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Thickened renewal processes

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Let $\{t_i, i \geqslant 0\}$ be an ordinary renewal process and assume the lifetime distribution function has the form $F(x) = cx^\alpha(1+\lambda x + o(x))$ near 0+. The asymptotic conditional distribution as $n \to \infty$ of $\{nt_i, i \geqslant 0\}$, given that $t_n \leqslant 1$, is that of a renewal process with a gamma lifetime distribution depending only on α.

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renewal processes * conditioned weak limits * busy periods

1. Introduction and the main theorem

Let $\{t_n, n \geqslant 0\}$ be an ordinary renewal process on $[0, \infty)$ with associated distribution function F, that is, $t_0 \equiv 0$, and $t_n = \sum_{i=1}^n \theta_i$ for $n > 0$, where the 'lifetimes' $\theta_1, \theta_2, \ldots$ are independent positive random variables with common distribution function F. We assume throughout that

$$F(0) = 0 \quad \text{and} \quad F(t) > 0 \quad \text{for all } t > 0. \tag{1.1}$$

Our main result concerns the limiting behaviour of the renewal process on the time interval $(0, b]$, given that the number of renewals in $(0, b]$, denoted by

$$\eta := \max\{n: t_n \leqslant b\}, \tag{1.2}$$

goes to infinity. We refer to this conditioning process as 'thickening', to contrast it with the more usual operation of thinning. It is clear that the θ_i's cannot be independent under a condition of the form $\eta \geqslant n$ or $\eta = n$. Nevertheless, asymptotically, they *are* independent and furthermore they have a gamma distribution, provided F is nicely behaved near 0+. For the record, we say a random variable X has a $\Gamma(\alpha, \beta)$ distribution if the distribution of X has probability density function (p.d.f.)

$$f(x) := (\Gamma(\alpha))^{-1}\beta^\alpha x^{\alpha-1}\,\mathrm{e}^{-x\beta} \quad \text{for } x > 0. \tag{1.3}$$

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Our exact result is:

Theorem 1. *Suppose F has the form*

$$F(x)=cx^{\alpha}\{1+\lambda x+\mathrm{o}(x)\} \quad as\ x\downarrow 0, \tag{1.4}$$

where $\alpha>0$, $c>0$ *and* $\lambda\in\mathbb{R}$. *Then for every* $k\in\mathbb{N}$ *and* $x_1\geqslant 0, x_2\geqslant 0,\dots,x_k\geqslant 0$,

$$\mathbb{P}\left[\bigcap_{i=1}^{k}\{n\theta_i\leqslant x_i\}\,\middle|\,\eta\geqslant n\right]\to\prod_{i=1}^{k}\mathbb{P}[\sigma_i\leqslant x_i] \quad as\ n\to\infty, \tag{1.5}$$

where $\sigma_1,\sigma_2,\dots$ *are independent random variables with a* $\Gamma(\alpha,b^{-1}\alpha)$ *distribution. The same result holds if "*$\eta\geqslant n$*" is replaced by "*$\eta=n$*" in* (1.5).

By making a linear change of variables, we see that it suffices to consider the case $b=1$; henceforth we assume this holds.

The gamma renewal processes (by which we mean renewal processes with gamma lifetimes) are the only nondegenerate processes which we know can arise as limits of the type used here. At the end of the paper, we do show that for some F we get the degenerate limit for which $\mathbb{P}[\sigma_i=1]=1$ for all i.

If (1.4) holds, with $\alpha=1$, then the limiting lifetime distribution is exponential, so the renewal process is Poisson. It is noteworthy that every other gamma renewal process is also a possible limit (with $b=1$, we must have $E\sigma_i=1$).

A key ingredient in the proof of Theorem 1 is a collection of results taken from Section 2 of Doney and O'Brien (1991), hereafter referred to as "the shot noise paper". We present these results in Section 2, using the notation of this paper. They are interesting in their own right in the renewal theory context.

