Asymptotically Most Powerful Rank Tests for Multivariate Randomness against Serial Dependence

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A class of linear serial multirank statistics is introduced for the problem of testing the null hypothesis that a multivariate series of observations is white noise (with unspecified density function) against alternatives of ARMA dependence. The asymptotic distributional properties of these statistics are investigated, both under the null as well as local alternative hypotheses. These statistics are shown to provide permutationally distribution-free tests that are asymptotically most powerful against specified local alternatives of ARMA dependence. In particular, a test of the van der Waerden type is shown to be asymptotically as powerful as the corresponding normal theory parametric test, based on classical sample autocovariances.

INTRODUCTION

The problem of testing for white noise against serial dependence is perhaps one of the most fundamental in statistical inference. The assumption of serial independence plays a crucial role indeed in most classical statistical procedures and, whenever this independence assumption is relaxed (e.g., in time series analysis), a number of hypotheses of interest (typically, all those appearing in model fitting problems) are closely related to hypotheses of white noise, to which they reduce after some adequate manipulations.

However, in spite of the importance and long history of the problem, in spite of the tremendous development of time series analysis, and in spite of the recognized need for robust and nongaussian procedures in this latter domain, the introduction of rank-based methods in the area has received little attention (for a commented bibliography, see Hallin et al. [5]).

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In recent papers by the authors (Hallin et al. [5–7] and Hallin and Puri [9]), optimal rank-based procedures are derived for testing univariate white noise against alternatives of ARMA dependence and for the problem of testing a specified univariate ARMA model against other ARMA models. However, whereas early attempts to introduce rank methods in the problem of testing univariate white noise (with unspecified density) against serial dependence can be traced back as far as Wald and Wolfowitz [20]—the corresponding multivariate problem has never been considered so far.

This latter fact is by no means surprising, and it certainly cannot be attributed to a lack of interest in multivariate problems; indeed, it is due to the much greater complexity inherent in the nature of multivariate variables. Thus, the development of multivariate theories—especially in rank-order and time series analysis—typically has been much slower and less complete than that of their univariate counterparts.

The purpose of this paper is to provide rank-based, permutationally distribution-free (hence similar) tests which are asymptotically most powerful against local alternatives of multivariate ARMA dependence. These tests are thus multivariate analogues of the tests proposed in Hallin et al. [5, 7]. In order to obtain these tests, we first investigate the asymptotic distribution, under local alternatives of ARMA dependence, of a class of linear serial multirank statistics. The word multirank in this terminology emphasizes the fact that our statistics are univariate functions of vectors of ranks $\mathbf{R}^{(n)} = (R_{1,1}^{(n)} \ldots R_{t,n}^{(n)})$ (for a more precise definition, see Section 2 below), whereas the statistics considered in the existing literature are vector-valued variables, each component of which is a function of exactly one rank component $R_{i,j}^{(n)}$ of a rank vector $\mathbf{R}^{(n)}$; the word serial refers to the fact that the scores used here are functions of several consecutive vectors of ranks (viz, $R_{1,1}^{(n)}, \ldots, R_{t-p}^{(n)}$, $p > 0$).

The asymptotic distributional properties of linear serial multirank statistics under randomness (with unspecified density function) and under local alternatives of ARMA dependence are studied in Section 1 (where contiguity is established for the distributions of certain sequences of ARMA processes and white noise) and Section 2. Asymptotic normality is derived through permutational arguments (Sections 2.2 and 2.5), equivalence with sequences of U-statistics (Section 2.3), a central-limit theorem for weakly dependent processes and, finally, LeCam’s lemmas (Section 2.4). These distributional results allow for constructing permutationally distribution-free tests for randomness, based on linear serial multirank statistics, computing their asymptotic powers against local alternatives and deriving asymptotic relative efficiencies.

Section 3 is devoted to optimality considerations. It must be noted that although the estimation theory for multiple time series has been considered
by a number of authors, hypothesis testing procedures have been discussed in less detail; moreover, optimality results in the area (in a parametric approach) always refer to gaussian types of situations (even if strict gaussian assumptions need not be made—see, e.g., Hosking [11, 12] or Poskitt and Tremayne [15]). If \( f \) denotes the density of the generating white noise under the alternative of ARMA dependence, we show here that the key role is played by what we define as the rank autocovariance matrices associated with \( f \), which appear to be nongaussian, rank-based, alternatives for the classical sample cross-covariance matrices, to which they actually asymptotically reduce in the case of a gaussian \( f \) and score functions of the van der Waerden type. The asymptotic joint normality of these matrices is established under a rather weak generalized strong unimodality condition on the density function \( f \) (Section 3.2). Finally, asymptotically locally most powerful tests are shown to be provided by appropriate linear combinations of the entries of the rank autocovariance matrices.

The optimality properties of such tests are, of course, of a local nature, and hold against the alternatives they were designed for. However, whatever the (local or nonlocal) alternatives, they have the great advantage to provide tests which are (i) of asymptotic size \( \alpha \) for any density \( f \), (ii) permutationally distribution-free—which, in view of classical similarity and Neyman-structure arguments, is a very important property, (iii) at least as powerful, asymptotically and locally, as their parametric counterparts, and (iv) rank-based, hence fairly robust against possible outliers.

Finally, in Section 4, we treat in some detail the case of gaussian scores. The asymptotically locally most powerful parametric test (based on usual sample autocovariances) is derived, and is shown to be equivalent to the corresponding rank-based test of the van der Waerden type.

1. CONTIGUOUS HYPOTHESES OF RANDOMNESS AND ARMA DEPENDENCE

1.1. Notation

In what follows, \( f(x) \) and \( F(x), x \in \mathbb{R}^m \) denote an \((m\text{-variate})\) probability density function, and the corresponding \((m\text{-variate})\) distribution function. \( f_i(x) \) and \( F_i(x), x \in \mathbb{R}, i = 1, \ldots, m \) denote the marginal probability density functions and the corresponding marginal distribution functions. We assume that \( \int x f(x) \, dx = 0 \) and that \( \int x x' f(x) \, dx = \Sigma \), a (finite) strictly positive definite covariance matrix with diagonal terms \((\Sigma)_{ii} = \sigma_i^2, i = 1, \ldots, m\). Letting \( F_i^{-1}(u) = \inf \{ x \mid F_i(x) \geq u \}, u \in (0, 1) \), define

\[
F(x) = \begin{pmatrix}
F_1(x_1) \\
\vdots \\
F_m(x_m)
\end{pmatrix}, \quad x = (x_1 \cdots x_m)' \in \mathbb{R}^m,
\]
and
\[ F^{-1}(u) = \left( \begin{array}{c} F_1^{-1}(u_1) \\ \vdots \\ F_m^{-1}(u_m) \end{array} \right), \quad u = (u_1 \cdots u_m)' \in (0, 1)^m. \]

1.2. ARMA Dependence

The null hypothesis we are interested in is the hypothesis of randomness under which the observations \( X^{(n)}_t \) in an \( m \)-variate series \( X^{(n)} = (X^{(n)}_1, \ldots, X^{(n)}_t, \ldots, X^{(n)}_n) \) are independently and identically distributed, according to some (specified or unspecified) density function \( f(x) \). Let us denote by \( H_f^{(n)} \) and \( H^{(n)} \) respectively this null hypothesis of randomness according as \( f \) is completely specified, or remains unspecified. The likelihood function for \( X^{(n)} \) under \( H_f^{(n)} \) is, obviously,
\[
L_f^{(n)}(X^{(n)}) = \prod_{i=1}^{n} f(X^{(n)}_i). \tag{1.1}
\]

The serial dependence alternatives we shall consider are alternatives under which \( X^{(n)} \) is generated by some specified ARMA model, driven by some white noise with specified or unspecified density function. The testing procedures we are deriving in this paper are devised against all types of serial dependencies; however, as far as optimality is concerned, some parametrization of serial dependence has to be introduced, of which ARMA dependencies certainly constitute the most convenient and intuitively appealing one. Our purpose here is to derive locally asymptotically most powerful tests against ARMA dependence. Nonlocal optimality results (such as an investigation of exact Bahadur efficiency) certainly would be of great interest. However, such results seem by no means easy to derive; by the way, they are not available even for the much simpler cases of univariate linear models (with independent observations). It should be noted, also, that whereas asymptotically locally most powerful rank tests can be found in the literature for univariate linear models (see, e.g., Puri and Sen, [17]), the rank-based tests available in the multivariate case are not asymptotically locally most powerful.

In order to obtain the local optimality result, we shall use the contiguity approach introduced by LeCam [13] but popularized mostly by Hájek and Šidák [3].

