Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

Journal of Functional Analysis 239 (2006) 631–637

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

A variation embedding theorem and applications

Peter Friz [a,*], Nicolas Victoir

[a] *Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, UK*

Received 28 November 2005; accepted 20 December 2005

Available online 3 February 2006

Communicated by Paul Malliavin

Abstract

Fractional Sobolev spaces, also known as Besov or Slobodetski spaces, arise in many areas of analysis, stochastic analysis in particular. We prove an embedding into certain q-variation spaces. Applications include a new route to a regularity result by Kusuoka for stochastic differential equations, integration against Besov-paths, a regularity criterion for rough paths and a new regularity result for Cameron–Martin paths associated to fractional Brownian motion.

Keywords: Fractional Sobolev, Besov spaces; q-Variation embedding; Rough paths; Regularity of the Ito-map; Regularity of Cameron–Martin

1. Fractional Sobolev spaces

For a real valued measurable path $h:[0,1]\to\mathbb{R}$ and $\delta\in(0,1)$ and $p\in(1,\infty)$ we define the fractional Sobolev (semi-)norm

$$|h|_{W^{\delta,p}}=\left(\iint\limits_{[0,1]^2}\frac{|h_t-h_s|^p}{|t-s|^{1+\delta p}}\,ds\,dt\right)^{1/p}\in[0,+\infty].$$

* Corresponding author.
E-mail address: p.k.friz@statslab.cam.ac.uk (P. Friz).

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2005.12.021

For $\delta = 1$ and $p \in (1, \infty)$, writing $\dot{h}$ for the weak derivative, we set

$$|h|_{W^{1,p}} = \left(\int_0^1 |\dot{h}_t|^p \, dt \right)^{1/p} \in [0, +\infty].$$

Define $W^{\delta,p}$ as the set of h for which $|h|_{L^p} + |h|_{W^{\delta,p}} < \infty$. They are known to be Banach-spaces. For $1 \geqslant \delta > 1/p > 0$ one can assume that h is continuous; compare with the embedding theorems below. It then makes sense to consider the closed subspace

$$W_0^{\delta,p} = \left\{ h \in W^{\delta,p} \colon h(0) = 0 \right\}$$

which is Banach under $|\cdot|_{W^{\delta,p}}$. We finally remark that the space $W^{1,p}$ is precisely the set of absolutely continuous paths on $[0, 1]$ with (a.e. defined) derivative in $L^p[0, 1]$. The space $W_0^{1,2}$ is the usual Cameron–Martin space for Brownian motion. We recall some well-known continuous respectively compact embeddings[1] [1–3],

$$p \in (1, \infty), \quad 1 \geqslant \tilde{\delta} > \delta \geqslant 0 \quad \Rightarrow \quad W^{\tilde{\delta},p} \Subset W^{\delta,p}, \tag{1.1}$$

$$1 < p \leqslant q < \infty, \quad \delta \equiv 1 - 1/p + 1/q > 0 \quad \Rightarrow \quad W^{1,p} \subset W^{\delta,q}. \tag{1.2}$$

2. A q-variation embedding

Theorem 1. *Let $p \in (1, \infty)$ and $\alpha = 1 - 1/p > 0$. Then the variation of any $h \in W^{1,p}$ is controlled by the control function*[2]

$$\omega(s, t) = |h|_{W^{1,p};[s,t]} (t - s)^{\alpha}, \quad 0 \leqslant s \leqslant t \leqslant 1$$

and we have the continuous embeddings

$$W^{1,p} \subset C^{\alpha\text{-Hölder}} \quad \textit{and} \quad W^{1,p} \subset C^{1\text{-var}}.$$

Proof. By absolute continuity and Hölder's inequality with conjugate exponents p and $1/\alpha$

$$|h_{s,t}| = \int_s^t |\dot{h}_r| \, dr \leqslant (t - s)^{\alpha} \left(\int_s^t |\dot{h}_r|^p \, dr \right)^{1/p} = |h|_{W^{1,p};[s,t]} (t - s)^{\alpha}.$$

We now show that the variation of h is controlled by the control function

$$\omega(s, t) = |h|_{W^{1,p};[s,t]} (t - s)^{\alpha}, \quad t \geqslant s.$$

Only super-additivity, $\omega(s, t) + \omega(t, u) \leqslant \omega(s, u)$ with $s \leqslant t \leqslant u$, is non-trivial. Note $p \in (1, \infty)$. From Hölder's inequality with conjugate exponents p and $p/(p - 1) = 1/\alpha$ we obtain