We prove Theorem 1 in Section 3 and add some remarks in Section 4.

2. Results from the shot noise paper

The shot noise process is constructed by summing the contributions associated with each 'shot' or renewal, via a function h. If $h=1_{[0,1)}$, then the process just counts the number of renewals in an interval of length 1, so that statements about shot noise can be expressed in terms of the renewal process. For readers who refer to the shot noise paper we note that the quantities $N(x)$, $S(x)$ and ξ from that paper have the following interpretations here (with $h=1_{[0,1)}$): $N(x)=\lfloor x\rfloor$ if $t_{\lfloor x\rfloor}\leqslant 1$, $N(x)=\infty$ otherwise; $S(x)=t_{\lfloor x\rfloor}$ if $t_{\lfloor x\rfloor}\leqslant 1$, $S(x)=\infty$ otherwise, and $\xi=\eta+1$. With this translation, it is possible to read most of Section 2 of the shot noise paper without reference to the other sections.

The proofs of Section 2 of the shot noise paper mostly become only a little simpler in the renewal theory context, and we do not want to reproduce them here. The main exception is Lemma 1 which is implied by the following obvious statement about renewal processes: for any $y>0$ and conditional on $t_n\leqslant y$, $\theta_1,\theta_2,\dots,\theta_n$ have the same distributions. (This fact may be compared with the so-called inspection paradox and its variations, as discussed in Kremers (1988).)

The following two results are translations of Lemmas 3 and 4 in the shot noise paper. They assert that if $t_n \leqslant y$ for large n, then most likely t_n is close to y and most likely $\theta_1, \theta_2, \ldots, \theta_n$ are all small. These results are certainly plausible but, it seems, rather tricky to prove.

Lemma 1. *For each* $\gamma > 0$,

$$\mathbb{P}[t_n \leqslant y-\gamma \mid t_n \leqslant y] \to 0 \quad \textit{as } n \to \infty, \tag{2.1}$$

uniformly over $y \in (0, 1]$. □

Lemma 2. *Let* $B_n := \max\{\theta_1, \ldots, \theta_n\}$. *Then for each fixed* $t > 0$,

$$\mathbb{P}[B_n > t \mid t_n \leqslant y] \to 0 \quad \textit{as } n \to \infty, \tag{2.2}$$

uniformly over $y \in (0, 1]$. □

Another result of the shot noise paper which is of interest here is that the tail of the distribution of η gets small very rapidly in the following sense:

$$\lim_{n\to\infty} \mathbb{P}[\eta > n+1 \mid \eta \geqslant n] = 0. \tag{2.3}$$

To see this, note that the condition $\eta \geqslant n$ is equivalent to the condition $t_n \leqslant 1$. By Lemma 1, t_n is therefore most likely close to 1 for large n. Since $t_{n+1} - t_n$ and $\{t_1, \ldots, t_n\}$ are independent this means that t_{n+1} probably exceeds 1.

As a final remark, we note that Lemmas 1 and 2 and formula (2.3) also hold for stationary renewal processes.

3. Proof of Theorem 1

We will use the following notation:

$$\bar{\lambda} := \lambda(\alpha+1)\alpha^{-1}, \quad \bar{c} := \alpha c \Gamma(\alpha), \quad F_n(y) := \mathbb{P}[t_n \leqslant y]. \tag{3.1}$$

The first major step is to obtain a good approximation for $F_n(y)$ for large n.

Lemma 3. *Under the hypothesis* (1.4),

$$F_n(y) \sim e^{\bar{\lambda} y}(\bar{c} y^{\alpha})^n / \Gamma(n\alpha+1) \quad \textit{as } n \to \infty, \tag{3.2}$$

uniformly over $y \in [0, 1]$.