Consider therefore the sequence of stochastic difference equations
\[
Z_t - n^{-1/2} \sum_{i=1}^{p_1} A_i Z_{t-i} = \varepsilon_t + n^{-1/2} \sum_{i=1}^{p_2} B_i \varepsilon_{t-i}, \quad t \in \mathbb{Z}, \ n \in \mathbb{N}^+, \tag{1.2}
\]
where \( A_1, \ldots, A_{p_1}, B_1, \ldots, B_{p_2} \) are real \( m \times m \) matrices, \( A_{p_1} \) and \( B_{p_2} \) are of full rank, and \( \{ \varepsilon_t ; t \in \mathbb{Z} \} \) is an \( m \)-variate white noise process, i.e., a sequence of
i.i.d. random variables with common density function \( f(x) \). For \( n \) sufficiently large, the stationary and invertibility conditions (cf., e.g., Hannan [10]) are satisfied, and (1.2) admits a uniquely defined stationary solution \( \{Z_i^{(n)}; i \in \mathbb{Z}\} \), say: an ARMA \((p_1, p_2)\) process admitting the ARMA \((p_1, p_2)\) representation (1.2). Moreover, denoting by \( \mathcal{G}_u^{(n)}, u \in \mathbb{Z} \), the Green’s matrices (see Hallin [4] or Hallin and Puri [9, Appendix 1] for a univariate version) associated with the moving average difference operator in (1.2), and by \( \mathcal{C}^{(n)}(i) \) the Casorati matrix associated with the fundamental system \( \{\mathcal{G}_i, \mathcal{G}_{i+1}, ..., \mathcal{G}_{i+p_2-1}\} \), we may write, for \( t \geq 1, \)

\[
\begin{align*}
\varepsilon_t - Z_t^{(n)} &= -n^{-1/2} \sum_{i=1}^{p_1} A_i Z_{t-i}^{(n)} + \sum_{u=1}^{i-1} \mathcal{G}_u^{(n)} \left( Z_{t-u}^{(n)} - n^{-1/2} \sum_{i=1}^{p_1} A_i Z_{t-u-i}^{(n)} \right) \\
&+ \left( \mathcal{G}_t^{(n)} \mathcal{G}_{t+1}^{(n)} \cdots \mathcal{G}_{t+p_2-1}^{(n)} \right) \left[ \mathcal{C}^{(n)}(0) \right]^{-1} \left( \varepsilon_{t-p_2+1} \cdots \varepsilon_{t-1} \right)'.
\end{align*}
\]

with

\[
\begin{bmatrix} \mathcal{C}^{(n)}(0) \end{bmatrix}^{-1} = \\
\begin{bmatrix}
\mathcal{G}_0^{(n)} & \mathcal{G}_1^{(n)} & \cdots & \mathcal{G}_{p_2-1}^{(n)} \\
\mathcal{G}_1^{(n)} & \mathcal{G}_0^{(n)} & \cdots & \mathcal{G}_{p_2-2}^{(n)} \\
\vdots & \vdots & \ddots & \vdots \\
\mathcal{G}_{p_2-1}^{(n)} & \mathcal{G}_{p_2-2}^{(n)} & \cdots & \mathcal{G}_0^{(n)}
\end{bmatrix}^{-1} = n^{-1/2} \\
\begin{bmatrix}
1 & B_1 & \cdots & B_{p_2-1} \\
0 & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & 1
\end{bmatrix}.
\]

Accordingly, under the alternative—denote it by \( K_{(n)}(A, B) \)—that \( X^{(n)} \) constitutes a finite realization of length \( n \) of \( \{Z_i^{(n)}\} \), we have the likelihood

\[
L_{A, B, f}^{(n)}(X^{(n)}) = \int_{\mathbb{R}^{p_1+p_2}} A_{A, B, f}^{(n)}(X^{(n)}; x_0, e_0) \, dG^{(n)}(x_0, e_0),
\]

where \( G^{(n)}(x_0, e_0) = G^{(n)}(x_0, x_{-1}, ..., x_{-p_1+1}, e_0, e_{-1}, ..., e_{-p_2+1}) \) stands for the joint distribution function of \( Z_0^{(n)}, ..., Z_{-p_1+1}, x_0, e_0, ..., e_{-p_2+1} \), and

\[
A_{A, B, f}^{(n)}(X^{(n)}; x_0, e_0) \]

\[
= \prod_{i=1}^{n} f \left\{ X_i^{(n)} - n^{-1/2} \sum_{i=1}^{p_1} A_i X_{i-i}^{(n)} \\
+ \sum_{u=1}^{i-1} \mathcal{G}_u^{(n)} \left( X_{i-u}^{(n)} - n^{-1/2} \sum_{i=1}^{p_1} A_i X_{i-u-i}^{(n)} \right) \\
+ \left( \mathcal{G}_t^{(n)} \mathcal{G}_{t+1}^{(n)} \cdots \mathcal{G}_{t+p_2-1}^{(n)} \right) \left[ \mathcal{C}^{(n)}(0) \right]^{-1} \left( \varepsilon_{t-p_2+1} \cdots \varepsilon_{t-1} \right)'.
\]
1.3. Contiguity

Consider the log-likelihood ratio

\[ \mathcal{L}_{A, B, f}(X^{(n)}) = \begin{cases} \log L_{A, B, f}^{(n)}(X^{(n)}) - \log L_{f}^{(n)}(X^{(n)}), & \text{if } L_{f}^{(n)}(X^{(n)}) > 0, \\ 0, & \text{if } L_{A, B, f}^{(n)}(X^{(n)}) = 0 = L_{f}^{(n)}(X^{(n)}) \\ -\infty, & \text{if } L_{A, B, f}^{(n)}(X^{(n)}) \neq 0 = L_{f}^{(n)}(X^{(n)}). \end{cases} \]  

(1.6)

In order to establish the contiguity of the sequence \( K^{(n)}_{f}(A, B) \) to the sequence \( H^{(n)}_{f} \) we shall use a corollary \([3, p. 204]\) to the so-called Le Cam's first lemma. To this end, we need the following regularity conditions (which are assumed to hold throughout the paper).

(i) Finite cross-moments up to the third order: \( \int x_i^2 x_j f(x) \, dx < \infty, \) \( i, j \in \{1, \ldots, m\} \) (a sufficient condition is \( \int x_i^4 f(x) \, dx < \infty, \) \( i = 1, \ldots, m \)).

(ii) Finite Fisher's information. \( f(x) \) is a.e. derivable; denoting \( \phi(x) = -\text{grad log } f(x) = (\phi_1(x) \cdots \phi_m(x))' \), we assume that \( \int |\phi_i(x)|^{2+\delta} f(x) \, dx < \infty \) for some \( \delta > 0 \). (Then \( f(x) \) has finite Fisher's information matrix \( \mathcal{F}(f) = \int \phi(x) \phi'(x) f(x) \, dx \).

(iii) The \( \phi_i's \) are a.e. derivable; denoting by \( \Phi(x) = (-\partial^2 \log f(x)/\partial x_i \partial x_j) \) its derivatives, assume them to satisfy (a.e.) a Lipschitz condition of the form \( \|\phi_i(x) - \phi_i(y)\| \leq k \|x - y\| \) \( (k \) is a constant depending on \( f; \|\cdot\| \) stands for the euclidean norm).

We then have the following result.

**Proposition 1.1.** Under the above assumptions, letting \( p = \max(p_1, p_2) \),

\[
\mathbf{D}_i = \begin{cases} \mathbf{A}_i + \mathbf{B}_i & \text{if } 1 \leq i \leq \min(p_1, p_2) \\ \mathbf{A}_i & \text{if } p_2 < i \leq p_1 \\ \mathbf{B}_i & \text{if } p_1 < i \leq p_2 \\ 0 & \text{if } p < i \end{cases}
\]  

(1.7)

and

\[ \mathcal{L}_{0}^{(n)} = n^{-1/2} \sum_{i=p+1}^{n} \sum_{i=1}^{p} \text{tr}[\Phi(X^{(n)}_i X^{(n)}_{-i} | \mathbf{D}_i)], \]  

(1.8)

the log-likelihood (1.6) can be written as

\[ \mathcal{L}_{A, B, f}^{(n)} = \mathcal{L}_{0}^{(n)} - \frac{1}{2} \sum_{i=1}^{p} \text{tr}[\mathcal{F}(f) \mathbf{D}_i \Sigma_i] + o_p(1), \]  

(1.9)
and is asymptotically normal with mean $-\frac{1}{2} \sum_{t=1}^p \text{tr}[\mathcal{J}(f) D_t \Sigma D_t']$ and variance $\sum_{t=1}^p \text{tr}[\mathcal{J}(f) D_t \Sigma D_t']$, which implies that $K^{(n)}(A, B)$ is contiguous to $H^{(n)}$.

