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The linear arboricity of planar graphs with no short cycles☆

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Abstract

The linear arboricity of a graph G is the minimum number of linear forests which partition the edges of G. Akiyama, Exoo and Harary conjectured that $\lceil \frac{\Delta(G)}{2} \rceil \leq la(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ for any simple graph G. In the paper, it is proved that if G is a planar graph with $\Delta \geq 7$ and without i-cycles for some $i \in \{4, 5\}$, then $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$.

Keywords: Planar graph; Linear arboricity; Cycle

1. Introduction

In this paper, all graphs are finite, simple and undirected. Any undefined notation follows that of Bondy and Murty [6]. For a real number x, $\lceil x \rceil$ is the least integer not less than x and $\lfloor x \rfloor$ is the largest integer not larger than x. Given a graph $G = (V, E)$. Let $N(v) = \{u \mid uv \in E(G)\}$ and $N_k(v) = \{u \mid u \in N(v) \text{ and } d(u) = k\}$, where $d(v) = |N(v)|$ is the *degree* of the vertex v. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum (vertex) degree and the minimum (vertex) degree, respectively. A k-, k^+- or k^--vertex is a vertex of degree k, at least k, or at most k, respectively.

A *linear forest* is a graph in which each component is a path. A map φ from $E(G)$ to $\{1, 2, \ldots, t\}$ is called a t-*linear coloring* if $(V(G), \varphi^{-1}(\alpha))$ is a linear forest for $1 \leq \alpha \leq t$. The *linear arboricity* $la(G)$ of a graph G defined by Harary [9] is the minimum number t for which G has a t-linear coloring. Given a t-linear coloring and a vertex v of G, let $C^i_\varphi(v) = \{j \mid \text{ the color } j \text{ appears } i \text{ times at } v\}$, where $i = 0, 1, 2$. Then $|C^0_\varphi(v)| + |C^1_\varphi(v)| + |C^2_\varphi(v)| = t$.

Akiyama, Exoo and Harary [2] conjectured that $la(G) = \lceil (\Delta(G) + 1)/2 \rceil$ for any regular graph G. It is obvious that $la(G) \geq \lceil \Delta(G)/2 \rceil$ for any graph G and $la(G) \geq \lceil (\Delta(G) + 1)/2 \rceil$ for every regular graph G. So the conjecture is equivalent to the following conjecture.

Conjecture A. *For any graph G,* $\lceil \frac{\Delta(G)}{2} \rceil \leq la(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$.

The linear arboricity has been determined for complete bipartite graphs [2], Halin graphs [11], series–parallel graphs [13], complete regular multipartite graphs [14], and regular graphs with $\Delta = 3, 4$ [2,3], 5, 6, 8 [7], 10 [8].

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Péroche [10] proved that the determination of the linear arboricity of a graph G is an **NP**-hard problem, even when $\Delta = 4$. Alon, Teague and Wormald [5] proved that there is an absolute constant $c > 0$ such that for every d-regular graph G, $la(G) \le \frac{d}{2} + cd^{2/3}(\log d)^{1/3}$ (A slightly weaker result has been proved in [4, p. 64]). AÏt-djafer [1] obtained some results for graphs with multiple edges. Wu, Liu and Wu [14] obtained an upper bound for the linear arboricity of composition of two graphs and proved that for a nonempty regular graph G and a null graph S_n, $la(G[S_n]) = \lceil(\Delta(G[S_n]) + 1)/2\rceil$ if $\Delta(G)$ is even and G has a Hamiltonian factorization orthogonal to a linear forest, or $\Delta(G)$ is odd and the graph by removing a 1-factor F from G has a Hamiltonian factorization orthogonal to a matching M such that $M \cup F$ is a linear forest.

Conjecture A has already been proved to be true for all planar graphs, see [12] and [15]. Wu also proved in [12] that for a planar graph G with maximum degree Δ, $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$ if $\Delta \ge 13$. In the same paper, he proved that if G is a planar graph with $\Delta \ge 7$ and without 3-cycles, then $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$. Here we obtain that the result is also true for a planar graph with $\Delta \ge 7$ and without i-cycles for some $i \in \{4, 5\}$.

2. Main results and their proofs

In the section, all graphs are planar graphs which have been embedded in the plane. For a planar graph G, the degree of a face f, denote by $d(f)$, is the number of edges incident with it, where each cut-edge is counted twice. A k-, k^+- or k^--face is face of degree k, at least k or at most k, respectively. First, let's prove some lemmas.