1 The symbol $\Subset$ means compact embedding.

2 A continuous, super-additive map $(s, t) \mapsto \omega(s, t) \in [0, \infty)$, defined for $0 \leqslant s \leqslant t \leqslant 1$.

$$
\begin{aligned}
&|h|_{W^{1,p};[s,t]}(t-s)^{\alpha}+|h|_{W^{1,p};[t,u]}(u-t)^{\alpha}\\
&\quad\leqslant \left(|h|^p_{W^{1,p};[s,t]}+|h|^p_{W^{1,p};[t,u]}\right)^{1/p}\left[(t-s)^{\alpha\frac{p}{p-1}}+(u-t)^{\alpha\frac{p}{p-1}}\right]^{(p-1)/p}\\
&\quad=|h|_{W^{1,p};[s,u]}(u-t)^{\alpha}.
\end{aligned}
$$

This shows that ω is super-additive and we conclude that for any $0\leqslant a<b\leqslant 1$,

$$|h|_{1\text{-var};[a,b]}\leqslant \omega(a,b)=|b-a|^{\alpha}|h|_{W^{1,p};[a,b]}.$$

In particular, we established $W^{1,p}\subset C^{\alpha\text{-Hölder}}$ and $W^{1,p}\subset C^{1\text{-var}}$. $\square$

Theorem 2. *Let $0<\delta<1$ and $p\geqslant 1$ such that*

$$\alpha=\delta-1/p>0.$$

Set $q=1/\delta$. Then the q-variation of any $h\in W^{\delta,p}$ is controlled by a constant multiple of the control function

$$\omega(s,t)=|h|^q_{W^{\delta,p};[s,t]}(t-s)^{\alpha q},\quad 0\leqslant s\leqslant t\leqslant 1,$$

and we have the continuous embeddings

$$W^{\delta,p}\subset C^{\alpha\text{-Hölder}}\quad\text{and}\quad W^{\delta,p}\subset C^{q\text{-var}}.$$

Proof. We have

$$|h|^p_{W^{\delta,p};[s,t]}\equiv F_{s,t}=\iint_{[s,t]^2}\frac{|h_{u,v}|^p}{|v-u|^{1+\delta p}}\,du\,dv=\iint_{[s,t]^2}\left(\frac{|h_{u,v}|}{|v-u|^{1/p+\delta}}\right)^p du\,dv.$$

The Garsia–Rodemich–Rumsey lemma with $\Psi(\cdot)=(\cdot)^p$ and $p(\cdot)=(\cdot)^{1/p+\delta}$ yields

$$
\begin{aligned}
|h_{s,t}|&\leqslant C\int_0^{t-s}\left(\frac{F_{s,t}}{u^2}\right)^{1/p}dp(u)=C|h|_{W^{\delta,p};[s,t]}\int_0^{t-s}u^{-2/p}\,dp(u)\\
&=C|h|_{W^{\delta,p};[s,t]}\int_0^{t-s}u^{-1/p+\delta-1}\,du=C|h|_{W^{\delta,p};[s,t]}(t-s)^{\delta-1/p},
\end{aligned}
$$

using $\alpha\equiv\delta-1/p>0$. We now show that the q-variation of h is controlled by the control function

$$\omega(s,t):=|h|^q_{W^{\delta,p};[s,t]}(t-s)^{\alpha q},\quad t\geqslant s.$$

Only super-additivity, $\omega(s,t)+\omega(t,u)\leqslant\omega(s,u)$ with $s\leqslant t\leqslant u$, is non-trivial. Note that $p/q=1/(p\alpha+1)\in(1,\infty)$. From Hölder's inequality with conjugate exponents p/q and $p/(p-q)$ we obtain

$$\begin{aligned}&|h|^q_{W^{\delta,p};[s,t]}(t-s)^{q\alpha}+|h|^q_{W^{\delta,p};[t,u]}(u-t)^{q\alpha}\\&\quad\leqslant\bigl(|h|^p_{W^{\delta,p};[s,t]}+|h|^p_{W^{\delta,p};[t,u]}\bigr)^{q/p}\bigl[(t-s)^{q\alpha\frac{p}{p-q}}+(u-t)^{q\alpha\frac{p}{p-q}}\bigr]^{(p-q)/p}.\end{aligned}$$

