle n, n\rangle\}$ is an infinite, locally finite family of non-empty open sets. Thus $Y_0 \times Y_1$ is not pseudocompact. (Another way to see it: The function f: $Y_0 \times Y_1 \to \mathbb{R}$ defined by $f(\langle n, n\rangle) = n$, $f \equiv 0$ elsewhere, is continuous and unbounded.) Since X_i retracts onto (a homeomorph of) Y_i, the space $X_0 \times X_1$ retracts onto $Y_0 \times Y_1$. Thus $X_0 \times X_1$ is not pseudocompact.

It remains to see that the spaces X_i are pseudocompact. The argument is in two stages. (1) First, Y_i^κ is pseudocompact for all cardinals κ. For a proof, based on the observation [21] that a product of spaces is pseudocompact if and only if each countable subproduct is pseudocompact, see Frolík [15]. (2) Secondly, X_i is a subset of Y_i^c with the property that $\pi_A X_i = Y_i^A$ for each countable $A \subset c$ (that much is obvious), and each such subspace of a pseudocompact product is itself pseudocompact. (This latter assertion is easily proved. See for example [6, 26] or [9, (3.3(b))] for generalizations.) $\square$

1.2. Let us note in passing that an attempt to use simply the argument of Theorem 1.1 to produce in ZFC two countably compact, homogeneous spaces X_0, X_1 such that $X_0 \times X_1$ is not pseudocompact cannot succeed.

Theorem. *If Y is an infinite space then $H(Y)$ is not countably compact.*

Proof. Indeed, $H(Y)$ is sequentially dense in Y^α (with $\alpha = |Y|$) in the sense that for every $f \in Y^\alpha$ there is a sequence $\{f_n : n < \omega\}$ in $H(Y)$ such that $f_n \to f$ in Y^α. To see this, let $\{A(y, n): y \in Y, n < \omega\}$ be a faithfully indexed partition of α into sets of cardinality α, and given $f \in Y^\alpha$ define $f_n \in Y^\alpha$ by

$$f_n(\eta) = \begin{cases} y & \text{if } \eta \in A(y, n), \\ f(\eta) & \text{otherwise.} \end{cases} \quad \square$$

Evidently we will have to extend Uspenskiĭ's construction before dealing successfully with countably compact spaces.

2. Universal topological properties

Following Van der Slot [43], we say that a property $\mathcal{P}$ possessed by some spaces is a *universal topological property* provided:

(a) every compact space has $\mathcal{P}$;

(b) the product of any set of spaces with $\mathcal{P}$ has $\mathcal{P}$; and

(c) every closed subspace of a space with $\mathcal{P}$ has $\mathcal{P}$.

Within our context (Tychonoff spaces), the universal topological properties are exactly the *topological extension properties* defined and studied by Woods [51].

The following theorem, a special case of the adjoint functor theorem of Freyd [13, 14], is accessible through arguments given by Kennison [31], Van der Slot [43, 44], Herrlich [23], Herrlich and Van der Slot [24], Franklin [12] and Woods [51]; a systematic exposition is given in [8].

2.1. Theorem. *Let $\mathcal{P}$ be a universal topological property, and for every space Y define*

$$\beta_{\mathcal{P}}(Y) = \bigcap\{Z\colon\ Y \subset Z \subset \beta Y,\ Z \text{ has } \mathcal{P}\}.$$

Then

(a) *$\beta_{\mathcal{P}}(Y)$ has $\mathcal{P}$;*

(b) *Y is dense in $\beta_{\mathcal{P}}(Y)$;*

(c) *for every continuous $f\colon Y \to X$ such that X has $\mathcal{P}$, the function $\bar{f}$ satisfies $\bar{f}[\beta_{\mathcal{P}}(Y)] \subset X$; and*

(d) *up to a homeomorphism fixing Y pointwise, $\beta_{\mathcal{P}}(Y)$ is the only space satisfying* (a), (b) *and* (c).