Proof. Fix $\varepsilon > 0$. For $i = 1, 2$ and $x > 0$, let

$$f^{(i)}(x) = \bar{c}\, e^{(1+\bar{\lambda}+\varepsilon_i)x} g(x), \tag{3.3}$$

where g is the p.d.f. of the $\Gamma(\alpha, 1)$ distribution and $\varepsilon_i = (-1)^i\varepsilon$. Let

$$F^{(i)}(x) = \int_0^x f^{(i)}(y)\,\mathrm{d}y, \quad i = 1, 2. \tag{3.4}$$

Integrating by parts twice and incorporating (3.1), we obtain

$$\begin{aligned} F^{(i)}(x) &= \bar{c}(\alpha\Gamma(\alpha))^{-1}\Big[x^\alpha\,\mathrm{e}^{(\bar\lambda+\varepsilon_i)x}\{1-(\bar\lambda+\varepsilon_i)x/(\alpha+1)\} \\ &\qquad + (\bar\lambda+\varepsilon_i)^2(\alpha+1)^{-1}\int_0^x \mathrm{e}^{(\bar\lambda+\varepsilon_i)y}y^{\alpha+1}\,\mathrm{d}y\Big] \\ &= cx^\alpha\,\mathrm{e}^{\alpha(\bar\lambda+\varepsilon_i)x/(\alpha+1)}(1+\mathrm{o}(x)) \\ &= cx^\alpha\,\mathrm{e}^{\lambda x}[1+\varepsilon_i\alpha x(\alpha+1)^{-1}+\mathrm{o}(x)], \end{aligned}$$

as $x \to 0+$. A comparison of this formula with (1.4) yields the existence of an $x_0 > 0$ such that

$$F^{(1)}(x) \leqslant F(x) \leqslant F^{(2)}(x) \quad \text{for } 0 \leqslant x \leqslant x_0. \tag{3.5}$$

By reducing x_0 if necessary, we also have $F^{(i)}$, $i = 1, 2$, nondecreasing and $[0, 1]$-valued in $[0, x_0]$. Now define distribution functions (d.f.'s).

$$\hat{F}^{(1)} \leqslant \hat{F}^{(0)} \leqslant \hat{F}^{(2)}, \tag{3.6}$$

which are equal to $F^{(1)}, F$ and $F^{(2)}$ respectively on $[0, x_0)$ and are all identically one on $[x_0, \infty)$.

For $i = 0, 1, 2$ we may by (3.6) construct sequences $\{\theta_n^{(i)}, n \geqslant 1\}$ of independent random variables all with d.f. $\hat{F}^{(i)}$ and with the three sequences coupled in such a way that

$$\theta_n^{(1)} \geqslant \theta_n^{(0)} \geqslant \theta_n^{(2)} \quad \text{for all } n.$$

Let $\{t_n^{(i)}, n \geqslant 0\}$ be the three corresponding renewal processes and let

$$B_n^{(i)} := \max\{\theta_1^{(i)}, \ldots, \theta_n^{(i)}\}, \quad i = 0, 1, 2, \ n \geqslant 1.$$

Fix $x_1 \in (0, x_0)$. We then have

$$\begin{aligned} \mathbb{P}[t_n^{(1)} \leqslant y, B_n^{(1)} \leqslant x_1] &\leqslant \mathbb{P}[t_n^{(0)} \leqslant y, B_n^{(0)} \leqslant x_1] \\ &= \mathbb{P}[t_n \leqslant y, B_n \leqslant x_1] \\ &\leqslant \mathbb{P}[t_n^{(2)} \leqslant y, B_n^{(2)} \leqslant x_1]. \end{aligned} \tag{3.7}$$

Now choose $\gamma > 0$ such that

$$\mathrm{e}^{-(1+\bar\lambda-\varepsilon)\gamma} \geqslant 1-\varepsilon \quad \text{and} \quad \mathrm{e}^{-(1+\bar\lambda+\varepsilon)\gamma} \leqslant (1-\varepsilon)^{-1}. \tag{3.8}$$