The first factor is easily estimated:

$$\bigl(|h|^p_{W^{\delta,p};[s,t]}+|h|^p_{W^{\delta,p};[t,u]}\bigr)^{q/p}\leqslant|h|^q_{W^{\delta,p};[s,u]}.$$

To estimate the second factor note that the exponent of $t-s$, respectively $u-t$, equals one; indeed

$$q\alpha\frac{p}{p-q}=1\quad\Longleftrightarrow\quad q=\frac{p}{p\alpha+1}$$

and the second factor equals

$$(u-s)^{(p-q)/p}=(u-s)^{q\alpha}.$$

This shows that ω is super-additive and we conclude that for any $0\leqslant a<b\leqslant 1$,

$$|h|_{q\text{-var};[a,b]}\leqslant C\omega(a,b)^{1/q}=C|b-a|^{\alpha}|h|_{W^{\delta,p};[a,b]}.$$

In particular, we have established continuity of the embeddings

$$W^{\delta,p}\subset C^{\alpha\text{-Hölder}}\quad\text{and}\quad W^{\delta,p}\subset C^{q\text{-var}}.\qquad\Box$$

The case $p=2$ deserves special attention. The assumptions of Theorem 2 are then satisfied for any $\delta\in(1/2,1)$.

Remark 1. In [7], Kusuoka discusses differentiability of SDE solution beyond the usual Malliavin sense. In particular, he shows the existence of a nice version of the Itô-map which has derivatives in directions $W_0^{\delta,2}\supset W_0^{1,2}$ for $\delta\in(1/2,1)$. Since $W_0^{\delta,2}\subset C^{q\text{-var}}$ with $q=1/\delta<2$ this result is now explained by Lyons' theory of rough paths [8,9]. Note that in Lyons' continuity statements the modulus ω is preserved. This implies that after perturbation a Brownian path in a $W_0^{\delta,2}$-direction the solution maintains α-Hölder regularity with $\alpha=\delta-1/2$. (Clearly, this is not true for an arbitrary perturbation in $C^{q\text{-var}}$!) We can then extend Gateaux-differentiability to suited $W_0^{\delta,p}$-spaces as long as $\delta-1/p>0$ and even apply this to rough path differential equations driven by enhanced fBM. We note that Kusuoka's full statement is on Fréchet-smoothness in starting point and perturbations in $W_0^{\delta,2}$. It should be possible to recover this by a careful application of Lyons' universal limit theorem, noting that all estimates are uniform over bounded sets, but this is not the aim of this paper. (In [10] smoothness in starting point and perturbations is discussed separately.)

Remark 2. Integrals of form $\int f\,dg$ for $f,g\in W^{\delta,2}$ are discussed in [12]. Theorem 2 reveals them as normal Young-integral. Following [10] its continuity properties of $(f,g)\mapsto\int f\,dg$ are conveniently expressed in terms of the modulus ω. In particular, the modulus of continuity of the indefinite integral $\int f\,dg$ is immediately controlled by the $W^{\delta,2}$-Sobolev-norms of f and g and we can easily extend this to $W^{\delta,p}$ provided $\delta - 1/p > 0$. On the other hand, we have no control of the $W^{\delta,2}$-norm of the indefinite integral.

Remark 3. When $\delta < 1$ the notion of $W^{\delta,p}$ makes perfect sense for paths with values in a metric space (E,d). Theorem 2 still holds with the same proof.[3] The case of the free step-N nilpotent group $(G^N(\mathbb{R}^d),\otimes)$ with Carnot–Caratheodory norm $\|\cdot\|$ and distance $d(x,y)=\|x^{-1}\otimes y\|$ is of particular importance: Theorem 2 is a criterion for variation and Hölder regularity of a $G^N(\mathbb{R}^d)$-valued path, a fundamental aspect in Lyons' theory of rough paths [8]. To illustrate the idea we give a simple application to enhanced Brownian motion $\mathbf{B}$, see [4,5]. Then[4]

$$\mathbb{E}\|\mathbf{B}\|^p_{W^{\delta,p};[0,1]} = \iint_{[0,1]^2} \frac{\mathbb{E}\|\mathbf{B}_{s,t}\|^p}{|t-s|^{1+\delta p}}\,ds\,dt = \mathbb{E}\|\mathbf{B}_{0,1}\|^p \iint_{[0,1]^2} |t-s|^{p/2-1-\delta p}\,ds\,dt.$$