The following simple result has been noted and used, at least for special instances of $\mathcal{P}$, by many authors.

2.2. Lemma. *Let $\mathcal{P}$ be a universal topological property and let h be a homeomorphism of a space Y onto Y. Then h extends to a homeomorphism of $\beta_{\mathcal{P}}(Y)$ onto $\beta_{\mathcal{P}}(Y)$.*

Proof. Let $j = \bar{h}|\beta_{\mathcal{P}}(Y)$ and $k = \overline{h^{-1}}|\beta_{\mathcal{P}}(Y)$. By Theorem 2.1(c) both $k \circ j$ and $j \circ k$ map $\beta_{\mathcal{P}}(Y)$ continuously into itself, and both are the identity on Y. Thus both $k \circ j$ and $j \circ k$ are the identity on $\beta_{\mathcal{P}}(Y)$, so j is a homeomorphism of the kind required. $\square$

In the following lemma we associate with each $\mathcal{P}$-space Y an enveloping $\mathcal{P}$-space $I(Y)$ to which Y-homeomorphisms extend.

2.3. Lemma. *Let $\mathcal{P}$ be a universal topological property. For every space Y with $\mathcal{P}$ there is a space $I(Y)$, containing a homeomorph $\bar{Y}$ of Y, such that:*

(a) *$I(Y)$ has $\mathcal{P}$;*

(b) *$I(Y)$ admits a retraction onto $\bar{Y}$;*

(c) *every homeomorphism of $\bar{Y}$ onto $\bar{Y}$ extends to a homeomorphism of $I(Y)$ onto $I(Y)$; and*

(d) *for every* p, $q \in \bar{Y}$ *there is a homeomorphism* h *of* $I(Y)$ *onto* $I(Y)$ *such that* $h(p)=q$.

Proof. Set $I(Y)=\beta_{\mathscr{P}}(Y\times H(Y))$. Fix $f\in H(Y)$, and set $\bar{Y}=Y\times\{f\}$.

Statement (a) is given by Theorem 2.1(a). According to Theorem 2.1(c) the retraction $r: Y\times H(Y)\to\bar{Y}$ given by $\langle y, g\rangle\to\langle y, f\rangle$ satisfies $\bar{r}[I(Y)]=\bar{Y}$; this proves (b).

We prove (c). Let h be a homeomorphism of $\bar{Y}$ onto $\bar{Y}$ and denote again by h its natural extension (given by $\langle y, g\rangle\to\langle h(y,f), g\rangle$) from $Y\times H(Y)$ onto $Y\times H(Y)$. An appeal to Lemma 2.2 completes the proof.

For (d) use Theorem 0.4(a), (b) to find a homeomorphism h of $Y\times H(Y)$ onto $Y\times H(Y)$ such that $h(p)=q$, and again apply Lemma 2.2. □

2.4. We remark that for universal topological properties $\mathscr{P}$ and spaces Y, Z the relation $\beta_{\mathscr{P}}(Y\times Z)=\beta_{\mathscr{P}}(Y)\times\beta_{\mathscr{P}}(Z)$ can fail, even when Y has $\mathscr{P}$. Indeed if $\mathscr{P}$ is the most important of all such properties, compactness, and Y is an infinite compact space, then according to Glicksberg's theorem [21] an infinite space Z satisfies $\beta_{\mathscr{P}}(Y\times Z)=Y\times\beta_{\mathscr{P}}(Z)$ if and only if Z is pseudocompact.

2.5. It would be nice to embed every $\mathscr{P}$-space as a retract into a homogeneous $\mathscr{P}$-space, but our methods appear to be inadequate to this task. The argument we give in Theorem 2.6 applies just when $\mathscr{P}$ is cardinally determined in the sense of the following definition.