Also, let

$$\begin{aligned} A &:= A(n, x) \\ &:= \{(s_1, \ldots, s_{n-1}) \in [0, x_1]^{n-1} : x - x_1 \leqslant s_1 + \cdots + s_{n-1} \leqslant x\}. \end{aligned}$$

Finally, let $\{t_n^*, n \geqslant 0\}$ be the renewal process whose lifetimes $\theta_n^*, n \geqslant 1$, have a $\Gamma(\alpha, 1)$ distribution and let

$$B_n^* := \max\{\theta_1^*, \ldots, \theta_n^*\}.$$

By (3.7) and (3.8),

$$\begin{aligned}
F_n(y) &\geqslant \mathbb{P}[t_n^{(1)} \leqslant y, B_n^{(1)} \leqslant x_1] \\
&\geqslant \mathbb{P}[y-\gamma \leqslant t_n^{(1)} \leqslant y, B_n^{(1)} \leqslant x_1] \\
&= \int_{y-\gamma}^{y} \Bigg[\int \cdots \int_A \bar{c}^n \, \mathrm{e}^{(1+\bar{\lambda}-\varepsilon)(s_1+\cdots+s_{n-1})} g(s_1) \cdots g(s_{n-1}) \\
&\qquad \cdot \mathrm{e}^{(1+\bar{\lambda}-\varepsilon)(x-s_1-\cdots-s_{n-1})} g(x-s_1-\cdots-s_{n-1}) \, \mathrm{d}s_1 \cdots \mathrm{d}s_{n-1} \Bigg] \mathrm{d}x \\
&= \bar{c}^n \, \mathrm{e}^{(1+\bar{\lambda}-\varepsilon)y} \int_{y-\gamma}^{y} \mathrm{e}^{(1+\bar{\lambda}-\varepsilon)(x-y)} \\
&\qquad \cdot \Bigg[\int \cdots \int_A g(s_1) \cdots g(s_{n-1}) \\
&\qquad \cdot g(x-s_1-\cdots-s_{n-1}) \, \mathrm{d}s_1 \cdots \mathrm{d}s_{n-1} \Bigg] \mathrm{d}x \\
&\geqslant \bar{c}^n \, \mathrm{e}^{(1+\bar{\lambda}-\varepsilon)y} (1-\varepsilon) \mathbb{P}[y-\gamma \leqslant t_n^* \leqslant y, B_n^* \leqslant x_1]. \qquad (3.9)
\end{aligned}$$

This is valid even if $y < \gamma$. We now make use of Lemmas 1 and 2 for $\{t_n^*, n \geqslant 0\}$ to obtain the bound

$$F_n(y) \geqslant \bar{c}^n \, \mathrm{e}^{(1+\bar{\lambda}-\varepsilon)y} (1-\varepsilon)^2 \mathbb{P}[t_n^* \leqslant y], \qquad (3.10)$$

for $n \geqslant N$, where N may be chosen independently of $y \in [0, 1]$. By Lemma 2, we also have

$$\begin{aligned}
F_n(y) &= \mathbb{P}[t_n \leqslant y, B_n \leqslant x_1] + \mathbb{P}[t_n \leqslant y, B_n > x_1] \\
&\leqslant \mathbb{P}[t_n \leqslant y, B_n \leqslant x_1] + \varepsilon F_n(y),
\end{aligned}$$

for large enough n. By (3.7), Lemma 1 and (3.8), we now obtain as in (3.9) that

$$\begin{aligned}
F_n(y) &\leqslant (1-\varepsilon)^{-1} \mathbb{P}[t_n \leqslant y, B_n \leqslant x_1] \\
&\leqslant (1-\varepsilon)^{-2} \mathbb{P}[y-\gamma \leqslant t_n^{(2)} \leqslant y, B_n^{(2)} \leqslant x_1] \\
&\leqslant \bar{c}^n \, \mathrm{e}^{(1+\bar{\lambda}+\varepsilon)y} (1-\varepsilon)^{-3} \mathbb{P}[y-\gamma \leqslant t_n^* \leqslant y, B_n^* \leqslant x_1] \\
&\leqslant \bar{c}^n \, \mathrm{e}^{(1+\bar{\lambda}+\varepsilon)y} (1-\varepsilon)^{-3} \mathbb{P}[t_n^* \leqslant y], \qquad (3.11)
\end{aligned}$$