For every $\alpha < 1/2$ and $\delta \in (\alpha, 1/2)$ there exists $p_0(\delta)$ such that for all $p \geqslant p_0$ the double integral is bounded by 1. Thus for all p large enough,

$$\mathbb{E}\|\mathbf{B}\|^p_{W^{\delta,p};[0,1]} \leqslant \mathbb{E}\|\mathbf{B}_{0,1}\|^p.$$

Is is well known, [5], that $\|\mathbf{B}_{0,1}\|$ has a Gaussian tail and it follows that $\|\mathbf{B}\|_{W^{\delta,p}}$ has a Gaussian tail, provided $p \geqslant p_0(\delta)$. For p large enough we have $\alpha \leqslant \delta - 1/p$ and we conclude that $\|\mathbf{B}\|_{\alpha\text{-Hölder}}$ has a Gaussian tail, too. For a direct proof see [4]. Note that the law of $\mathbf{B}$ is not Gaussian and there are no Fernique-type results. Finally, a similar proof can be given for enhanced fractional Brownian motion.

Remark 4. Potential spaces, see [2] and the references therein, are a popular alternative to fractional Sobolev spaces. But only the latter adapt easily to (E,d)-valued paths as required in rough path analysis.

Remark 5. The $W^{\delta,p}$-embedding of Theorem 2 has two different regimes:

(1) For p large one has $q = 1/\delta \sim 1/\alpha$. Since every α-Hölder path has finite $1/\alpha$-variation (the converse not being true) one can forget about q-variation.
(2) When p is small, the variation parameter $q = 1/\delta$ can be considerably smaller than $1/\alpha$ and q-variation is an essential part of the regularity. Elementary examples show that q-variation does not imply any Hölder regularity and therefore one should not forget about α-Hölder regularity. The fractional Sobolev space $W^{\delta,p}$ respectively the modulus ω are tailor-made to keep track of both regularity aspects. Finally, we note that any finite $1/\delta$-variation path can be reparametrized to a δ-Hölder path. In comparison, without reparametrization one has only Hölder regularity of exponent $\alpha = \delta - 1/p$.

[3] Simply write $h_{s,t} \equiv d(h_s,h_t)$ and note that the Garsia–Rodemich–Rumsey lemma works for (E,d)-valued continuous functions.

[4] Note $\|\mathbf{B}_{s,t}\| \overset{\mathcal{D}}{=} |t-s|^{1/2}\|\mathbf{B}_{0,1}\|$.

3. Cameron–Martin space of fBM

We consider fractional Brownian motion with $H \in (0, 1/2)$. Call $\mathcal{H}^H$ the associated Cameron–Martin space.

Theorem 3. *Let* $1/2 < \delta < H + 1/2$. *Then* $\mathcal{H}^H \Subset W_0^{\delta,2}$.

Proof. From [2] and the references therein we know that $\mathcal{H}^H$ is continuously embedded in the potential space $I^+_{H+1/2,2}$ which we need not define here. Then, [2,3], $I^+_{H+1/2,2} \subset W^{\delta,2}$ so that

$$\mathcal{H}^H \subset W^{\delta,2}. \tag{3.1}$$

The compact embeddings is obtained by a standard squeezing argument: replace δ by $\tilde{\delta} \in (\delta, H + 1/2)$, repeat the argument for $\tilde{\delta}$ and then use (1.1). $\square$

Corollary 1. *For* $\alpha \in (0, H)$ *and* $1/(H + 1/2) < q < \infty$ *we have*

$$\mathcal{H}^H \Subset C^{\alpha\text{-Hölder}}, \qquad \mathcal{H}^H \Subset C^{q\text{-var}}.$$

Remark 6. From $\mathcal{H}^H \subset I^+_{H+1/2,2}$ it follows that $\mathcal{H}^H \subset C^{H\text{-Hölder}}$, this is well known [2].

Remark 7. For any $H \in (0, 1/2)$ we can find $1/(H + 1/2) < q < 2$. This has useful consequences. For instance, for $h, g \in \mathcal{H}^H$ that integral $\int h\,dg$ makes sense as classical Young integral with all its continuity properties. In particular, the lift of $h \in \mathcal{H}^H$ to a geometric p-rough paths $p > 1/H$, see [11], is well defined and convergence of piecewise-linear approximations, uniformly over bounded sets in $\mathcal{H}^H$, is an easy consequence. Such a result leads to a quick proof of a large deviations principle for enhanced fractional Brownian motion, see [6] for details.