Definition. Let $\mathscr{P}$ be a property possessed by some spaces and let α be an infinite cardinal. Then $\mathscr{P}$ is *α-determined* if, for each space Y, the following condition is satisfied: Y has $\mathscr{P}$ if and only if every closed subset F of Y such that $|F|\leq\alpha$ has $\mathscr{P}$. If $\mathscr{P}$ is α-determined for some infinite cardinal α then $\mathscr{P}$ is *cardinally determined.*

We illustrate this notion by mentioning some properties which are cardinally determined and some which are not. (We remind the reader that all spaces in this paper are Tychonoff spaces. Thus for Y a space and $A\subset Y$ we have always $|\mathrm{cl}_Y A|\leq 2^{2^{|A|}}$.)

(a) Compactness and realcompactness, both universal topological properties, are not cardinally determined. To see that neither property is α-determined when $\alpha\geq\omega$, let D be a discrete space with $|D|>\alpha$ and define

$$Y=\bigcup\{\mathrm{cl}_{\beta D}A\colon A\subset D, |A|\leq\alpha\}.$$

If F is a closed subset of Y such that $|F|\leq\alpha$ then for $y\in F$ there is $A_y\subset D$ such that $|A_y|\leq\alpha$ and $y\in\mathrm{cl}_{\beta D}A_y$. Writing $A=\bigcup_{y\in F}A_y$ we have $|A|\leq\alpha$ and $F\subset\mathrm{cl}_{\beta D}A\subset Y$; thus F is compact (and hence realcompact). But Y itself is not realcompact. To show this it is sufficient, according to Hewitt [25], to show that for some $p\in\beta Y\setminus Y$ no continuous $f\colon\beta Y\to[0,1]$ satisfies $f(p)=0$ and $f>0$ on Y. (For proofs of this and of other characterizations of realcompactness, the interested reader should consult Gillman and Jerison [19] or Walker [49].) Indeed, for no $p \in \beta Y \backslash Y$ does such a function f exist. For if f is given then since D is dense in Y for $n < \omega$ there is $x_n \in D$ such that $f(x_n) < 1/n$, and then with $B = \{x_n : n < \omega\}$ we have that $\mathrm{cl}_{\beta D} B$ is compact, $f > 0$ on $\mathrm{cl}_{\beta D} B$ since $\mathrm{cl}_{\beta D} B \subset Y$, and $\inf f[\mathrm{cl}_{\beta D} B] = 0$.

(b) For an infinite cardinal γ, a space Y is said to be *γ-bounded* if $\mathrm{cl}_Y A$ is compact for each $A \subset Y$ such that $|A| \leq \gamma$. It is easy to see (and it has been noted by several mathematicians) that γ-boundedness is a universal topological property ($\gamma \geq \omega$). An argument much like that of the preceding paragraph shows for each $\gamma \geq \omega$ that γ-boundedness is cardinally determined (in fact, γ-boundedness is α-determined with $\alpha = 2^{2^\gamma}$).

(c) Countable compactness, though not a universal topological property, is $2^{\mathfrak{c}}$-determined. In Section 3 for $p \in \beta(\omega)$ we will define p-compactness; it will be clear that each of these is $2^{\mathfrak{c}}$-determined.

In the following construction, given a universal topological property $\mathcal{P}$ we associate with each space Z with $\mathcal{P}$ the space $I(Z)$ given by Lemma 2.3; in the interest of notational simplicity we identify Z with its homeomorph $\bar{Z} \subset I(Z)$.

2.6. Let $\mathcal{P}$ be a universal topological property and let Y be a space with $\mathcal{P}$. For use in the following theorem we define by transfinite recursion: a space X_0 and a function $r_0 : X_0 \to Y$; for each successor ordinal $\xi + 1$, a space $X_{\xi+1}$ and a function $r_{\xi+1} : X_{\xi+1} \to Y$; and for each limit ordinal ξ, spaces X'_ξ and X_ξ and functions $r'_\xi : X'_\xi \to Y$ and $r_\xi : X_\xi \to Y$.