**Proof.** The proof follows by considering the asymptotic behavior of a second-order Taylor expansion of $\log A_{(n):f}^{(n)}(X^{(n)}) - \sum_{i=1}^n \log f(X^{(n)}_i)$. Using the fact that

$$\max_{u=pu+1}^{\infty} \sum_{j,k} \max_{u=pu+1}^{\infty} |(\mathcal{U}^{(n)}_{u,j,k})| \leq 2(p \max_{u=pu+1}^{\infty} |(B_{u,j,k})| n^{-1/2})^{p+1} = o(n^{-p/2})$$

for any $v \in \mathbb{N}$, $n$ sufficiently large (this property of Green's matrices can easily be adapted from the corresponding property of Green's functions, Hallin et al. [5, Appendix 1]), the first-order term in this expansion can be written in the form

$$n^{-1/2} \sum_{t=p+1}^n \phi'(X^{(n)}_t) \left( \sum_{i=1}^p A_i - n^{-1/2} \sum_{i=1}^p \mathcal{U}^{(n)}_i X^{(n)}_{t-i} \right) + o_p(1)$$

$$= n^{-1/2} \sum_{t=p+1}^n \sum_{i=1}^p \phi'(X^{(n)}_i) D_t X^{(n)}_{t-i} + o_p(1)$$

$$= n^{-1/2} \sum_{t=p+1}^n \sum_{i=1}^p \text{tr}[D_t X^{(n)}_{t-i} \phi'(X^{(n)}_t)] + o_p(1)$$

$$= \mathcal{Q}_0^{(n)}(X^{(n)}) + o_p(1).$$

As a sum of $p$-dependent identically distributed variables of the form $n^{-1/2} \zeta_t$, $\mathcal{Q}_0^{(n)}$ is asymptotically normal (see, e.g., [1, p. 427]), with mean

$$E(\zeta_t) = \sum_{i=1}^p \text{tr}[D_i E X^{(n)}_{t-i} E \phi'(X^{(n)}_i)] - 0$$

and variance

$$E(\zeta_t^2) + 2 \sum_{i=1}^p E(\zeta_t \zeta_{t+i})$$

$$= E \left\{ \phi'(X^{(n)}_t) \left( \sum_{i=1}^p D_i X^{(n)}_{t-i} \right) \left( \sum_{i=1}^p D_i X^{(n)}_{t-i} \right)' \phi(X^{(n)}_t) \right\} + 0$$

$$= \sum_{i=1}^p E \left\{ \phi'(X^{(n)}_i) D_i \Sigma D_i' \phi(X^{(n)}_i) \right\}$$

$$= \sum_{i=1}^p \text{tr}[\mathcal{J}(f) D_i \Sigma D_i'].$$
The second-order term can be treated along the same lines as in the univariate case, using the Lipschitz property of $\phi(X_i^{(n)})$ and the finiteness (under $H_f^{(n)}$) of $X_i^{(n)}$'s third-order moments. This second-order term then takes the form

\[
-\frac{1}{2n} \sum_{i=p+1}^{n} \left[ \left( \sum_{i=1}^{p_1} A_i - n^{1/2} \sum_{i=1}^{p_2} \mathcal{G}_i^{(n)} \right) X_{i-1}^{(n)} \right] \phi(X_i^{(n)}) \\
\times \left[ \left( \sum_{i=1}^{p_1} A_i - n^{1/2} \sum_{i=1}^{p_2} \mathcal{G}_i^{(n)} \right) X_{i-1}^{(n)} \right] + o_p(1) \\
= -\frac{1}{2n} \sum_{i=p+1}^{n} \left( \sum_{i=1}^{p} D_i X_{i-1}^{(n)} \right)' \phi(X_i^{(n)}) \left( \sum_{i=1}^{p} D_i X_{i-1}^{(n)} \right) + o_p(1).
\]

This second-order term thus converges in probability to

\[
-\frac{1}{2} E \left\{ \left( \sum_{i=1}^{p} D_i X_{i-1}^{(n)} \right)' \phi(X_i^{(n)}) \left( \sum_{i=1}^{p} D_i X_{i-1}^{(n)} \right) \right\} \\
= -\frac{1}{2} \sum_{i=1}^{p} E \left\{ (D_i X_{i-1}^{(n)})' \mathcal{S}(f)(D_i X_{i-1}^{(n)}) \right\} \\
= -\frac{1}{2} \sum_{i=1}^{p} \text{tr}[\mathcal{S}(f) D_i \Sigma D_i^t].
\]

To complete the proof, note that

\[
\mathcal{L}_{A, B, f}^{(n)}(X^{(n)}) = \log E \left[ \mathcal{L}_{A, B, f}^{(n)}(X^{(n)}; X_0, \epsilon_0) \mid X^{(n)} \right] - \sum_{i=1}^{n} \log f(X_i^{(n)}),
\]

where the expectation is taken with respect to the joint distribution $G^{(n)}(x_0, \epsilon_0)$ of $(X_0, \epsilon_0)$ considered independent of $X^{(n)}$ itself; hence

\[
\mathcal{L}_{A, B, f}^{(n)}(X^{(n)}) = \mathcal{L}_{0}^{(n)}(X^{(n)}) - \frac{1}{2} \sum_{i=1}^{p} \text{tr}[\mathcal{S}(f) D_i \Sigma D_i^t] \\
+ \log E[\exp(\mathcal{R}^{(n)}) \mid X^{(n)}],
\]

where the remainder term $\mathcal{R}^{(n)}$ converges to zero in quadratic mean. This in turn implies (1.9). Contiguity then follows from Le Cam's first lemma.
2. DISTRIBUTION THEORY OF LINEAR SERIAL MULTIRANK STATISTICS

2.1. LINEAR SERIAL MULTIRANK STATISTICS

Let \( R_j^{(n)} \) be the rank of \( X_j^{(n)} \) among \( \{ X_{i,j}^{(n)} \} \), \( R_i^{(n)} = (R_{i,1}^{(n)} \ldots R_{i,m}^{(n)})' \) and \( R^{(n)} = (R_1^{(n)} \ldots R_n^{(n)}) \). \( R^{(n)} \) is called the rank-collection matrix of the series.

It is well known that (unless \( m = 1 \)) the rank vectors \( R_i^{(n)} \) are not distribution-free, even under the null hypothesis of randomness. If, however, we denote, as usual, by \( R^{(n)}_* \) the rank-collection matrix obtained from \( R^{(n)} \) by rearranging the columns in such a manner that the first row has elements 1, 2, ..., \( m \) (in ascending order), the \( n! \) possible permutations of the columns of \( R^{(n)}_* \) have the same probability: \( (n!)^{-1} \) times the probability of \( R^{(n)} \) itself. Consequently, the rank vectors \( R_j^{(n)} \) are conditionally distribution-free, given the "ordered" rank-collection matrix \( R^{(n)}_* \); we also say that they are permutationally distribution-free.

Define a linear serial multirank statistic of order \( p \) as a statistic of the form

\[
S^{(n)}(R^{(n)} = (n-p)^{-1} \sum_{i-p+1}^n a^{(n)}(R_i^{(n)}, R_{i-1}^{(n)}, \ldots, R_{i-p}^{(n)}),
\]

where \( R^{(n)} \) is a rank-collection matrix of dimension \( m \times n \), and the \( a^{(n)}(\cdot, \ldots, \cdot) \)'s are a set of \( (n(n-1) \cdot \ldots \cdot (n-p))^{m} \) scores, one for each possible \( (p+1) \)-tuple of rank vectors. Related to these scores, assume the existence of a real-valued function \( J(u_1, u_2, \ldots, u_{p+1}) \), with \( u_i \in (0, 1)^m \), satisfying

\[
\int_{R^{(p+1)}} |J(F(x_1), \ldots, F(x_{p+1}))|^{2+\delta} dF(x_1) \cdot \ldots \cdot dF(x_{p+1}) < \infty \quad (2.2)
\]

for some \( \delta > 0 \), and such that, under \( H_j^{(n)} \),

\[
\lim_{n \to \infty} E\{ [J(F(X_i^{(n)}), \ldots, F(X_{i-p}^{(n)})) - a^{(n)}(R_i^{(n)}, \ldots, R_{i-p}^{(n)})]^2 \} = 0. \quad (2.3)
\]

The function \( J(u_1, \ldots, u_{p+1}) \) is called a score-generating function associated with (2.1).

Note that (2.1), as a univariate function of rank vectors (whence the term multirank statistic), is not of the type usually considered in the literature on multivariate linear models (see Puri and Sen, [17]), where multivariate linear statistics, each component of which however is a function of one rank at a time are usually preferred. Our statistics are thus of a more general
nature, which unlike the (nonserial) existing ones allows for asymptotically locally most powerful tests.

2.2. Some Permutational Properties of Linear Serial Multirank Statistics

Before proceeding to the study of the asymptotic distribution of linear serial rank statistics, we first establish some of their permutational properties. Denote by $E^*(\cdot)$, $\operatorname{Var}^*(\cdot)$ and $\operatorname{Cov}^*(\cdot, \cdot)$ respectively the permutation expectations, variances, and covariances, i.e., the conditional quantities $E(\cdot \mid R_{*}^{(n)})$, $\operatorname{Var}(\cdot \mid R_{*}^{(n)})$, and $\operatorname{Cov}(\cdot, \cdot \mid R_{*}^{(n)})$. The lemmas below will be used in the subsequent sections. Since under $H_f^{(n)}$, the conditional distribution of the rank vectors given $R_{*}^{(n)}$ does not depend on the underlying $f$, these lemmas essentially rely on combinatorial arguments. Also, since the scores $a^{(n)}(\cdots)$ depend on several consecutive rank vectors, the enumeration techniques we have to use in the proofs are much more tedious than in the classical nonserial case, where the scores depend only on one rank vector.

