ELSEVIER
Discrete Applied Mathematics 51 (1994) 27–37
DISCRETE APPLIED MATHEMATICS

On edge-hamiltonian Cayley graphs

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(Received 1 October 1991; revised 30 January 1992)

Abstract

Chen (1988) conjectured that every finite hamiltonian Cayley graph is edge-hamiltonian. We prove some hamiltonian Cayley graphs to be edge-hamiltonian and some Cayley graphs to be hamiltonian. Results of Alspach and Zhang (1989) on Hamilton cycles in Cayley graphs on dihedral groups are generalized.

1. Introduction

The well-known conjecture that every Cayley graph on a finite group of order at least three has a Hamilton cycle is still open. So far it is only known that the conjecture is true for special classes of Cayley graphs. The problem is solved for all abelian groups (see [4, 6]), but the situation is evidently more complicated for nonabelian groups (see [1, 10]). Now it is interesting to search for properties of Hamilton cycles in hamiltonian Cayley graphs. Chen [3] conjectured that every hamiltonian Cayley graph is edge-hamiltonian, in the sense that every edge of the graph lies on a Hamilton cycle. Quimpo [7] proved some Cayley graphs to be edge-hamiltonian, Chen and Quimpo [4] proved all Cayley graphs on abelian groups of order at least three to be edge-hamiltonian, and Chen [3] obtained the result that all Cayley graphs with 2^k ($k > 1$) vertices also have this property.

In the present paper Cayley graphs on dihedral groups are considered. Dihedral groups are of some importance in the search for Hamilton cycles in Cayley graphs on finite groups with cyclic commutator subgroups [2]. But so far the existence of a Hamilton cycle in every Cayley graph on a dihedral group has not been proved, although there has been some effort (see [2, 10]). It will be shown that the known results on Cayley graphs of dihedral groups can be generalized.

SSDI 0166-218X(92)E0196-5

2. Cayley graphs

Throughout this paper undirected connected Cayley graphs on finite groups will be considered. Let H denote a finite group and w a generating set of H satisfying $w^{-1} = w$ and not containing the unit element. Then a Cayley graph $G(H, w)$ is defined to be the graph as follows. Its vertices are the elements of H, and two vertices x, y are joined by an edge $\{x, y\}$ iff $x^{-1}y$ is an element of w.

Now colours are assigned to the edges of $G(H, w)$ to formulate the results on Hamilton cycles of the graph in a clear manner. For this purpose, the elements of w are enumerated $h_1, h_2, \ldots, h_m$, and an edge $\{x, y\}$ is coloured α_i iff $x^{-1}y = h_i$, where $\alpha_i = \alpha_j$ iff $h_i = h_j$ or $h_i^{-1} = h_j$. In this way a partition of the edge set of $G(H, w)$ into colour classes is obtained: A colour class $\{\{x, xh_i\}: x \in H\}$, consisting of all edges of $G(H, w)$ coloured α_i, is a perfect matching whenever the element h_i of w has order 2. Otherwise, the colour class generated by h_i determines a set of vertex-disjoint cycles

$$(\{x, xh_i\}, \{xh_i, xh_i^2\}, \{xh_i^2, xh_i^3\}, \ldots) \quad (x \in H)$$

covering the vertex set of $G(H, w)$, where the length of each of these cycles is equal to the order of h_i.

3. Edge-hamiltonian property of Cayley graphs

In investigating whether a given hamiltonian Cayley graph has the property that each of its edges lies on a Hamilton cycle, the following result of Chen can be applied effectively.

Observation 3.1 (Chen [3]) *If an edge coloured α_i is contained in a Hamilton cycle of $G(H, w)$, then every edge coloured α_i is contained in some Hamilton cycle of $G(H, w)$.*

Often graph automorphisms of $G(H, w)$ can be used to find classes of edges which are contained in Hamilton cycles. If there is a Hamilton cycle in $G(H, w)$ containing an edge coloured α_i and an automorphism of the graph mapping this edge onto an edge coloured α_j, then every edge coloured α_j lies on a Hamilton cycle in $G(H, w)$ by Observation 3.1. Note that group automorphisms of H mapping w onto itself are graph automorphisms of $G(H, w)$ [5].

The next lemma can be applied to characterize all edge-hamiltonian cubic Cayley graphs.