We proceed as follows. Let $X_0 = I(Y)$ and let r_0 be the retraction given by Lemma 2.3(b); for the successor ordinal $\xi+1$, X_ξ and r_ξ having been defined, let $X_{\xi+1} = I(X_\xi)$, let $s : X_{\xi+1} \to X_\xi$ be the retraction given by Lemma 2.3(b), and let $r_{\xi+1} = r_\xi \circ s$; and for a limit ordinal ξ, X_ζ and r_ζ having been defined for all $\zeta < \xi$, let $X'_\xi = \bigcup_{\zeta<\xi} X_\zeta$ topologized so that a subset U of X'_ξ is open if and only if $U \cap X_\zeta$ is open in X_ζ for each $\zeta < \xi$, let $r'_\xi = \bigcup_{\zeta<\xi} r_\zeta$, let $X_\xi = I(X'_\xi)$ and, noting that r'_ξ is continuous from X'_ξ onto Y, let r_ξ be the restriction to X_ξ of the function $\overline{r'_\xi} : \beta(X'_\xi) \to \beta Y$.

Theorem. *Let $\mathcal{P}$ be a universal topological property and let Y be a space with $\mathcal{P}$. Then the spaces and functions X_ξ, X'_ξ, r_ξ, r'_ξ defined above satisfy*:

(a) *each X_ξ has $\mathcal{P}$, and r_ξ is a retraction of X_ξ onto Y;*

(b) *each X'_ξ (ξ a limit ordinal) is homogeneous, and r'_ξ is a retraction of X'_ξ onto Y;*

(c) *if $\mathcal{P}$ is α-determined ($\alpha \geq \omega$) then X'_{α^+} has $\mathcal{P}$.*

Proof. For each ξ there is Z such that $X_\xi = \beta_{\mathcal{P}}(Z)$, so X_ξ has $\mathcal{P}$. The statements about r_ξ and r'_ξ are clear by induction, using (for limit ξ) Theorem 2.1(c) and the fact that Y has $\mathcal{P}$.

To see that X'_ξ is homogeneous for limit ξ, let p, $q \in X'_\xi$ and find $\eta < \xi$ such that p, $q \in X_\eta$. By Lemma 2.3(d) applied to X_η there is a homeomorphism $h_{\eta+1}$ of $X_{\eta+1}$

onto $X_{\eta+1}$ such that $h_{\eta+1}(p)=q$. A simple inductive argument will yield a homeomorphism h of X'_ξ onto X'_ξ such that $h_{\eta+1}\subset h$: Use Lemma 2.3(d) for non-limit ordinals $\zeta+1$ with $\eta\leqslant\zeta<\xi$ to define a homeomorphism $h_{\zeta+1}$ of $X_{\zeta+1}$ onto $X_{\zeta+1}$ such that $h_\zeta\subset h_{\zeta+1}$, and for limit ordinals ζ with $\eta<\zeta\leqslant\xi$ let $h'_\zeta=\bigcup_{i<\zeta}h_i: X'_\zeta\to X'_\zeta$ and define $h_\zeta: X_\zeta\to X_\zeta$ using Lemma 2.3(d); finally, take $h=h_\xi$.

To verify (c) it is enough to show that if $\mathcal{P}$ is α-determined and F is a closed subset of X'_{α^+} then F has $\mathcal{P}$. This is clear because there is $\xi<\alpha^+$ such that $F\subset X_\xi$, X_ξ has $\mathcal{P}$, and F is closed in X_ξ.

The proof is complete. $\square$

3. Homogeneous spaces with large Souslin number

Before passing in Section 4 to the construction in ZFC+MA of homogeneous, countably compact spaces whose product is not pseudocompact, we pause briefly to note some consequences of the foregoing theorem. It is convenient to have some other universal topological properties at our disposal, as follows.

Definition. Let $p\in\beta(\omega)$. A space X is *p-compact* if for every $f:\omega\to X$ the continuous extension $\bar{f}:\beta(\omega)\to\beta X$ satisfies $\bar{f}(p)\in X$.