Theorem 1. *Suppose that d is an integer with $d \ge 4$ and G is a planar graph with maximum degree $\Delta \le 2d$ and without i-cycles for some $i \in \{4, 5\}$. Then G has a d-linear coloring.*

Proof. Let $G = (V, E)$ be a minimal counterexample to the theorem. First, we prove some lemmas for G.

Lemma 2. *For any edge $uv \in E(G)$, $d_G(u) + d_G(v) \ge 2d + 2$.*

Proof. Suppose that G has an edge uv with $d_G(u) + d_G(v) \le 2d + 1$. Then $G' = G - uv$ has a d-linear coloring φ by the minimality of G. Let $S = C^2_\varphi(u) \cup C^2_\varphi(v) \cup (C^1_\varphi(u) \cap C^1_\varphi(v))$. Since $d_{G'}(u) + d_{G'}(v) = d(u) + d(v) - 2 \le 2d - 1$, $|S| < d$. Let $\varphi(uv) \in \{1, 2, \ldots, d\}\backslash S$. Thus φ is extended to a d-linear coloring of G, a contradiction. Hence the lemma holds. □

By Lemma 2, we have
(1) $\delta(G) \ge 2$, and
(2) any two 4^--vertices are not adjacent, and
(3) any 3-face is incident with three 5^+-vertices, or at least two 6^+-vertices.

Lemma 3. *G has no even cycle $v_0v_1 \cdots v_{2n-1}v_0$ such that $d(v_1) = d(v_3) = \cdots = d(v_{2n-1}) = 2$ and $\max_{0 \le i < n} |N_2(v_{2i})| \ge 3$.*

Proof. Suppose it does contain such an even cycle. Without loss of generality, let $N_2(v_0) \ge 3$. This implies that v_0 is adjacent to at least three 2-vertices. Let $u \in N_2(v_0)\backslash\{v_{2n-1}, v_1\}$ and $v \in N(u)\backslash\{v_0\}$. By the induction hypothesis, $G^* = G - \{v_1, \ldots, v_{2n-1}\} - uv_0$ has a d-linear coloring φ. Now we construct directly a d-linear coloring σ of G as follows.

First of all, if $C^0_\varphi(v_0) \ne \emptyset$, let $\sigma(uv_0) = \sigma(v_0v_1) \in C^0_\varphi(v_0)$. Otherwise, $|C^1_\varphi(v_0)| \ge 3$, let $\sigma(uv_0) \in C^1_\varphi(v_0)\backslash\varphi(uv)$ and $\sigma(v_1v_0) \in C^1_\varphi(v_0)\backslash\sigma(uv_0)$. After that, let $\sigma(v_0v_{2n-1}) \in (C^1_\varphi(v_0) \cup C^0_\varphi(v_0))\backslash\{\sigma(uv_0), \sigma(v_0v_1)\}$. So $\sigma(v_0v_1) \ne \sigma(v_0v_{2n-1})$. Furthermore, for $i = 1, 2, \ldots, n-1$, if $\sigma(v_0v_{2n-1}) \in C^1_\varphi(v_{2i})$, let $\sigma(v_{2i-1}v_{2i}) = \sigma(v_0v_{2n-1})$. Otherwise, let $\sigma(v_{2i-1}v_{2i}) \in (C^1_\varphi(v_{2i})\backslash\sigma(v_{2i-2}v_{2i-1})) \cup C^0_\varphi(v_{2i})$. $\sigma(v_{2i}v_{2i+1}) \in C^1_\varphi(v_{2i})\backslash\sigma(v_{2i-1}v_{2i})) \cup C^0_\varphi(v_{2i})$. Finally, the uncolored edges of G are colored the same colors as in φ of G^*. This contradiction proves the lemma. □

Let G_2 be the subgraph induced by edges incident with 2-vertices. Since G does not contain two adjacent 2-vertices, G_2 does not contain any odd cycle. So it follows from Lemma 3 that any component of G_2 is either an even cycle or a tree. So it is easy to find a matching M in G saturating all 2-vertices. Thus if $uv \in M$ and $d(u) = 2$, v is called a 2-*master* of u. Note that every 2-vertex has a 2-master and each vertex of degree at least d can be the 2-master of at most one 2-vertex.