Lemma 3.2. (Smith [9]) *Every hamiltonian cubic graph has at least three Hamilton cycles.*

Corollary 3.3. *Every hamiltonian cubic Cayley graph is edge-hamiltonian.*

Proof. Let C denote a Hamilton cycle in a cubic Cayley graph $G(H, w)$, and let e be an arbitrary edge of the graph not in C. If an edge of C has the same colour as e, then e lies on a Hamilton cycle of $G(H, w)$ by Observation 3.1 and we are done. Otherwise, all edges not in C have the same colour α_i because they form a perfect matching. Lemma 3.2 implies that $G(H, w)$ has a Hamilton cycle containing an edge coloured α_i. Since the colour of e is α_i, too, e lies on a Hamilton cycle of $G(H, w)$ by Observation 3.1. □

4. Cayley graphs on dihedral groups

First we give some elementary facts on dihedral groups. A dihedral group D_n ($n > 2$) is a group generated by two elements x, y of order 2 where the product $x \cdot y$ is an element of order n. Now $y(xy)^i$ and $(xy)^i$ for $i = 1, 2, \ldots, n$ are the elements of D_n.

The elements $(xy)^i$ are called rotations, and the elements $y(xy)^i$ are called reflections. Reflections are elements of order 2. The rotation over $\varphi = \pi$ is also an element of order 2. A rotation has order >2 iff it is not in the centre of D_n. If n is odd, then the centre of D_n only contains the unit element. Otherwise, the unit element and the rotation over $\varphi = \pi$ constitute the centre of the group. The set of all rotations constitutes a normal cyclic subgroup C_n of D_n. The commutator subgroup of D_n is a cyclic group generated by $(xy)^2$.

Now Hamilton cycles in Cayley graphs on dihedral groups will be investigated.

Every Cayley graph $G(D_p, w)$ on a dihedral group D_p of order $2p$ (p prime, $p > 2$) is edge-hamiltonian [7].

The generalization of this result to all Cayley graphs $G(D_n, w)$ on dihedral groups D_n is still unknown. Even the existence of Hamilton cycles in arbitrary Cayley graphs on dihedral groups is yet unknown.

Theorem 4.1. *Every hamiltonian Cayley graph $G(D_n, w)$ on a dihedral group D_n is edge-hamiltonian.*

Proof. Let C denote a Hamilton cycle in $G(D_n, w)$ and e an arbitrary edge (coloured α_i) not in C. If C contains an edge coloured α_i, then e lies on a Hamilton cycle by Observation 3.1. Otherwise, distinguish between the cases that the order of h_i is 2 or greater than 2.

Case 1. Let h_i be an element of order 2. The edges coloured α_i constitute a perfect matching M, and the edges of C and M generate a hamiltonian cubic graph G'. By Lemma 3.2, there is a Hamilton cycle in G' containing an edge coloured α_i. Now it follows from Observation 3.1 that e lies on a Hamilton cycle of $G(D_n, w)$.

Case 2. Let h_i be an element of order >2. In this case we construct a Hamilton cycle in $G(D_n, w)$ containing an edge coloured α_i (note that the construction can also be applied if $G(D_n, w)$ is not supposed to be hamiltonian).

Consider the subgraph $\bar{G}$ of $G(D_n, w)$ determined by all edges $\{x, y\}$ where $x^{-1}y \in C_n$ (defined before as the normal cyclic subgroup of D_n). Denote the components of $\bar{G}$ by $G_1, G_2, \ldots, G_k$. The edge e occurs in one of these components. Note that $G_1, G_2, \ldots, G_k$ are isomorphic graphs which are isomorphic with a Cayley graph on a cyclic group. There is a result of Chen and Quimpo [4] showing that every Cayley graph on an abelian group of order at least three is edge-hamiltonian. Hence every edge of G_j lies on a Hamilton cycle of G_j for $j = 1, 2, \ldots, k$.

Consider a Hamilton cycle C_1 in G_1 containing edges coloured α_i. The cycle C_1 (of length m, say) can be described by the sequence

$$(x_1, x_1 h_{t(1)}, x_1 h_{t(1)} h_{t(2)}, \ldots, x_1 h_{t(1)} h_{t(2)} \cdots h_{t(m-1)})$$

of covered vertices.

It is clear that every vertex z in a component G_j of $\bar{G}$ determines a cycle $C_{j,z}$ of the same length m given by the sequence

$$(z, zh_{t(1)}, zh_{t(1)} h_{t(2)}, \ldots, zh_{t(1)} h_{t(2)} \cdots h_{t(m-1)})$$

of vertices.