The concept of p-compactness for $p\in\beta(\omega)$ was discovered, investigated and exploited by Bernstein [1] in connection with problems in the theory of non-standard analysis; it may be noted that the p-limit concept of Bernstein [1] coincides in important special cases with the 'producing' relation introduced by Frolík [16, 17], with one of the orderings considered by Katětov [29, 30], and with a definition given independently by Saks [41].

The following result is due to Bernstein [1]. Proofs are available also in [20, 42, 47].

3.1. Lemma ([1]). *For $p\in\beta(\omega)\backslash\omega$, p-compactness is a universal topological property.*

When $\mathcal{P}$ is p-compactness and Y is a space, we write $\beta_p(Y)$ in place of $\beta_{\mathcal{P}}(Y)$.

3.2. Corollary. *Let Y be a compact space. Then:*

(a) *Y embeds as a retract in a σ-compact (hence, Lindelöf) homogeneous space;*

(b) *for every cardinally determined universal topological property $\mathcal{P}$, Y embeds as a retract in a homogeneous space with $\mathcal{P}$; and*

(c) *for $p\in\beta(\omega)\backslash\omega$, Y embeds as a retract in a p-compact (hence, countably compact), homogeneous space.*

Proof. The compact space Y has every universal topological property. Statement (a) then follows from Theorem 2.6; the required enveloping σ-compact space is the space X_ω of Theorem 2.6. Statement (b) follows similarly from Theorem 2.6, and (c) is a special case of (b) since p-compactness is $2^{\mathfrak{c}}$-determined. □

3.3. It has come to our attention that Corollary 3.2(a) has been proved already by Okromeshko [37]. For a number of properties Q (e.g., the Lindelöf property, paracompactness and the like) he is able to embed every space with Q as a retract of a homogeneous space with Q. His methods are quite different from ours, and so far as we can determine the only point of overlap or duplication is Corollary 3.2(a) above.

Definition. Let Y be a space. A *cellular family* in Y is a family of pairwise disjoint, non-empty open subsets of Y.

The *Souslin number* of Y, denoted $S(Y)$, is the cardinal number

$$S(Y) = \min\{\alpha : \text{every cellular family } \mathcal{U} \text{ of } Y \text{ has } |\mathcal{U}| < \alpha\}.$$

A closely related concept, the *cellular number* of Y, denoted $c(Y)$, is the cardinal number

$$c(Y) = \sup\{\alpha : \text{some cellular family } \mathcal{U} \text{ of } Y \text{ has } |\mathcal{U}| = \alpha\}.$$

For references to the literature concerning these numbers and for proofs of many theorems concerning them, see [7, 9, 27].

3.4. Corollary. *For every cardinal number α there are homogeneous, σ-compact spaces, and there are homogeneous, countably compact spaces, whose Souslin number exceeds α.*

Proof. The relation $S(X) \geq S(Y)$ holds whenever Y is the image of X under a continuous surjection. Thus the two statements follow from parts (a) and (c) respectively of Corollary 3.2, upon taking for Y (for example) the space βD with D discrete, $|D| = \alpha$. □

4. On the product of countably compact, homogeneous spaces

The *Rudin–Keisler (pre-) order* $\leq$ on $\beta(\omega)\backslash\omega$ is defined as follows: $r \leq p$ if there is $f: \omega \to \omega$ such that $\bar{f}(p) = r$. In the proof of Theorem 4.1 we need the existence of elements p in $\beta(\omega)\backslash\omega$ which are $\leq$-minimal in the sense that if $r \in \beta(\omega)\backslash\omega$ and $r \leq p$ then $T(r) = T(p)$. (For a discussion of $\leq$ and of several conditions equivalent to $\leq$-minimality, see [7, (9.6)].) Kunen [32] has shown that the existence of

$\leqslant$-minimal $p \in \beta(\omega)\backslash\omega$ cannot be proved in ZFC, but it is known that such p exist in any model of ZFC satisfying CH [3; 4; 7, (9.13)]] or MA [2, 39].