**Lemma 2.1.** (i)

$$E^*(S^{(n)}) = [n(n-1) \cdots (n-p)]^{-1} \sum_{1 \leq t_1 \neq \cdots \neq t_{p+1} \leq n} a^{(n)}(R_{t_1}^{(n)}, \ldots, R_{t_{p+1}}^{(n)}).$$

(ii)

$$\operatorname{Var}^*(S^{(n)}) = (n-p)^{-1} \operatorname{Var}^*(a^{(n)}(R_{1}^{(n)}, \ldots, R_{p+1}^{(n)}))$$

$$+ 2(n-p)^{-2} \sum_{i=1}^{p} (n-p-i) \times \operatorname{Cov}^*(a^{(n)}(R_{1}^{(n)}, \ldots, R_{p+1}^{(n)}), a^{(n)}(R_{i}^{(n)}, \ldots, R_{p+1+i}^{(n)}))$$

$$+ (n-2p)(n-2p-1)(n-p)^{-2} \times \operatorname{Cov}^*(a^{(n)}(R_{1}^{(n)}, \ldots, R_{p+1}^{(n)}), a^{(n)}(R_{p+2}^{(n)}, \ldots, R_{2p+2}^{(n)}))$$

$$\leq (2p+1)(n-p)^{-1} \operatorname{Var}^*(a^{(n)})$$

$$+ \frac{n^2}{(n-p)^2} \operatorname{Cov}^*(a^{(n)}(R_{1}^{(n)}, \ldots, R_{p+1}^{(n)}), (R_{p+2}^{(n)}, \ldots, R_{2p+2}^{(n)})).$$  (2.4)

**Proof.** (i) is obvious; (ii) follows from expanding $E^*[(\sum a^{(n)}(\cdots))^2]$.

**Lemma 2.2.**

$$|\operatorname{Cov}^*(a^{(n)}(R_{1}^{(n)}, \ldots, R_{p+1}^{(n)}), a^{(n)}(R_{p+2}^{(n)}, \ldots, R_{2p+2}^{(n)}))| \leq Kn^{-1} E^*[(a^{(n)})^2].$$
Proof. The proof is similar to that of Lemmas 3 and 4 in Hallin et al. [5, Appendix 3].

Lemma 2.3.

\[
E^*\left[a^{(n)}(R_1, ..., R_{p+1}) \mid R_{p+2}, ..., R_{n+1+k}^{(n)}\right] = [n(n-1) \cdots (n-p)]^{-1} E^*(a^{(n)}) + \sum_{l=1}^{k} \sum_{p+2 < \beta_1 < \cdots < \beta_l < p+1+k} \sum_{l \leq \alpha_i \leq \beta_i} (-1)^l \cdot \frac{(n-l)(n-l-1) \cdots (n-p)}{[(n-k)(n-k-1) \cdots (n-k-p)]^{1/2}} \times E^*[a^{(n)}(R_1, ..., R_{p+1}^{(n)}, R_{p+2}^{(n)}, ..., R_{n+1+k}^{(n)}) | R_{p+1}^{(n)}, R_{p+2}^{(n)}, ..., R_{n+1+k}^{(n)}]
\]

for \( k = 1, ..., n-p-1 \) and with the notational convention \( (n-l)(n-l-1) \cdots (n-p) = 0 \) for \( l \geq p+1 \).

Proof. The expectation in the left-hand side of (2.5) is actually \([n(n-1) \cdots (n-p)]^{-1} \sum a^{(n)}(R_{p+1}^{(n)}, ..., R_{p+1+k}^{(n)})\), where the summation extends over all the \((p+1)\)-tuples \((R_1^{(n)}, ..., R_{n+1+k}^{(n)})\) of distinct columns of \( R^{(n)} \) not belonging to \( \{R_2^{(n)}, ..., R_{n+1+k}^{(n)}\} \). This sum decomposes into \( \Sigma^0 - \Sigma^1 \), where \( \Sigma^0 \) extends over all possible \((p+1)\)-tuples of distinct columns of \( R^{(n)} \), whereas \( \Sigma^1 \) extends over all such \((p+1)\)-tuples having at least one element in the set \( \{R_2^{(n)}, ..., R_{n+1+k}^{(n)}\} \). \( \Sigma^1 \) in turn decomposes into \( \Sigma^{(1)} \Sigma^{(2)} \), where \( \Sigma^{(1)} \) extends over all possible ordered \((p+1)\)-tuples in \( \Sigma^1 \), and \( \Sigma^{(2)} \) over the \((p+1)!\) permutations of these \((p+1)\)-tuples.

Now, \( \Sigma^{(1)} \) can be written in the form

\[
\Sigma^{(1)}(\ldots) = -\sum_{l=1}^{k} (-1)^l \left[ \sum_{p+2 < \beta_1 < \cdots < \beta_l < p+1+k} \sum_{(\beta_1, ..., \beta_l)} \Sigma^{(1)}(\ldots) \right],
\]

where the sum between brackets concerns all the ordered \((p+1)\)-tuples of distinct columns of \( R^{(n)} \) containing at least \( l \) of the \( k \) columns \( R_{p+2}^{(n)}, ..., R_{p+1+k}^{(n)} \), and \( \Sigma^{(1)}(\beta_1, ..., \beta_l) \) runs over all such \((p+1)\)-tuples that include \( R_{\beta_1}^{(n)}, ..., R_{\beta_l}^{(n)} \).

A closer look at the sums \( \Sigma^{(1)}(\beta_1, ..., \beta_l) \Sigma^{(2)} \) shows that they extend over all possible choices of \( l \) subscripts \( 1 \leq \alpha_i \neq \cdots \neq \alpha_i \leq p+1 \), all the possible choices of an ordered \((p+1-l)\)-tuple of \( R_i^{(n)} \)'s in \( R^{(n)} \) not belonging to \( \{R_{\beta_1}^{(n)}, ..., R_{\beta_l}^{(n)}\} \) and finally all the permutations of this \((p+1-l)\)-tuple, i.e.,
\[
\sum_{1 \leq \gamma_1 \neq \cdots \neq \gamma_{l_1} \leq p} \cdots \sum_{1 \leq \gamma_l \neq \cdots \neq \gamma_{l_l} \leq p} \sum_{\beta \neq \gamma_1} \cdots \sum_{\beta \neq \gamma_l} a^{(n)}(R^{(n)}_{l_1}, \ldots, R^{(n)}_{l_{l_1} - 1}, R^{(n)}_{l_{l_1} + 1}, R^{(n)}_{l_1}, \ldots, R^{(n)}_{l_{l_1} + l_1 - 1}) \\
= \sum_{1 \leq \gamma_1 \neq \cdots \neq \gamma_{l_1} \leq p} \cdots \sum_{1 \leq \gamma_l \neq \cdots \neq \gamma_l \leq p} (n-l)(n-l-1) \cdots (n-p) \\
\times E^*\left[ a^{(n)}(R^{(n)}_{1, \ldots, n_1 - 1}, R^{(n)}_{n_1}, R^{(n)}_{n_1 + 1}, \ldots, R^{(n)}_{p_1 + 1}) | R^{(n)}_{1}, \ldots, R^{(n)}_{l} \right],
\]

whereas \( \sum^0 \) reduces to \( n(n-1) \cdots (n-p) E^*(a^{(n)}) \). This completes the proof.

A similar type of argument can be used to establish the next lemma.

**Lemma 2.4.**

\[
E^\star\left[ a^{(n)}(R^{(n)}_{1, \ldots, n_1 - 1}, R^{(n)}_{n_1}, R^{(n)}_{n_1 + 1}, \ldots, R^{(n)}_{p_1 + 1}) | R^{(n)}_{1}, \ldots, R^{(n)}_{l_1}, R^{(n)}_{l_1}, \ldots, R^{(n)}_{p_1 + 1} \right] \\
= \frac{(n-r) \cdots (n-p)}{(n-k) \cdots (n-k-p+p)} \\
\times E^\star\left[ a^{(n)}(R^{(n)}_{1, \ldots, n_1 - 1}, R^{(n)}_{n_1}, R^{(n)}_{n_1 + 1}) | R^{(n)}_{1}, \ldots, R^{(n)}_{l_1} \right] \\
+ \sum_{l=1}^{k} \frac{(n-l) \cdots (n-p)}{(n-k) \cdots (n-k-p+p)} \\
\times E^\star\left[ a^{(n)}(R^{(n)}_{1, \ldots, n_1 - 1}, R^{(n)}_{n_1}, R^{(n)}_{n_1 + 1}) | R^{(n)}_{1}, \ldots, R^{(n)}_{l_1} \right],
\]

for \( k = 1, \ldots, n-p-1 \) (with the same notational convention as in Lemma 2.3 above).