Now let $\{h_1, h_2, \ldots, h_d\}$ denote the set of reflections of D_n contained in the generating set w.

First consider the subgraph $\bar{G}'$ of $G(D_n, w)$ determined by all edges $\{x, y\}$ where $x^{-1}y \in C_n$ or $x^{-1}y = h_1$. Denote the components of $\bar{G}'$ by $G_1', G_2', \ldots$. Let a vertex u of a component G_p of $\bar{G}$ be joined with a vertex v of a component G_q of $\bar{G}$ by an edge $\{u, v\}$ where $u^{-1}v = h_1$. Note that $\{u, v\}$ is an edge of $\bar{G}'$ not in $\bar{G}$. Then each vertex c of G_p (use that $c = hu$ for some element h) is joined with some vertex d of G_q by an edge $\{c, d\}$ where $c^{-1}d = h_1$ (choose $d = hv$). This means that the number of components of $\bar{G}'$ is half the number of components of $\bar{G}$.

If $\{u, u'\}$ is an edge in G_p and $u^{-1}u' = h_s \in C_n$ ($h_s \neq h_i$), then the vertices $v = uh_1$, $v' = u'h_1$ are the neighbours of u and u', respectively, in G_q and $\{v, v'\}$ is an edge of G_q satisfying $v^{-1}v' = h_1^{-1}(u^{-1}u')h_1 = (u^{-1}u')^{-1} = h_s^{-1}$ (note that h_1 is a reflection of the dihedral group D_n, satisfying $h_1^{-1}h_s h_1 = h_s^{-1}$ for every element h_s of C_n).

Now it is possible to join together cycles of length m in G_p and G_q such that a cycle of length $2m$ (running through all vertices of G_p and G_q and containing at least two edges coloured α_i) is obtained (see Fig. 1). Therefore there is a set of vertex-disjoint cycles of length $2m$ covering the vertex set of $G(D_n, w)$ where each of these cycles contains at least two edges coloured α_i.

If h_2 is not contained in the group generated by the elements of C_n and h_1, then consider the subgraph $\bar{G}''$ of $G(D_n, w)$ determined by all edges $\{x, y\}$ where $x^{-1}y \in C_n$ or $x^{-1}y \in \{h_1, h_2\}$. It is clear that $\bar{G}'' \neq \bar{G}'$.

Let $\{u_1, u_1'\}$ be an edge of a component G_p' of $\bar{G}'$ satisfying $u_1^{-1}u_1' \in C_n$ (and $u_1^{-1}u_1' \in w$). The vertices u_1, u_1' have neighbours $v_1 = u_1 h_2$ and $v_1' = u_1' h_2$, respectively, in a component G_q' ($p \neq q$) of $\bar{G}'$. Moreover, $\{v_1, v_1'\}$ is an edge of G_q' since $v_1^{-1}v_1' = h_2^{-1}(u_1^{-1}u_1')h_2 = (u_1^{-1}u_1')^{-1} \in w$.

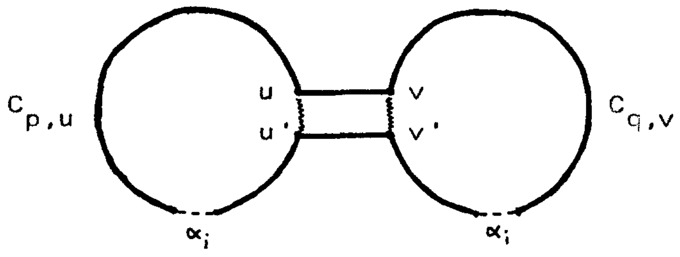

Fig. 1.

Join together cycles of length $2m$ in components of $\bar{G}''$ as described before. Applying this method Hamilton cycles in the components of $\bar{G}''$ can be constructed. Each of these Hamilton cycles contains at least two edges coloured α_i.

In this way, a Hamilton cycle of $G(D_n, w)$ containing edges coloured α_i is obtained (use all reflections contained in w).

By Observation 3.1, e lies on a Hamilton cycle of $G(D_n, w)$, too. Hence $G(D_n, w)$ is an edge-hamiltonian graph. □

5. Construction of Hamilton cycles

In the following some Cayley graphs shall be proved to be hamiltonian.