again for all $n \geqslant N$ where N may be chosen independently of $y \in [0, 1]$. Integrating by parts, we now calculate

$$\begin{aligned}
\mathbb{P}[t_n^* \leqslant y] &= (\Gamma(n\alpha))^{-1} \int_0^y x^{n\alpha-1} \, \mathrm{e}^{-x} \, \mathrm{d}x \\
&= (\Gamma(n\alpha+1))^{-1} \left[y^{n\alpha} \, \mathrm{e}^{-y} + \int_0^y \mathrm{e}^{-x} x^{n\alpha} \, \mathrm{d}x \right] \\
&= (\Gamma(n\alpha+1))^{-1} y^{n\alpha} \, \mathrm{e}^{-y} [1 + f(n, y)],
\end{aligned}$$

say, where $0 \leqslant f(n, y) \leqslant y^{-n\alpha} e^{y} y^{n\alpha+1}(n\alpha+1)^{-1} \to 0$ uniformly over $y \in [0, 1]$ as $n \to \infty$. Substituting this into (3.10) and (3.11), we obtain (3.2). □

Lemma 4. *For fixed k and d,*

$$\frac{F_{n-k}(1-z/n)}{F_n(1)} \sim (\bar{c})^{-k} e^{-\alpha z}(n\alpha)^{\alpha k} \quad \textit{as } n \to \infty,$$

uniformly over $z \in [0, d]$.

Proof. Applying Lemma 3 and Stirling's formula,

$$\begin{aligned}\frac{F_{n-k}(1-z/n)}{F_n(1)} &\sim \frac{e^{\bar{\lambda}(1-z/n)}\{\bar{c}(1-z/n)^{\alpha}\}^{n-k}\Gamma(n\alpha+1)}{e^{\bar{\lambda}}\bar{c}^{n}\Gamma((n-k)\alpha+1)} \\ &\sim \frac{\Gamma(n\alpha+1)}{\Gamma((n-k)\alpha+1)} \frac{e^{-\alpha z}}{\bar{c}^{k}} \\ &\sim (n\alpha)^{k\alpha} e^{-\alpha z}(\bar{c})^{-k}. \qquad \square\end{aligned}$$

Proof of Theorem 1. We are now in position to complete the proof of Theorem 1. By Lemma 4, we have

$$\begin{aligned}&\mathbb{P}\left[\bigcap_{i=1}^{k}\{n\theta_i \leqslant x_i\} \,\Big|\, t_n \leqslant 1\right] \\ &= (F_n(1))^{-1}\int_0^{n^{-1}x_1}\cdots\int_0^{n^{-1}x_k} \mathbb{P}[t_n \leqslant 1 \,|\, \theta_1 = y_1, \ldots, \theta_k = y_k] \\ &\qquad \cdot F(\mathrm{d}y_1)\cdots F(\mathrm{d}y_k) \\ &= (F_n(1))^{-1}\int_0^{x_1}\cdots\int_0^{x_k} F_{n-k}(1-n^{-1}(z_1+\cdots+z_k)) \\ &\qquad \cdot F(\mathrm{d}(n^{-1}z_1))\cdots F(\mathrm{d}(n^{-1}z_k)) \\ &\sim (n\alpha)^{k\alpha}(\alpha c\Gamma(\alpha))^{-k}\int_0^{x_1}\cdots\int_0^{x_k} e^{-\alpha(z_1+\cdots+z_k)} \\ &\qquad \cdot F(\mathrm{d}(n^{-1}z_1))\cdots F(\mathrm{d}(n^{-1}z_k)) \\ &= \prod_{i=1}^{k}\left\{(n\alpha)^{\alpha}(\alpha c\Gamma(\alpha))^{-1}\int_0^{x_i} e^{-\alpha z_i}F(\mathrm{d}(n^{-1}z_i))\right\}. \end{aligned} \tag{3.12}$$