Let F be the set of faces of G. By Euler's formula $|V| - |E| + |F| = 2$, we have

$$\sum_{v\in V}(2d(v) - 6) + \sum_{f\in F}(d(f) - 6) = -6(|V| - |E| + |F|) = -12 < 0.$$

We define ch to be the *initial charge*. Let $ch(x) = 2d(x) - 6$ for each $x \in V(G)$ and $ch(x) = d(x) - 6$ for each $x \in F(G)$. In the following, we will reassign a new charge denoted by $ch'(x)$ to each $x \in V(G) \cup F(G)$ according to the discharging rules. Since our rules only move charges around, and do not affect the sum, we have

$$\sum_{x\in V(G)\cup F(G)} ch'(x) = \sum_{x\in V(G)\cup F(G)} ch(x) = -12. \tag{$*$}$$

We'll show that $ch'(x) \geq 0$ for each $x \in V(G) \cup F(G)$, a contradiction to $(*)$, completing the proof.

First, we assume that G contains no 4-cycles. Then the discharging rules are defined as follows.

R1-1. Each 2-vertex receives 2 from its 2-master.

R1-2. Each 3-face f receives $\frac{3}{2}$ from each of its incident 5^+-vertex.

R1-3. Each 5-face f receives $\frac{1}{3}$ from each of its incident 5^+-vertex.

Let f be a face of G. Clearly, $ch'(f) = ch(f) = d(f) - 6 \geq 0$ if $d(f) \geq 6$. Suppose $d(f) = 3$. By (3), $ch'(f) \geq ch(f) + 2 \times \frac{3}{2} = 0$. If $d(f) = 5$, then f is incident with at least three 5^+-vertices and it follows that $ch'(f) \geq ch(f) + 3 \times \frac{1}{3} = 0$.

Let v be a vertex of G. Since G contains no 4-cycle, v is incident with at most $\lfloor\frac{d(v)}{2}\rfloor$ 3-faces. If $d(v) = 2$, then $ch'(v) = ch(v) + 2 = 0$ by R1-1. If $d(v) = 3$ or 4, then $ch'(v) = ch(v) \geq 0$. If $d(v) = 5$, then $ch'(v) \geq ch(v) - 2 \times \frac{3}{2} - 3 \times \frac{1}{3} = 0$. If $d(v) = 6$ or 7, then $ch'(v) \geq ch(v) - 3 \times \frac{3}{2} - 4 \times \frac{1}{3} > 0$. If $d(v) \geq 8$, then $ch'(v) \geq ch(v) - \lfloor\frac{d(v)}{2}\rfloor \times \frac{3}{2} - (d(v) - \lfloor\frac{d(v)}{2}\rfloor) \times \frac{1}{3} > 0$. Hence we complete the proof of the case that G contains no 4-cycles.

Now assume that G contains no 5-cycles. Let's prove the following lemma.

Lemma 4. *Suppose that a planar graph G contains no 5-cycles and $\delta(G) \geq 2$. Then any of the following results holds.*

(a) *Any vertex v is incident with at most $\lfloor\frac{2d(v)}{3}\rfloor$ 3-faces.*

(b) *A 3-face is incident with a 4-face if and only if the two faces are incident with a common 2-vertex.*

(c) *If a face is adjacent to two nonadjacent 3-faces then the face must be 6^+-face.*

(d) *If a $d(\geq 7)$-vertex v is incident with a 3-face, then v is incident with at most $d - 2$ 4^--faces.*

Proof. Since if there are three 3-faces f_1, f_2, f_3 such that they are incident with a common vertex and f_2 is incident with f_1 and f_3, then vertices incident with them form a 5-cycle, so (a) holds. If a 3-face is incident with a 4-face, then all three vertices incident with the 3-face f must be incident with the 4-face, too. So there is a vertex just incident with these two faces and it follows that the vertex is a 2-vertex. Hence (b) holds. For (c), suppose that a face f is adjacent to two nonadjacent 3-faces. It is obvious that f is not a 3-face for otherwise a 5-cycle is appeared. By (b), f is not a 4-face. So f must be a 6^+-face and (c) holds.

For (d), suppose that a $d(\geq 7)$-vertex v is incident with a 3-face. If v is a cut vertex, then (d) is obvious. So assume that v is not a cut vertex. Let $f_1, f_2, \ldots, f_d$ be faces incident with v clockwise, and $v_1, v_2, \ldots, v_d$ be vertices incident with v clockwise, and v_i be incident with $f_i, i = 1, 2, \ldots, d$, and v_d be incident with f_d and f_1. Assume that f_1 be the 3-face. Then by (a), f_1 or f_d is not a 3-face. Without loss of generality, assume that f_d is not a 3-face.