Theorem 5.1. *Let $G(D_n, w)$ be a Cayley graph on a dihedral group D_n where w contains at least one element of order >2. Then $G(D_n, w)$ has a Hamilton cycle.*

Proof. Since D_n is a dihedral group, its generating set w does not only contain elements of order >2, but also at least one reflection. The elements of an order different from 2 generate a cyclic subgroup C_n of D_n.

Now a Hamilton cycle of $G(D_n, w)$ can be constructed applying the method described in the proof of Theorem 4.1 (case 2). □

By Theorem 5.1, all further considerations on Cayley graphs $G(D_n, w)$ can be restricted to the case that w contains only elements of order 2.

At first cubic Cayley graphs $G(D_n, w)$ shall be investigated. Let w contain three elements of order 2. Alspach and Zhang [2] characterized the isomorphism classes of such graphs and proved every cubic Cayley graph on a dihedral group to be hamiltonian.

If no element of $w = \{a, b, c\}$ is a rotation of D_n, then the edges $\{x, y\}$ with $x^{-1}y \in \{a, b\}$ constitute a set of l vertex-disjoint cycles of length $2m$ covering the vertex set (note that m denotes the order of the product $a \cdot b$ and that $2n = l \cdot 2m$). There is a result of Alspach and Zhang [2] showing that $G(D_n, w)$ is isomorphic with a graph $\bar{G}(l, m)$ constructed as follows.

The vertex set of $\bar{G}(l, m)$ is defined to be the disjoint union of the l sets $V_i = \{v_{i,0}, v_{i,1}, \ldots, v_{i,2m-1}\}$ for $i = 1, 2, \ldots, l$ (for technical reasons the first subscript of $v_{i,j}$ is given reduced up to congruence modulo l and the second modulo $2m$).

The edge set $\bar{E}$ of $\bar{G}(l, m)$ is given by

$$\begin{aligned}\bar{E} = \bigcup_{i=1}^{l} & \{\{v_{i,j}, v_{i,j+1}\}: j = 0, 1, \ldots, 2m - 1\} \\ & \cup \{\{v_{i,j}, v_{i+1,j}\}: i \operatorname{odd}(i < l), j = 1, 3, \ldots, 2m - 1\} \\ & \cup \{\{v_{i,j}, v_{i+1,j}\}: i \operatorname{even}(i < l), j = 0, 2, \ldots, 2m - 2\} \\ & \cup \{\{v_{1,j+j_0}, v_{l,j}\}: j = 1, 3, \ldots, 2m - 1 \text{ if } l \text{ is odd}, \\ & \qquad\qquad j = 0, 2, \ldots, 2m - 2 \text{ if } l \text{ is even}\}.\end{aligned}$$

(Note that j_0 is even if l is even and that j_0 is odd if l is odd.)

The following results can be applied in the search for Hamilton cycles in Cayley graphs $G(D_n, w)$ with a 4-element generating set w consisting of elements of order 2.

Lemma 5.2. *Let $G(D_n, w)$ be a cubic Cayley graph on a dihedral group D_n where w is a generating set $\{h_i, h_j, h_k\}$ of D_n satisfying*

$$\begin{aligned}& h_i, h_j, h_k \text{ are reflections;} \\ & h_j h_k \text{ is an element of order } p\ (p \text{ prime}).\end{aligned} \tag{$*$}$$

Then $G(D_n, w)$ has a Hamilton cycle containing all edges coloured α_i.

Proof. Note that every cubic Cayley graph $G(D_n, w)$ under consideration has an edge set consisting of pairwise disjoint cycles of length $2p$ (p prime), covering the vertex set, and a perfect matching. The graph $G(D_n, w)$ is isomorphic with some graph $\bar{G}(l, p)$ defined above. All edges of $\bar{G}(l, p)$ joining vertices of different cycles of length $2p$ are assigned to an element h_i of order 2 of D_n. Now a Hamilton cycle containing all edges coloured α_i will be constructed.