Integrating by parts twice, we see from (1.4) that

$$\begin{aligned}\int_0^{x} e^{-\alpha z}F(\mathrm{d}(n^{-1}z)) &= e^{-\alpha x}F(n^{-1}x) + \alpha\int_0^{x} e^{-\alpha z}F(n^{-1}z)\,\mathrm{d}z \\ &\sim e^{-\alpha x}cn^{-\alpha}x^{\alpha} + \alpha cn^{-\alpha}\int_0^{x} e^{-\alpha z}z^{\alpha}\,\mathrm{d}z \\ &= \alpha cn^{-\alpha}\int_0^{x} z^{\alpha-1}e^{-\alpha z}\,\mathrm{d}z.\end{aligned}$$

Thus, the ith factor on the right side of (3.12) is

$$\sim\alpha^{\alpha}(\Gamma(\alpha))^{-1}\int_0^{x_i} z_i^{\alpha-1}\,e^{-\alpha z_i}\,dz_i,$$

which is the d.f. of the $\Gamma(\alpha, \alpha)$ distribution, as required to prove the first conclusion of Theorem 1. The last statement of Theorem 1 now follows by (2.3). □

4. Complements

There are many F which satisfy (1.1) but do not satisfy (1.4). There is not much we can say about whether the left side of (1.5) has a limit in these cases nor, if it does, about what the limit is. Noting that the limit in (1.5) is independent of λ, one is tempted to conjecture that the theorem might extend to the case $F(x)=cx^{\alpha}(1+O(x))$ as $x\to 0+$, but we have no idea if this is true. Our proof depends heavily on the fact that F can be approximated by a gamma distribution function (if $\lambda<0$) or something similar, in order that we may approximate the n-fold convolution of F with itself for large n.

The one other case where we do have some results is when $F(x)\to 0$ very fast as $x\to 0+$. We have not attempted to give the most general result possible, opting instead for the following result because of its simple proof.

Theorem 2. *Suppose F satisfies*

$$nF(n^{-1}-\varepsilon n^{-2})=o((F(n^{-1}))^n), \tag{4.1}$$

as $n\to\infty$, for all $\varepsilon>0$. Then the conditional distribution of each $n\theta_i$, given $\eta\geq n$, *converges to the degenerate distribution with unit mass at* 1.

Proof. Let $\varepsilon>0$. If $\eta\geq n$ and $\theta_i>(1+\varepsilon)n^{-1}$ for some $n\geq i$, then at least one of $\theta_1, \theta_2, \ldots, \theta_n$ must be less than $n^{-1}-\varepsilon n^{-2}$. Also,

$$\begin{aligned}\mathbb{P}[\text{any of } \theta_1,\ldots,\theta_n<n^{-1}-\varepsilon n^{-2}]&\leq nF(n^{-1}-\varepsilon n^{-2})\\&=o(\mathbb{P}[\theta_1\leq n^{-1},\ldots,\theta_n\leq n^{-1}]).\end{aligned}$$

This inequality is maintained when the probabilities are both conditioned on $\eta\geq n$. It follows that

$$\mathbb{P}[n\theta_i>1+\varepsilon\,|\,\eta\geq n]\to 0.$$

Also, (4.1) implies

$$\mathbb{P}[n\theta_i<1-\varepsilon\,|\,\eta\geq n]\to 0. \qquad \square$$

As an example, suppose the distribution of θ_i has support $\{n^{-1}: n=1,2,\ldots\}$ and that $p_n := \mathbb{P}[\theta_i = n^{-1}]$ satisfies

$$np_{n+1} = \mathrm{o}(p_n^n).$$

Then the degenerate limit of Theorem 2 is obtained.

References

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