Define the “conditionally centered” scores \( a^{(n)}_* \) as

\[
a^{(n)}_*(R^{(n)}_{1, \ldots, p_1 + 1}) \\
= a^{(n)}(R^{(n)}_{1, \ldots, p_1 + 1}) \\
- \frac{(n-1)(n-p)}{n(n-p+1)} \sum_{i=1}^{p} E^\star\left[ a^{(n)}(R^{(n)}_{1, \ldots, h_1}, R^{(n)}_{h_1}, \ldots, R^{(n)}_{p_1}) | R^{(n)}_{i} \right] \\
- \frac{(n-1)}{n(n-p+1)} \sum_{1 \leq \gamma \neq \beta \leq p+1} E^\star\left[ a^{(n)}(R^{(n)}_{1, \ldots, h_1}, R^{(n)}_{h_1}, R^{(n)}_{h_1 + 1}, \ldots, R^{(n)}_{p_1 + 1}) | R^{(n)}_{1}, \ldots, R^{(n)}_{h_1}, R^{(n)}_{h_1 + 1}, \ldots, R^{(n)}_{p_1 + 1} \right] \\
\quad \left[ np/(n-p-1) \right] E^\star(a^{(n)}).
\]

These \( a^{(n)}_* \) however, cannot really be considered as score functions (of
order $p$) anymore, since they depend on all the columns of $R^{(n)}_*$. We shall use them merely as a notational convenience.

**Lemma 2.5.** The conditionally centered scores $a^{(n)}_*$ are such that

(i) $E^*(a^{(n)}_*) = 0$.

(ii) $E^*[a^{(n)}_*(R^{(n)}_1, \ldots, R^{(n)}_{a-1}, R^{(n)}_a, R^{(n)}_{a+1}, \ldots, R^{(n)}_{p+1}) | R^{(n)}_*] = 0$, \hspace{1cm} $\alpha = 1, \ldots, p+1$, $\beta = 1, 2, \ldots, n$.

(iii) $(n-p)^{-1} \sum_{i=p+1}^{n} a^{(n)}_*(R^{(n)}_i, R^{(n)}_{i-1}, \ldots, R^{(n)}_{i-p}) = S^{(n)} - E^*(S^{(n)}) + o_p(1)$.

**Proof.** (i) is a straightforward consequence of (ii). In order to prove (ii), first note that on account of Lemma 2.4, the value of

$$E^*[E^*[a^{(n)}_*^{(R^{(n)}_i, R^{(n)}_{i-1}, R^{(n)}_{i-2})} | R^{(n)}_i]]$$

where the conditional expectation $E^*[\cdot | R^{(n)}_*]$ is computed under $R^{(n)}_* = R^{(n)}_\beta$ is $(n/(n-1)) E^*[a^{(n)}_*(R^{(n)}_1, \ldots, R^{(n)}_{a-1}, R^{(n)}_a, R^{(n)}_{a+1}, \ldots, R^{(n)}_{p+1}) | R^{(n)}_\beta]$, if $b \neq \alpha$, and $E^*[a^{(n)}_*(R^{(n)}_1, \ldots, R^{(n)}_{a-1}, R^{(n)}_\beta, R^{(n)}_{a+1}, \ldots, R^{(n)}_{p+1}) | R^{(n)}_\beta]$ if $b = \alpha$. Therefore,

$$E^*[a^{(n)}_*(R^{(n)}_1, \ldots, R^{(n)}_{a-1}, R^{(n)}_\beta, R^{(n)}_{a+1}, \ldots, R^{(n)}_{p+1}) | R^{(n)}_\beta] = \frac{n-1}{n(n-p-1)} \frac{1}{n-1} \sum_{i=1}^{p+1} E^*[a^{(n)}_*(R^{(n)}_1, \ldots, R^{(n)}_{a-1}, R^{(n)}_a, R^{(n)}_{a+1}, \ldots, R^{(n)}_{p+1}) | R^{(n)}_\beta]$$

$$- \frac{n-1}{n(n-p-1)} \frac{1}{n-1} \sum_{i=1}^{p+1} E^*[a^{(n)}_*(R^{(n)}_1, \ldots, R^{(n)}_{a-1}, R^{(n)}_\beta, R^{(n)}_{a+1}, \ldots, R^{(n)}_{p+1}) | R^{(n)}_\beta]$$

$$- \frac{n-1}{n(n-p-1)} \frac{1}{n-1} \sum_{i=1}^{p+1} E^*[a^{(n)}_*(R^{(n)}_1, \ldots, R^{(n)}_{a-1}, R^{(n)}_\beta, R^{(n)}_{a+1}, \ldots, R^{(n)}_{p+1}) | R^{(n)}_\beta]$$

$$- \frac{n-1}{n(n-p-1)} \frac{1}{n-1} \sum_{i=1}^{p+1} E^*[a^{(n)}_*(R^{(n)}_1, \ldots, R^{(n)}_{a-1}, R^{(n)}_\beta, R^{(n)}_{a+1}, \ldots, R^{(n)}_{p+1}) | R^{(n)}_\beta]$$

$$+ \frac{np}{n(n-p-1)} E^*[a^{(n)}_*(R^{(n)}_1, \ldots, R^{(n)}_{a-1}, R^{(n)}_\beta, R^{(n)}_{a+1}, \ldots, R^{(n)}_{p+1}) | R^{(n)}_*] = 0.$$
To establish (iii), note that $a^{(n)}_\bullet$ is a sum of four terms. After summing over $t$, the first one gives $S^{(n)}$. The second one gives

$$
- \frac{(n-1)}{n(n-p-1)} \sum_{r=1}^{n} \sum_{i=\max(1, p+2-r)}^{\min(p+1, n+1-1)} E^* \left[ a^{(n)}(R_i^{(n)}, ...) \right]
$$

$$
- \frac{(p+1)(n-1)}{(n-p-1)} E^*(S^{(n)}) + o_p \left( \frac{1}{n} \right).
$$

The third one gives

$$
- \frac{(n-1)(p+1)p}{(n-p-1)(n-p)} E^*(S^{(n)}) + O_p(n^{-2}) = o_p(n^{-1/2}).
$$

Regrouping with the fourth one yields

$$
S^{(n)} - \left[ \frac{(p+1)(n-1)(n-p) + (n-1)(p+1)p}{(n-p)(n-p-1)} - \frac{np}{n-p-1} \right] E^*(S^{(n)})
$$

$$
+ o_p(n^{-1/2}) = S^{(n)} - E^*(S^{(n)}) + o_p(n^{-1/2}).
$$

Note that (iii) would hold exactly (without any $o_p(1)$ term) if a circular version of $S^{(n)}$ were adopted.

**Lemma 2.6.**

$$
\text{Var}^*(S^{(n)}) = (n-p)^{-1} \text{Var}^*(a^{(n)}_\bullet)
$$

$$
+ 2(n-p)^{-2} \sum_{i=1}^{p} (n-p-i) \times \text{Cov}^*(a^{(n)}_\bullet(R_1^{(n)}, ..., R_{p+1}^{(n)}), a^{(n)}_\bullet(R_1^{(n+1)}, ..., R_{p+1}^{(n+1)}))
$$

$$
+ \sum_{I=1}^{p} \sum_{1 \leq \beta_1 < \cdots < \beta_I \leq p+1} \sum_{1 \leq \alpha_1 < \cdots < \alpha_I \leq p+1} (-1)^I \times \frac{(n-l) \cdots (n-p+1)}{(n-p) \cdots (n-2p+1)}
$$

$$
\times E^*[a^{(n)}_\bullet(R_{\{\beta_1\}}^{(n)}, ..., R_{\{\beta_I\}}^{(n)})]
$$

$$
\times E^*[a^{(n)}_\bullet(R_{\{\alpha_1\}}^{(n)}, ..., R_{\{\alpha_I\}}^{(n)})]
$$

$$
+ o_p(n^{-1/2}).
$$
Proof. The proof follows by using Lemma 2.5(iii), Lemma 2.1(ii), then Lemma 2.3, and finally Lemma 2.5(i).

2.3. Linear Serial Multirank Statistics as U-Statistics

In this section, we show that $S^{(n)} - E^*(S^{(n)})$ is asymptotically equivalent (under $H_f^{(n)}$ and up to $o_p(n^{-1/2})$ terms) to a U-statistic $\mathcal{S}^{(n)} - \mathcal{E}^{(n)}$, where

$$\mathcal{S}^{(n)} = (n-p)^{-1} \sum_{i=1}^{n} J(F(X_{i}^{(n)}), \ldots, F(X_{i-p}^{(n)}))$$

and

$$\mathcal{E}^{(n)} = [n(n-1) \cdots (n-p)]^{-1} \sum_{1 \leq i_1 < \cdots < i_p+1 \leq n} J(F(X_{i_1}^{(n)}), \ldots, F(X_{i_{p+1}}^{(n)})).$$

First, we establish that $\mathcal{S}^{(n)} - \mathcal{E}^{(n)}$ is, asymptotically, a U-statistic (still under $H_f^{(n)}$). Beginning with $\mathcal{E}^{(n)}$, define the estimable parameter

$$\mathcal{E}^{(n)} = \left\{ \right\}$$

where $F(y)$ is the distribution function of the $m(p+1)$-dimensional random vector \( Y_{i}^{(n)} = (F'(X_{i}^{(n)}) \cdots F'(X_{i-p}^{(n)}))' \), \( t = p+1, \ldots, n \) (under $H_f^{(n)}$: $F(y) = \prod_{t=0}^{p} F[F^{-1}((Y_{i_1}^{(n)})_{m+1}, \ldots, Y_{i_m}^{(n)})]$). The kernel $y^{(p+1)}$ is defined as

$$y^{(p+1)}(y_1, \ldots, y_{p+1}, \ldots, y_{p+1}) = c(p+1) \left( \prod_{i=1}^{p+1} c(i) \right)$$

The corresponding U-statistic is then

$$\mathcal{U}^{(n)} = \left( \begin{array}{c}
(\begin{array}{c}
(n-p) \\
p+1
\end{array}) \\
p+1
\end{array} \right)^{-1} \sum_{p+1 \leq i_1 < \cdots < i_{p+1} \leq n} y^{(p+1)}(Y_{i_1}^{(n)}, \ldots, Y_{i_{p+1}}^{(n)})$$

It follows that $(n-p)^{-1/2} (\mathcal{S}^{(n)} - \mathcal{E}^{(n)}) = o_p(1)$.
We now prove that $A(n) = (S(n) - E*(S(n))) - (S(n) - \sigma(n))$ is $o_p(n^{-1/2})$.