First, let l be even. Then the sequence

$$\begin{aligned}(& v_{1,0}, v_{1,1}, v_{2,1}, v_{2,0}, \\ & v_{3,0}, v_{3,1}, v_{4,1}, v_{4,0}, \\ & \vdots \\ & v_{l-1,0}, v_{l-1,1}, v_{l,1}, v_{l,0})\end{aligned}$$

of $2l$ vertices describes a path q_0 in $\bar{G}(l, p)$. Further paths q_s can be obtained from q_0 by replacing each vertex $v_{i,j}$ by $v_{i,j+j_0 s}$ for $s = 1, 2, \ldots, p - 1$. It is easy to see that the

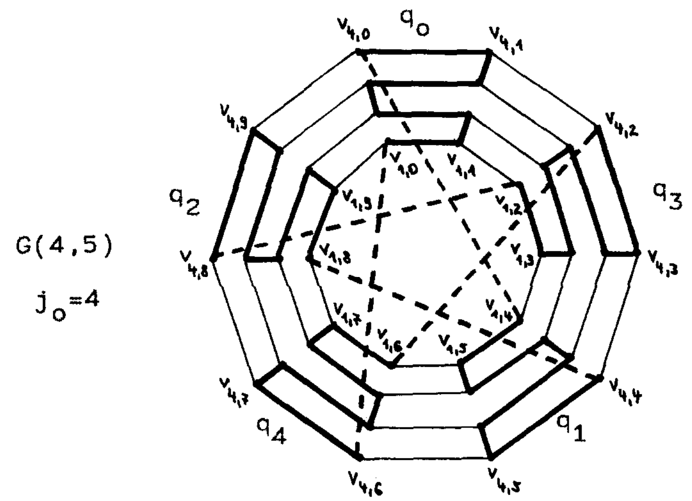

Fig. 2.

paths q_s ($s = 0, 1, \ldots, p-1$) are pairwise vertex-disjoint. They determine a Hamilton cycle in $\bar{G}(l, p)$, since the end vertex of each path q_i is adjacent to the start vertex of q_{i+1} for $i = 0, 1, \ldots, p-2$ and the end vertex of q_{p-1} is adjacent to the start vertex of q_0 (see Fig. 2). Moreover, this Hamilton cycle contains all edges joining vertices v_{i_1, j_1} and v_{i_2, j_2} with $i_1 \neq i_2$ (note that adjacent vertices $v_{i_1, j_1}, v_{i_2, j_2}$ with $i_1 \neq i_2$ satisfy $j_1 = j_2$). Therefore the graph $G(D_n, w)$ isomorphic with $\bar{G}(l, p)$ has a Hamilton cycle containing all edges coloured α_i.

Now let l be odd. In this case, the sequence

$$
\begin{aligned}
&(v_{1,0}, v_{1,1}, v_{2,1}, v_{2,0},\\
&v_{3,0}, v_{3,1}, v_{4,1}, v_{4,0},\\
&\vdots\\
&v_{l-2,0}, v_{l-2,1}, v_{l-1,1}, v_{l-1,0},\\
&v_{l,0}, v_{l,1})
\end{aligned}
$$

of $2l$ vertices describes a path q_0' in $\bar{G}(l, p)$. Further paths can be obtained from q_0' by replacing each vertex $v_{i,j}$ by $v_{i,j+(j_0+1)s}$ for $s = 1, 2, \ldots, p-1$. These paths $q_0', q_1', \ldots, q_{p-1}'$ are pairwise vertex-disjoint. Now, as in the case that l is even, a Hamilton cycle of $\bar{G}(l, p)$ can be obtained by the concatenation of $q_0', q_1', \ldots, q_{p-1}'$ (see Fig. 3). This Hamilton cycle contains all edges joining vertices v_{i_1, j_1} and v_{i_2, j_2} with $i_1 \neq i_2$. Therefore the graph $G(D_n, w)$ has a Hamilton cycle containing all edges coloured α_i. $\square$

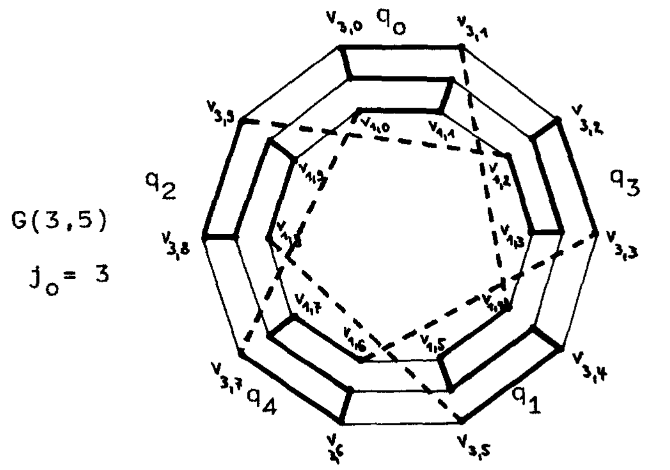

Fig. 3.