4.1. Theorem. [ZFC+MA] *There are countably compact, homogeneous spaces* X_0, X_1 *such that* $X_0 \times X_1$ *is not pseudocompact.*

Proof. From MA there are by [2, 39] $\leqslant$-minimal $p_0, p_1 \in \beta(\omega)\backslash\omega$ such that $T(p_0) \cap T(p_1) = \emptyset$. We set $Y(i) = \beta_{p_i}(\omega)$ and, noting that p_i-compactness is a 2^c-determined universal topological property, we use Theorem 2.6 to define spaces $X(i)$ (of the form $X'_{(2^c)^+}$) such that $X(i)$ is p_i-compact, $X(i)$ is homogeneous, and $X(i)$ retracts onto $Y(i)$ $(i = 0, 1)$. Since $X(0) \times X(1)$ then retracts onto $Y(0) \times Y(1)$, to complete the proof it is enough to show that $Y(0) \times Y(1)$ is not pseudocompact.

For this as in Theorem 1.1 it is enough to show $\{\langle n, n\rangle: n < \omega\}$ is closed in $Y(0) \times Y(1)$, i.e., that $Y(0) \cap Y(1) = \omega$. We show in fact, writing

$$E(i) = \omega \cup \bigcup \{\mathrm{cl}_{\beta(\omega)} A: A \subset T(p_i), |A| \leqslant \omega\}, \quad (i = 0, 1),$$

that (a) $E(i)$ is p_i-compact (hence, $Y(i) \subset E(i)$) and (b) $E(0) \cap E(1) = \omega$.

(a) Let $f: \omega \to E(i)$. If $\{n < \omega: f(n) \in \omega\} \in p_i$ then $\bar{f}(p_i) \leqslant p_i$ and hence

$$\bar{f}(p_i) \in T(\bar{f}(p_i)) = T(p_i) \subset E(i)$$

from the $\leqslant$-minimality of p_i. If $\{n < \omega: f(n) \in \omega\} \notin p_i$ then with $F = \{n < \omega: f(n) \in E(i)\backslash\omega\}$ we have $F \in p_i$ and for $n \in F$ there is countable $A_n \subset T(p_i)$ such that $f(n) \in \mathrm{cl}_{\beta(\omega)} A_n$; then with $A = \bigcup_{n \in F} A_n$ we have $|A| \leqslant \omega$ and hence

$$\bar{f}(p_i) \in \mathrm{cl}_{\beta(\omega)} f[F] \subset \mathrm{cl}_{\beta(\omega)} A \subset E(i),$$

as required.

(b) Let $A_i \subset T(p_i)$ with $|A_i| \leqslant \omega$, $i = 0, 1$. Each point of A_i is a P-point of $\beta(\omega)\backslash\omega$ (see [7], for example) and hence $A_0 \cap \mathrm{cl}_{\beta(\omega)} A_1 = \mathrm{cl}_{\beta(\omega)} A_0 \cap A_1 = \emptyset$. It then follows, as noted in a related context by Frolík [17, 18] (see also [19, (14N.5); 49, (1.64); 7, (16.15); 5, (2.10)] that $\mathrm{cl}_{\beta(\omega)} A_0 \cap \mathrm{cl}_{\beta(\omega)} A_1 = \emptyset$, as required. $\square$

5. Some questions

Here we list some questions which our arguments appear unable to settle.

5.1.(a) Is the conclusion of Theorem 4.1 available in ZFC? Are there in ZFC countably compact, homogeneous spaces X_0, X_1 such that $X_0 \times X_1$ is not countably compact?

(b) Clearly the argument of Theorem 4.1 applies whenever one has two cardinally determined universal topological properties $\mathcal{P}_0$ and $\mathcal{P}_1$, each implying countable compactness, for which there exist spaces $X(i)$ with $\mathcal{P}_i$ $(i=0,1)$ such that $X(0)\times X(1)$ is not pseudocompact. Can one find such properties $\mathcal{P}_i$ in ZFC?