We have

$$E[(A(n))^2] = E[E[(A(n))^2 | R_p, F_*^r]] = E[E*[((A(n))^2 | F_*^r)],$$

where $F_*^r$ is the matrix whose columns are the vectors $F(X_1)$ rearranged in such a way that the first row has elements in ascending order (i.e., the same permutation of columns $(1, ..., t, ..., n)$ as in $R_1$). In this conditional distribution, $S(n) - \sigma(n)$ can be written as a (permutationally distribution-free) linear serial multirank statistic of order $p$,

$$S(n) - \sigma(n) = (n - p)^{-1} \sum_{i=p+1}^{n} [a(n)(R_i, ..., R_{i-p}) - J(F(X_1), ..., F(X_{i-p+1}))].$$

Moreover, $E*[((A(n))^2 | F_*^r)]$ is precisely the conditional permutation variance of $(S(n) - \sigma(n))$: indeed,

$$E*[((S(n) - \sigma(n)) | F_*^r)] = E*(S(n)) - [n(n-1)...(n-p)]^{-1} \sum_{1 \leq i_1 \neq ... \neq i_{p+1} \leq n} J(F_*^r), i_1, ..., (F_*^r), i_{p+1}.$$ 

Thus

$$E[(A(n))^2] = E[E*[((A(n))^2 | F_*^r)] - E[\text{Var}*[(S(n) - \sigma(n)) | F_*^r]].$$

(2.10)

Using inequality (2.4) and Lemma 2.2 we find that (2.10) is bounded by

$$(2p+1)(n-p)^{-1} E[\text{Var}*[a(n)(R_1, ..., R_{p+1}) - J(F(X_1), ..., F(X_{p+1})) | F_*^r)]$$

$$+ n^2(n-p)^{-2} E[\text{Cov}*[a(n)(R_1, ..., R_{p+1}) - J(F(X_1), ..., F(X_{p+1}))]],$$

$$(a(n)(R_{p+2}, ..., R_{p+2}) - J(F(X_{p+2}), ..., F(X_{p+2}))) | F_*^r])$$

$$\leq [(2p+1)(n-p) + K \cdot n](n-p)^{-2} E*[a(n)(R_1, ..., R_{p+1})]$$

$$- J(F(X_1), ..., F(X_{p+1})))^2 | F_*^r)]$$

$$= [(2p+K+1)n^{-1} + o(n^{-1})] E[(a(n)(R_1, ..., R_{i-p}))$$

$$- J(F(X_1), ..., F(X_{i-p}))^2].$$

This latter quantity in view of (2.3) is $o(n^{-1})$, which establishes the desired equivalence of $S(n) - E*(S(n))$ and $\sigma(n)$. 
2.4. Asymptotic Normality of Linear Serial Multirank Statistics

We are now in a position to derive the asymptotic normality of linear serial multirank statistics, both under the null hypothesis and under contiguous alternatives, by using weak convergence results for U-statistics in weakly dependent processes. We therefore consider first the joint distribution under $H_f^{(n)}$ of the log-likelihood $\mathcal{L}_{A,B,f}^{(n)}$ and $(n-p)^{1/2} \left( S^{(n)} - E^*(S^{(n)}) \right)$.

**Proposition 2.1.** Under $H_f^{(n)}$, $(\mathcal{L}_{A,B,f}^{(n)}(X^{(n)}) (n-p)^{1/2} \left( S^{(n)} - E^*(S^{(n)}) \right))' \left( \begin{array}{c} \mathcal{H}(f) \mathcal{D} \Sigma \mathcal{D}' \mathcal{H} \end{array} \right)$ is asymptotically normal, with mean and covariance matrix

$$ \left( \begin{array}{c} -\frac{1}{2} \sum_{i=1}^{p} \text{tr}[\mathcal{H}(f) \mathcal{D} \Sigma \mathcal{D}' \mathcal{H}] \\ 0 \end{array} \right) $$

and

$$ \left( \begin{array}{cc} \sum_{i=1}^{p} \text{tr}[\mathcal{H}(f) \mathcal{D} \Sigma \mathcal{D}' \mathcal{H}] & \sum_{i=1}^{p} \text{tr}[\mathcal{C} \mathcal{D}' \mathcal{C}] \\ \sum_{i=1}^{p} \text{tr}[\mathcal{C} \mathcal{D}' \mathcal{C}] & \nu^2 \end{array} \right) $$

respectively, where

$$ \mathcal{C}_i = \sum_{j=0}^{p-i} \left[ J^*(F(x_1), ..., F(x_{p+1})) \phi(x_{j+1}) x_{j+1+i} dF(x_1) \cdots dF(x_{p+1}) \right] \quad (2.11) $$

and

$$ \nu^2 = \int J^{*2}(F(x_1), ..., F(x_{p+1})) dF(x_1) \cdots dF(x_{p+1}) + 2 \sum_{i=1}^{p} \int J^*(F(x_1), ..., F(x_{p+1})) $$

$$ \times J^*(F(x_{i+1}), ..., F(x_{i+p+1})) dF(x_1) \cdots dF(x_{p+1+i}) \quad (2.12) $$

with

$$ J^*(u_1, ..., u_{p+1}) $$

$$ = J(u_1, ..., u_{p+1}) $$

$$ - \sum_{i=1}^{p+1} \int J(F(x_1), ..., F(x_{i-1}), u_1, F(x_i), ..., F(x_p)) dF(x_1) \cdots dF(x_p) $$

$$ + p \int J(F(x_1), ..., F(x_{p+1})) dF(x_1) \cdots dF(x_{p+1}). \quad (2.13) $$
Proof. Consider the kernel of order $p + 1$ (same notation as in Section 2.3)

$$
\gamma^{(p+1)}(y_1, \ldots, y_{p+1}) = (p + 1)^{-1} \sum_{j=1}^{p+1} \sum_{i=1}^{p} \left[ \Phi(F^{-1}((y_{j,1} \cdots y_{j,m})) \right] \\
\times (F^{-1}((y_{j,m+1} \cdots y_{j,(i+1)m})))' D_i',
$$

and denote by $\gamma^{(p+1)}$ the corresponding U-statistic. It is then easy to see that (under $H_f^{(n)}$) $\mathcal{L}_0^{(n)}(X^{(n)}) = n^{1/2} \gamma^{(p+1)} + O_p(1)$. Hence, considering an arbitrary linear combination $\alpha n^{-1/2} \mathcal{L}_0^{(n)} + \beta (S^{(n)} - E^*(S^{(n)}))$, it is asymptotically equivalent to a U-statistic with kernel $\gamma^{(p+1)} + \beta (\gamma^{(p+1)} - \gamma^{(p)})$. Now, under $H_f^{(n)}$, $(Y_{p+1}^{(n)} \cdots Y_n^{(n)}) = (F(X_{p+1}^{(n)}) \cdots F(X_n^{(n)}))$ obviously constitutes a finite realization of a $p$-dependent, $m(p+1)$-variate process and therefore satisfies all the usual mixing conditions. We may thus apply one of the many available central limit theorems for U-statistics under weakly dependent observations—e.g., Yoshihara’s [19] Theorem 1.