Appendix A

The proof of (3.1) appears somewhat spread out in the references. We present a direct argument which avoids potential spaces and fractional calculus and extends to other Volterra kernels.[5]

Step 1. $\mathcal{H}^H$ is the image of $L^2[0, 1]$ under the integral operator $K = K_1 + K_2$ where

$$K_1(t, s) = (t - s)^{H-1/2},$$

$$K_2(t, s) = s^{H-1/2} F_1(t/s), \quad F_1 = \int_0^{(\cdot)-1} u^{H-3/2}\big(1 - (u + 1)^{H-1/2}\big),$$

for $s < t$. Set $h_i = K_i g \equiv \int_0^{\cdot} K_i(\cdot, s) g(s)\,ds$ with $g \in L^2[0, 1]$, $i = 1, 2$.

Step 2. An elementary computation shows

[5] For instance, every kernel for which one can get estimates as those in Step 2 will lead to a fractional Sobolev embedding.

$$\sup_{u\in[0,1]} \int_0^{1-t} \left|K_1(s+t,u) - K_1(s,u)\right| ds = O\left(t^{H+1/2}\right),$$

$$\sup_{s\in[0,1-t]} \int_0^{1} \left|K_1(s+t,u) - K_1(s,u)\right| du = O\left(t^{H+1/2}\right).$$

From Cauchy–Schwartz and trivial sup-estimates,

$$(*) := \int_{s=0}^{1-t} \left|h_1(s+t) - h(s)\right|^2 ds = |g|_{L^2}^2 \cdot O\left(t^{1+2H}\right).$$

The $W^{\delta,2}$-norm of h_1 is equivalent to $\int dt(*)/t^{1+2\delta}$ which is less than $C|g|_{L^2}^2$ provided $1+2H-(1+2\delta) > -1$ and this happens precisely for $\delta < H + 1/2$.

Step 3. A straight-forward computation shows (one can assume $g \in C^1 \cap L^2$ for the computation) that $|\dot{h}_2| < C|g|_{L^2}^2$ provided $p < 1/(1-H)$ and hence $h_2 \in W^{1,p}$. From (1.2), $W^{1,1/(1-H)} \subset W^{H+1/2,2}$. Similarly, given $\delta < H + 1/2$ we can find $p < 1/(1-H)$, close enough to $1/(1-H)$ so that $W^{1,p} \subset W^{\delta,2}$.

References

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[2] L. Decreusefond, Stochastic calculus with respect to Volterra processes, Ann. Inst. Poincaré (2004).
[3] D. Feyel, A. de La Pradelle, On fractional Brownian processes, Potential Anal. 10 (3) (1999) 273–288.
[4] P. Friz, T. Lyons, D. Stroock, Lévy's area under conditioning, Ann. Inst. Poincaré Probab. Statist. 42 (1) (2006) 89–101.
[5] P. Friz, N. Victoir, Approximations of the Brownian rough path with applications to stochastic analysis, Ann. Inst. Poincaré Probab. Statist. 41 (4) (2005) 703–724.
[6] P. Friz, N. Victoir, Large deviation principle for enhanced Gaussian processes, preprint, 2005.
[7] S. Kusuoka, On the regularity of solutions to SDEs, in: K.D. Elworthy, N. Ikeda (Eds.), Asymptotic Problem in Probability Theory: Wiener Functionals and Asymptotics, in: Pitman Res. Notes Math. Ser., vol. 284, Longman Scientific, New York, 1993, pp. 90–106.
[8] T. Lyons, Differential equations driven by rough signals, Rev. Mat. Iberoamericana 14 (2) (1998) 215–310.
[9] T. Lyons, Z. Qian, Calculus of variation for multiplicative functionals, in: New Trends in Stochastic Analysis, Charingworth, 1994, World Scientific, River Edge, NJ, 1997, pp. 348–374.
[10] T. Lyons, Z. Qian, System Control and Rough Paths, Oxford Univ. Press, Oxford, 2002.
[11] A. Millet, M. Sanz-Solé, Large deviations for rough paths of the fractional Brownian motion, Ann. Inst. Poincaré Probab. Statist., in press. Available online 17 June 2005.
[12] M. Zähle, Integration with respect to fractal functions and stochastic calculus. II, Math. Nachr. 225 (2001) 145–183.