Lemma 5.3. *Let $G(D_n, w)$ be a cubic Cayley graph on a dihedral group D_n where w is a generating set of D_n containing the rotation over $\varphi = \pi$. Then $G(D_n, w)$ has a Hamilton cycle containing all edges coloured α_i for some h_i of order 2 of w.*

Proof. Let h_i denote the rotation over $\varphi = \pi$ contained in w. Then the order of the group generated by $w - \{h_i\}$ is equal to half the order of the group D_n. Hence the edge set of $G(D_n, w)$ consists of two cycles of even length n described by the sequences $(v_1, v_2, \dots, v_n)$ and $(v_1', v_2', \dots, v_n')$ of covered vertices and of the edges $\{v_1, v_1'\}, \{v_2, v_2'\}, \dots, \{v_n, v_n'\}$ determined by the element h_i of w.

Now it is easy to see that $G(D_n, w)$ has a Hamilton cycle containing all edges coloured α_i. $\square$

The results obtained above shall be used to prove the following result on Cayley graphs of degree 4.

Theorem 5.4. *Let $G(D_n, w)$ be a Cayley graph on a dihedral group D_n, where w is a minimal generating set $\{h_i, h_j, h_k, h_l\}$ of D_n satisfying*

h_i, h_j, h_k, h_l are elements of order 2;

h_i, h_j, h_k are reflections;

$h_i h_j$ is an element of order p (p prime). $(*')$

Then the graph $G(D_n, w)$ has a Hamilton cycle.

Proof. Consider the Cayley graph $G(D_m, \tilde{w})$ where $\tilde{w} = \{h_i, h_j, h_k\}$. It follows from Lemma 5.2 that this graph has a Hamilton cycle with a corresponding sequence f of colours of covered edges containing all edges coloured α_k. Consider the components of the subgraph $\tilde{G}$ of $G(D_n, w)$ determined by all edges $\{x, y\}$ where $x^{-1}y \in \tilde{w}$. Now $2m$ is the number of vertices in each of these components. It is clear that every component has a cycle of length $2m$ with f as corresponding sequence of colours. In f every element α_k is succeeded by α_i or α_j, and the successor to this element of $\{\alpha_i, \alpha_j\}$ is again α_k.

Now distinguish the cases that h_l is a rotation over $\varphi = \pi$ or a reflection.

In the first case, $\tilde{G}$ has exactly two components. The considered cycles of length $2m$ can be described by sequences $(v_1, v_2, \ldots, v_{2m})$ and $(v'_1, v'_2, \ldots, v'_{2m})$ of covered vertices, and for a suitable enumeration edges $\{v_1, v'_1\}, \{v_2, v'_2\}, \ldots, \{v_{2m}, v'_{2m}\}$ coloured α_l join vertices of these two cycles. Now it is clear that $G(D_n, w)$ has a Hamilton cycle.

Now let h_l be a reflection (satisfying $h_l^{-1}h_s h_l = h_s^{-1}$ for every element h_s of the normal cyclic subgroup C_n of D_n). Every vertex h' of $G(D_n, w)$ is incident with an edge coloured α_l. By minimality of w the end vertices of this edge are contained in different components of $\tilde{G}$ (h' in Q' and $h'h_l$ in Q''). The vertex h' lies on a cycle C' (of length $2m$) in Q' with f as corresponding sequence of colours. Now h' is incident with an edge $\{h', h'h_k\}$ on C' coloured α_k, and its end vertex $h'h_k$ determines an edge on C' with a colour different from α_k. Without loss of generality, this colour will be α_i and $\{h'h_k, h'h_kh_i\}$ is the appropriate edge. Moreover, $h'h_l$ is incident with an edge $\{h'h_l, h'h_lh_i\}$ of colour α_i, too. Its second end vertex determines an edge $\{h'h_lh_i, h'h_lh_ih_k\}$ coloured α_k. Now the edges $\{h'h_l, h'h_lh_i\}$, $\{h'h_lh_i, h'h_lh_ih_k\}$ determine a cycle C'' of length $2m$ in Q'' with f as corresponding sequence of colours. Furthermore, the vertices $h'h_kh_i$ and $h'h_lh_ih_k$ of C' and C'', respectively, are joined by an edge coloured α_l since $(h'h_kh_i)^{-1}(h'h_lh_ih_k) = (h_ih_k)h_l(h_ih_k) = h_l$ (note that h_l is a reflection, $h_ih_k \in C_n$ and $h_l^{-1}(h_ih_k)h_l = (h_ih_k)^{-1}$).