We have referred already to Kunen's result [32] that there is a model of ZFC in which no $p\in\beta(\omega)\backslash\omega$ is $\leqslant$-minimal. It is not known whether in ZFC one can find $p_0, p_1\in\beta(\omega)\backslash\omega$ such that no $q\in\beta(\omega)\backslash\omega$ satisfies $q\leqslant p_i$ $(i=0,1)$. (For a related problem see Van Mill [35, (5.2)].)

(c) It may be difficult to respond positively to Questions 5.1(a) and 5.1(b) without recourse to the theory of $\beta(\omega)\backslash\omega$. Indeed, Kannan and Soundararajan [28] have shown for a vast class of properties $\mathcal{P}$ closely related to the universal topological properties—the class of so-called PCS properties—that a space has $\mathcal{P}$ if and only if it is p-compact for each p in some set or class of ultrafilters (living not necessarily on ω but on various discrete spaces). This result has been extended and formulated in a categorical context by Hager [22]. For other results on spaces required to be p-compact simultaneously for various p, see Woods [51] and Saks [42].

5.2. It appears that neither our methods nor those of Okromeshko [37] are strong enough to furnish a positive response to the following question, which was brought to our attention some years ago by Eric Van Douwen: Is there for every cardinal number α a homogeneous, compact space X such that $S(X)>\alpha$? The Haar measure theorem shows $S(G)\leqslant\omega^+$ for every compact topological group G; Maurice [33] and Van Mill [34] have furnished in ZFC examples of compact, homogeneous spaces X such that $S(X)=c^+$. We do not even know whether in some or every model of ZFC there is a compact, homogeneous space X such that $S(X)>c^+$.

5.3. Is there a pseudocompact (or even: countably compact) homogeneous space X such that $X\times X$ is not pseudocompact?

5.4. Given an α-determined universal topological property $\mathcal{P}$ and a space Y with $\mathcal{P}$ such that $\gamma=w(Y)$ (the weight of Y), we have in Theorem 2.6 embedded Y as a retract into a homogeneous $\mathcal{P}$-space $X=X'_{\alpha^+}$ such that $|X|=\beth_{\alpha^+}(\gamma)=\beth_{\alpha^+}(|Y|)$. (Here as usual for each cardinal $\gamma\geqslant\omega$ the *beth cardinals* $\beth_\xi(\gamma)$ are defined by $\beth_0(\gamma)=\gamma$, $\beth_{\xi+1}(\gamma)=2^{\beth_\xi(\gamma)}$ for ordinals $\xi>0$, and $\beth_\xi(\gamma)=\sum_{\eta<\xi}\beth_\eta(\gamma)$ for limit ordinals ξ. When $\gamma=w(Y)$ the relation $\beth_\omega(\gamma)=\beth_\omega(|Y|)$, and hence $\beth_{\alpha^+}(\gamma)=\beth_{\alpha^+}(|Y|)$, results from the inequalities $|Y|\leqslant 2^\gamma$, $\gamma\leqslant 2^{|Y|}$.) It seems a long way from γ to $\beth_{\alpha^+}(\gamma)$. Problem: Given such $\langle\mathcal{P},\alpha,Y,\gamma\rangle$, find the minimal cardinal $\delta=\delta\langle\mathcal{P},\alpha,Y,\gamma\rangle$ for which Y embeds as a retract into a homogeneous $\mathcal{P}$-space X such that $|X|=\delta$.

5.5. Remark (added March, 1985). We have been told informally that S. Shelah has announced in conversation the following result: It is consistent with ZFC that

(*) for all $p_0, p_1\in\beta(\omega)\backslash\omega$ there is $r\in\beta(\omega)\backslash\omega$ such that $r\leqslant p_i$.

It is clear that in a model with (*) no spaces $X(i)$ as in Theorem 4.1 can exist of the form $X(i)=X'_{(2^c)^+}$ with $Y(i)=\beta_{p_i}(\omega)$, $p_i\in\beta(\omega)\backslash\omega$.

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