The $g_i(Y_j)$ function appearing in Yoshihara [19, Theorem 1] here decomposes into $\alpha g_i^{(p+1)}(Y_j) + \beta g_i^{(p+1)}(Y_j) - \beta g_i^{(p+1)}(Y_j)$, say, with

$$
g_i^{(p+1)}(Y_j) = \int \gamma^{(p+1)}(Y_j, F(x_1), \ldots, F(x_p)) dF(x_1) \cdots dF(x_p)
$$

$$
= (p + 1)^{-1} \left\{ \sum_{i=1}^{p} \text{tr}[\Phi(F^{-1}(Y_{i,1} \cdots Y_{i,m}))'] \\
\times F^{-1}(Y_{i,m+1} \cdots Y_{i,(i+1)m})' D_i' \right\},
$$

$$
g_i^{(p+1)}(Y_j) = \int \gamma^{(p+1)}(Y_j, F(x_1), \ldots, F(x_p)) dF(x_1) \cdots dF(x_p)
$$

$$
= (p + 1)^{-1} \{ J((Y_{i,1} \cdots Y_{i,m})', \ldots, (Y_{i,p+1} \cdots Y_{i,(p+1)m})') \\
+ p \int J(F(x_1), \ldots, F(x_{p+1})) dF(x_1) \cdots dF(x_{p+1}) \},
$$

and

$$
g^{(p+1)}(y_i) = \frac{p!}{(p + 1)!} \left\{ \sum_{i=1}^{p+1} \int J(F(x_1), \ldots, F(x_{i-1}), Y_i, F(x_i), \ldots, F(x_{p+1})) \\
\times dF(x_1) \cdots dF(x_{p+1}) \right\}.
$$

It follows, letting $y_i = (u_1^i, \ldots, u_{p+1}^i)'$, that

$$
g_1(y_i) = (p + 1)^{-1} \left\{ \alpha \sum_{i=1}^{p} \text{tr}[\Phi(F^{-1}(u_1^i)) F^{-1}(u_{i+1}^i) D_i'] + \beta J^*(u_1, \ldots, u_{p+1}) \right\}. \]
Since the expected value of $g_1(F(X_1), ..., F(X_{t-p}))$ is zero under $H_{j}^{(n)}$, its variance equals

$$(p + 1)^{-2} \left\{ \alpha^2 \sum_{i=1}^{p} \text{tr} [ \mathcal{S}(f) D_i \Sigma D_i'] \right.$$\
$$+ 2 \alpha \beta \sum_{i=1}^{p} \text{tr} \left[ J^*(F(x_1), ..., F(x_{p+1})) \right.$$
$$\times \phi(x_1) x_{i+1} dF(x_1) \cdots dF(x_{p+1}) D_i' \left. \right]$$
$$+ \beta^2 \int \left[ J^*(F(x_1), ..., F(x_{p+1})) \right]^2 dF(x_1) \cdots dF(x_{p+1}),$$

whereas the covariance between $g_1(F(X_1), ..., F(X_{t-p}))$ and $g_1(F(X_{t-j}), ..., F(X_{t-j-p}))$, $j = 1, ..., p$, is

$$(p + 1)^{-2} \left\{ \alpha^2 \cdot 0 + \alpha \beta \sum_{i=1}^{p-j} \text{tr} \left[ J^*(F(x_1), ..., F(x_{p+1})) \right.$$
$$\times \phi(x_{j+1}) x_{j+1+i} dF(x_1) \cdots dF(x_{2p+1}) D_i' \left. \right]$$
$$+ \beta^2 \int J^*(F(x_1), ..., F(x_{p+1}))$$
$$\times J^*(F(x_{j+1}), ..., F(x_{j+p+1})) dF(x_1) \cdots dF(x_{2p+1}) \right\}.$$

On account of (2.11) and (2.12), Yoshihara’s theorem ensures the asymptotic normality of $\sum_{n}^{(n)} \beta(S^{(n)} - E'(S^{(n)}))$, with mean zero and variance

$$(p + 1)^2 \left[ \text{Var}(g_1(F(X_1), ..., F(X_{p+1}))$$
$$+ 2 \sum_{j=1}^{p} \text{Cov}(g_1(F(X_1), ..., F(X_{p+1})), g_1(F(X_{j+1}), ..., F(X_{j+p+1})))) \right]$$

$$= \alpha^2 \sum_{i=1}^{p} \text{tr} [ \mathcal{S}(f) D_i \Sigma D_i'] + 2 \alpha \beta \sum_{i=1}^{p} \text{tr} [ \mathcal{C}_i D_i'] + \beta^2 V^2.$$

The proposition then readily follows from the usual Wold–Cramér argument.

Proposition 2.1 and Le Cam’s third lemma now allow us to give the asymptotic joint distribution, under $K_{A,B}^{(n)}$, of any $k$-tuple of linear serial multirank statistics. We are giving here this result for $k = 2.$
Proposition 2.2. Let $S_1^{(n)}$ and $S_2^{(n)}$ be two linear serial multirank statistics, or orders $\pi_j$ and with score-generating functions $J_j(u_1, \ldots, u_{\pi_j+1})$, $j = 1, 2$, respectively. Then
\[
((n - \pi_1)^{1/2} [S_1^{(n)} - E^*(S_1^{(n)})] (n - \pi_2)^{1/2} [S_2^{(n)} - E^*(S_2^{(n)})])
\]
is asymptotically bivariate normal with mean $(0, 0)'$ under $H_j^{(n)}$ and mean $(\sum_{i=1}^{\min(p, \pi_1)} \text{tr}[C_j^{(1)}, D_i] \sum_{i=1}^{\min(p, \pi_2)} \text{tr}[C_j^{(2)}, D_i])'$ under $K_j^{(n)}$, and with covariance matrix
\[
V^2 = \begin{pmatrix} V_1^2 & V_{12} \\ V_{12} & V_2^2 \end{pmatrix}
\]
under both, where $V_1^2, V_2^2, C_j^{(1)}, D_j$ are defined as in Proposition 2.1, substituting
\[
J_j(u_1, \ldots, u_{\pi_j+1}) = \int J_j(u_1, \ldots, u_{\pi_j+1}, F(x_{\pi_j+2}), \ldots, F(x_{\pi_j+1}))
\]
and
\[
\frac{1}{\pi_j} \sum_{i=1}^{\pi_j} \frac{1}{\pi_j} \left[ J_j(u_1, \ldots, u_{\pi_j+1}, F(x_{\pi_j+2}), \ldots, F(x_{\pi_j+1})) \right]
\]
for $j = 1, 2$, and where the asymptotic covariance is
\[
V_{12} = \int J_1^*(F(x_1), \ldots, F(x_{\pi_1+1})) J_2^*(F(x_1), \ldots, F(x_{\pi_1+1}))
\]
and
\[
+ \sum_{i=1}^{\pi_2} \left[ \int J_1^*(F(x_1), \ldots, F(x_{\pi_1+1})) J_2^*(F(x_{i+1}), \ldots, F(x_{i+p+1})) \right]
\]
and
\[
+ \int J_1^*(F(x_{i+1}), \ldots, F(x_{i+p+1})) J_2^*(F(x_1), \ldots, F(x_{\pi_1+1}))
\]
and
\[
\times dF(x_1) \cdots dF(x_{\pi_1+1}). \quad (2.14)
\]

Proof. The proof follows from applying Proposition 2.1 and Le Cam's third lemma to an arbitrary linear combination of $(n - p)^{-1} (n - \pi_1) S_1^{(n)}$ and $(n - p)^{-1} (n - \pi_2) S_2^{(n)}$.  

2.5. Convergence of Permutation Moments

The asymptotic results in the preceding section were all obtained under a null hypothesis of the form $H_j^{(n)}$ (i.e., where the underlying density $f$ is specified), and the parameters of the limiting normal distributions in Propositions 2.2 and 2.3 explicitly depend on $f$ (cf. the asymptotic variance $V^2$ in (2.12), the asymptotic covariance $V_{12}$ in (2.14), and the “centered”
score-generating functions $J^*$ in (2.13)). These limiting distributions thus cannot, as they stand, be used for building statistical tests for the null hypothesis of randomness $H^{(n)}$ (with unspecified underlying density $f$).

Genuinely permutationally (conditionally) distribution-free test statistics which (unconditionally) also are asymptotically distribution-free, however, can be obtained if, in accordance with the conditional nature of rank permutation tests, the (distribution-dependent) unknown asymptotic means, variances, and covariances are replaced with their permutational analogues. Moreover, the statistics thus obtained are asymptotically equivalent (under $H^{(n)}$ as well as under contiguous alternatives) to the former ones. This follows from the fact that, as shown below, permutational first- and second-order moments converge to their unconditional counterparts.

**Proposition 2.4.** Under $H^{(n)}$, as $n \to \infty$,

(i) $E^*(S^{(n)}) \to^P J(F(x_1), \ldots, F(x_{p+1})) dF(x_1) \cdots dF(x_{p+1})$.