Suppose that f_d is a 4-face. Then $d(v_d) = 2$ by (b). Thus f_2 must be a 3-face or a 6^+-face. If f_2 is a 3-face, then f_3 or f_4 must be a 6^+-face. So one of f_2, f_3, f_4 is a 6^+-face. Similarly, if f_{d-1} is a 4-face, then $d(v_{d-1}) = 2$. So one of f_d, f_{d-1}, f_{d-2} is a 6^+-face.

Suppose that f_d is a 6^+-face. If f_2 is a 3-face, then f_3 must be a 4-face or 6^+-face. If f_3 is a 4-face, then f_4 or f_5 must be a 6^+-face. So one of the faces in $\{f_2, f_3, f_4, f_5\}$ is a 6^+-face. Thus we prove (d). □

The discharging rules are defined as follows.

R2-1. Each 2-vertex receives 2 from its 2-master.

R2-2. For a 3-face f and its incident vertex v, f receives $\frac{1}{2}$ from v if $d(v) = 4$, 1 if $d(v) = 5$, $\frac{5}{4}$ if $d(v) = 6$ and $\frac{3}{2}$ if $d(v) \geq 7$.

R2-3. For a 4-face f and its incident vertex v, f receives $\frac{1}{2}$ from v if $4 \le d(v) \le 6$, 1 if $d(v) \ge 7$.

Let f be a face of G. Clearly, $ch'(f) = ch(f) = d(f) - 6 \ge 0$ if $d(f) \ge 6$. Suppose $d(f) = 3$. If f is incident with a 3^--vertex, then other incident vertices of f are 7^+ vertices and it follows that $ch'(f) \ge ch(f) + 2 \times \frac{3}{2} = 0$. If f is incident with a 4-vertex, then $ch'(f) \ge ch(f) + \frac{1}{2} + 2 \times \frac{5}{4} = 0$. If all vertices incident with f are 5^+-vertex, then $ch'(f) \ge ch(f) + 3 \times 1 = 0$. Suppose $d(f) = 4$. If f is incident with a vertex of degree at most 3, then f is incident with at least two 7^+-vertices and it follows that $ch'(f) \ge ch(f) + 2 \times 1 = 0$. Otherwise $ch'(f) \ge ch(f) + 4 \times \frac{1}{2} = 0$.

Let v be a vertex of G. If $d(v) = 2$, then $ch'(v) = ch(v) + 2 = 0$ by R2-1. If $d(v) = 3$, then $ch'(v) = ch(v) = 0$. If $d(v) = 4$, then $ch'(v) \ge ch(v) - 4 \times \frac{1}{2} = 0$. If $d(v) = 5$, then $ch'(v) \ge ch(v) - \max\{3 \times 1 + 2 \times \frac{1}{2}, 2 \times 1 + 3 \times \frac{1}{2}, 1 + 4 \times \frac{1}{2}\} = 0$. If $d(v) = 6$, then $ch'(v) \ge ch(v) - 4 \times \frac{5}{4} - 2 \times \frac{1}{2} = 0$. Suppose $d(v) = 7$. Then it is not incident with a 2-vertex and it is incident with at most four 3-faces. At the same time, if a 3-face f is incident with v, then there is at least one face of degree at least 6 which is incident with v and is adjacent to f. So $ch'(v) \ge ch(v) - \max\{4 \times \frac{3}{2} + 2 \times \frac{1}{2}, 7 \times 1\} = 0$. Suppose $d(v) = 8$. If v is not incident with a 3-face, then $ch'(v) = ch(v) - 2 - 8 \times 1 = 0$. So assume that v is incident with at least one 3-face. By (d), v is incident with at most two $d(v) - 2$ faces of degree at most 4. If v is incident with five 3-faces, then all 4^+-faces incident with v must be 6^+-faces by (b) and it follows that $ch'(v) = ch(v) - 2 - 5 \times \frac{3}{2} > 0$; otherwise $ch'(v) = ch(v) - 2 - 4 \times \frac{3}{2} - 2 \times 1 = 0$. Suppose $d(v) \ge 9$. Similarly, we have $ch'(v) \ge ch(v) - 2 - \lfloor \frac{2d(v)}{3} \rfloor \times \frac{3}{2} - (d(v) - 2 - \lfloor \frac{2d(v)}{3} \rfloor) \times \frac{1}{3} > 0$. Hence we complete the proof of the case that G contains no 5-cycles. □

Corollary 5. *If G is a planar graph with $\Delta \ge 7$ and without i-cycles for some $i \in \{3, 4, 5\}$, then $la(G) = \lceil \frac{\Delta(G)}{2} \rceil$.*

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