It follows immediately from the properties of f mentioned above that the graph generated by the edges of the considered cycles of length $2m$ and all edges of $G(D_n, w)$ coloured α_l is isomorphic with some graph $\bar{G}(q, m)$. As it was stated above, $\bar{G}(q, m)$ has a Hamilton cycle. Hence $G(D_n, w)$ is a hamiltonian graph. □

The construction used in the proofs above can also be applied to find Hamilton cycles in Cayley graphs of other groups.

Let H be a group with a generating set

$$w = \{h_i, h_j, h_k\}$$

satisfying

$$h_i^n = h_k^2 = e \ (n > 2);$$

$$h_i^{-1} = h_j;$$

$$h_k h_i^2 h_k \in \{h_i^2, h_i^{-2}\}. \qquad (*'')$$

Consider the Cayley graph $G(H, w)$.

If $h_k h_i h_k \in \{h_i, h_j\}$, then $G(H, w)$ is a cubic Cayley graph on an abelian group or on a dihedral group. Hence $G(H, w)$ has a Hamilton cycle in this case.

Now let $h_k h_i h_k \notin \{h_i, h_j\}$. The sequences $(h, hh_i, hh_i^2, \ldots, hh_i^{n-1})$ of vertices describe vertex-disjoint cycles of length n in $G(H, w)$ covering the vertex set. An edge $\{x, y\}$ is contained in one of these cycles iff $x^{-1}y = h_i$ or $x^{-1}y = h_j$.

If h_k is an element of the group generated by $\{h_i, h_j\}$, then H is a cyclic group and $G(H, w)$ is a hamiltonian graph.

Now suppose that $w - \{h_j\}$ is a minimal generating set of H.

Case 1. $h_k h_i^2 h_k = h_i^2$ holds. If $\{h, h'\}$ is an edge of $G(H, w)$ coloured α_k (satisfying $h^{-1}h' = h_k$), then all pairs $\{hh_i^{2t}, h'h_i^{2t}\}$ of vertices determine edges coloured α_k by $(hh_i^{2t})^{-1}(h'h_i^{2t}) = h_i^{-2t}h_k h_i^{2t} = h_k$ (use $h_i^{-2}h_k h_i^2 = h_k$). Moreover, the vertices h and h' determine disjoint cycles of length n given by the sequences

$$(h, hh_i, hh_i^2, \ldots, hh_i^{n-1})$$

and

$$(h', h'h_i, h'h_i^2, \ldots, h'h_i^{n-1})$$

of vertices. Now it is clear that $G(H, w)$ is isomorphic with a graph $\bar{G}(l, m)$ defined before.

Case 2. $h_k h_i^2 h_k = h_i^{-2}$ holds. If $\{h, h'\}$ is an edge of $G(H, w)$ coloured α_k, then all pairs $\{hh_i^{2t}, h'h_i^{-2t}\}$ of vertices determine edges coloured α_k by $(hh_i^{2t})^{-1}(h'h_i^{-2t}) = h_i^{-2t}h_k h_i^{-2t} = h_k$ (use $h^{-1}h' = h_k$ and $h_i^{-2}h_k h_i^{-2} = h_k$). These edges join vertices of vertex-disjoint cycles of length n given by the sequences

$$(h, hh_i, hh_i^2, \ldots, hh_i^{n-1})$$

and

$$(h', h'h_i^{-1}, h'h_i^{-2}, \ldots, h'h_i^{-(n-1)})$$

of covered vertices.

Now it is clear that $G(H, w)$ is isomorphic with a graph $\bar{G}(l, m)$ as in case 1.

Corollary 5.5. *Every Cayley graph $G(H, w)$ on a group H with a generating set $w = \{h_i, h_j, h_k\}$ satisfying $(*'')$ has a Hamilton cycle.*

Applying Observation 3.1, it is easy to see that the graphs mentioned in Corollary 5.5 are edge-hamiltonian.

Acknowledgement

The author wishes to thank the referees for their useful comments and for suggesting a number of improvements concerning the presentation.

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