(ii) Provided that $a^{(n)}(R_1^{(n)}, \ldots, R_{p+1}^{(n)}) a^{(n)}(R_t^{(n)}, \ldots, R_{t+p}^{(n)})$ converges in the quadratic mean to $J(F(X_1^{(n)}), \ldots, F(X_{p+1}^{(n)})) J(F(X_t^{(n)}), \ldots, F(X_{t+p}^{(n)}))$, $t = 1, 2, \ldots, p + 2$, then

$$(n-p) \text{Var}^*(S^{(n)}) \to^P V^2.$$

(iii) Provided that the condition in (ii) above is satisfied for $a^{(n)}(R_1^{(n)}, \ldots)$ and $a^{(n)}(R_t^{(n)}, \ldots)$, and provided further that $a^{(n)}(R_1^{(n)}, \ldots, R_{p+1}^{(n)}) a^{(n)}(R_t^{(n)}, \ldots, R_{t+p}^{(n)})$ converges in the quadratic mean to $J_{(1)}(F(X_1^{(n)}), \ldots, F(X_{p+1}^{(n)})) J_{(2)}(F(X_t^{(n)}), \ldots, F(X_{t+p}^{(n)}))$, then (same notation as in Proposition 2.3)

$$((n-p)(n-2))^{1/2} \text{Cov}^*(S^{(n)}_{(1)}, S^{(n)}_{(2)}) \to^P V_{(12)}.$$

**Proof.** (i). As a U-statistic, $\delta^{(n)}$ converges a.s. to $E[J(F(X_1^{(n)}), \ldots, F(X_{p+1}^{(n)}))]$. In order to prove (i), it is therefore sufficient to show that $E[\text{Var}^*(S^{(n)}) - \delta^{(n)}]^2$ converges to zero. This latter expectation takes the form

$$(n(n-1) \cdots (n-p))^{-2} \sum_{1 \leq i_1 \neq \cdots \neq i_{p+1} \leq n} \sum_{1 \leq i \neq \cdots \neq i_{p+1} \leq n} \sum_{i} \int E\left[\left[a^{(n)}(R_{i_1}^{(n)}, \ldots, R_{i_{p+1}}^{(n)}) - J(F(X_{i_1}^{(n)}), \ldots, F(X_{i_{p+1}}^{(n)}))\right]\right]$$

$$\times E\left[\left[a^{(n)}(R_{i_1}^{(n)}, \ldots, R_{i_{p+1}}^{(n)}) - J(F(X_{i_1}^{(n)}), \ldots, F(X_{i_{p+1}}^{(n)}))\right]\right]$$

in which the $[n(n-1) \cdots (n-p)]^2$ summations are equal to each other. It follows from the Cauchy–Schwarz inequality that this expression is
bounded by \( E\{[a^{(n)}(R^{(n)}_{1}, \ldots, R^{(n)}_{p+1}) - J(F(X^{(n)}_{1}), \ldots, F(X^{(n)}_{p+1}))]^2}\), whose limit, on account of assumption (2.3), is zero.

(ii). In view of (i), we may assume here, without any loss of generality, that \( E[J(F(X^{(n)}_{1}), \ldots, F(X^{(n)}_{p+1}))] = 0\). We then have the following lemmas.

**Lemma 2.7.**

\[
a^{(n)}_{*}(R^{(n)}_{1}, \ldots, R^{(n)}_{p+1})
\]

\[
= a^{(n)}(R^{(n)}_{1}, \ldots, R^{(n)}_{p+1})
\]

\[
- \sum_{r=1}^{p+1} E^{*}[a^{(n)}(R^{(n)}_{1}, \ldots, R^{(n)}_{r}, \ldots, R^{(n)}_{p+1}) | R^{(n)}_{r}] + o_{p}(1).
\]

**Lemma 2.8.**

\[
\text{Var}^{*}(a^{(n)}_{*}) \xrightarrow{p} \int J^{*2}(F(x_{1}), \ldots, F(x_{p+1})) dF(x_{1}) \cdots dF(x_{p+1}).
\]

**Lemma 2.9.**

\[
\text{Cov}^{*}(a^{(n)}_{*}(R^{(n)}_{1}, \ldots, R^{(n)}_{p+1}), a^{(n)}_{*}(R^{(n)}_{l_{1}}, \ldots, R^{(n)}_{l_{p+1}}))
\]

\[
\xrightarrow{p} \int J^{*}(F(x_{1}), \ldots, F(x_{p+1})) J^{*}(F(x_{1} + 1), \ldots, F(x_{p+1} + 1))
\]

\[
\times dF(x_{1}) \cdots dF(x_{p+1} + 1).
\]

It follows from Lemmas 2.6, 2.8, and 2.9 that \((n - p) \text{Var}^{*}(S^{(n)}) = (n - p)\)

\[
\sum_{l=1}^{p} \sum_{i} \sum_{\alpha}(\cdots), \text{where } \sum_{l=1}^{p} \sum_{i} \sum_{\alpha}(\cdots) \text{ is the last sum in the right-hand side expression in Lemma 2.6, converges to } V^{2} \text{ in probability. It remains thus to show that } \sum_{l=1}^{p} \sum_{i} \sum_{\alpha}(\cdots) = o_{p}(n^{-1}). \text{ Now}
\]

\[
E^{*}[a^{(n)}_{*}(R^{(n)}_{1}, \ldots, R^{(n)}_{p+1})]E^{*}[a^{(n)}_{*}(R^{(n)}_{l_{1}}, \ldots, R^{(n)}_{l_{p+1}})]
\]

\[
\leq [E^{*}(a_{*}^{(n)2})]^{1/2} [E^{*}((E^{*}[a^{(n)}_{*}(R^{(n)}_{1}, \ldots, R^{(n)}_{l_{1}})\
\quad R^{(n)}_{1}, R^{(n)}_{z_{1} + 1}, \ldots, R^{(n)}_{p+1}) | R^{(n)}_{l_{1}}, \ldots, R^{(n)}_{l_{p+1}})]^{2}]^{1/2}
\]

\[
\leq E^{*}(a_{*}^{(n)2}) \xrightarrow{p} \int J^{*2}(F(x_{1}), \ldots, F(x_{p+1})) dF(x_{1}) \cdots dF(x_{p+1}) < \infty.
\]

Since \(\sum_{l=1}^{p} \sum_{i} \sum_{\alpha}(\cdots) \) contains a finite number of terms with coefficients

\[
(n - l)(n - l - 1) \cdots (n - p + l)(n - p)^{-1} (n - p - 1)^{-1} \cdots (n - 2p + 1)^{-1}
\]
$= O(n^{-r-2})$, with $l \geq 1$ and $p \geq 1$, the proof of part (ii) of Proposition 2.4 follows.

(iii) Follows analogously.

**Proof of Lemma 2.7.** Without any loss of generality, we may assume here that $E(J(\cdots)) = 0$. Going back to the definition of $a_\xi^{(n)}$, it follows from Proposition 2.4(i), that $(np)(n - p - 1)^{-1} E^*(a^{(n)}(\cdot \cdot \cdot)) = (np)(n - p - 1)^{-1} E^*(S^{(n)})$ converges to $E(J(\cdots)) = 0$. As for

$$(n - 1)^{-1}(n - p - 1)^{-1/2} \times E\{E^*[a^{(n)}(\mathbf{R}_1^{(n)}, \ldots, \mathbf{R}_{n-1}^{(n)}, \mathbf{R}_p^{(n)}, \mathbf{R}_{p+1}^{(n)}, \ldots, \mathbf{R}_{p+1}^{(n)})]^2 | \mathbf{R}_p^{(n)}]\},$$

it is bounded by

$$(n - p - 1)^{-2} E[E^*[a^{(n)}(\mathbf{R}_1^{(n)}, \ldots, \mathbf{R}_i^{(n)}, \ldots, \mathbf{R}_{p+1}^{(n)}) | \mathbf{R}_j^{(n)}]^2$$

$$= (n - p - 1)^{-2} E[(a^{(n)}(\cdots))^2],$$

which also converges to zero.

**Proof of Lemma 2.8.** In view of Lemma 2.7, we have

$$\text{Var}^* a_\xi^{(n)} = E^*(a_\xi^{(n)2})$$

$$= E^*(a^{(n)2}) + \sum_{i=1}^{p+1} E^*[(E^*[a^{(n)}(\mathbf{R}_1^{(n)}, \ldots, \mathbf{R}_i^{(n)}, \ldots, \mathbf{R}_{p+1}^{(n)}) | \mathbf{R}_i^{(n)}]^2]$$

$$+ \sum_{1 \leq i \neq j \leq p+1} E^*[E^*[a^{(n)}(\mathbf{R}_i^{(n)}, \ldots, \mathbf{R}_j^{(n)}, \ldots, \mathbf{R}_{p+1}^{(n)}) | \mathbf{R}_i^{(n)}]]$$

$$\times E^*[a^{(n)}(\mathbf{R}_j^{(n)}, \ldots, \mathbf{R}_j^{(n)}, \ldots, \mathbf{R}_{p+1}^{(n)}) | \mathbf{R}_j^{(n)}]]$$

$$- 2 \sum_{i=1}^{p+1} E^*[a^{(n)}(\mathbf{R}_1^{(n)}, \ldots, \mathbf{R}_{p+1}^{(n)})$$

$$\times E^*[a^{(n)}(\mathbf{R}_i^{(n)}, \ldots, \mathbf{R}_i^{(n)}, \ldots, \mathbf{R}_{p+1}^{(n)}) | \mathbf{R}_i^{(n)}]] + o_p(1)$$

$$= E^*(a^{(n)2}) + (2) + (3) - (4) + o_p(1),$$

say.

$E^*(a^{(n)2})$ converges to $\int J^2(F(x_1), \ldots, F(x_{p+1})) \, dF(x_1) \cdots dF(x_{p+1})$. As for the second term, Lemma 2.4 (see also the end of the proof of part (ii) of Proposition 2.4 for the $o_p(1)$ term) implies that

$$E^*[a^{(n)}(\mathbf{R}_1^{(n)}, \ldots, \mathbf{R}_i^{(n)}, \ldots, \mathbf{R}_{p+1}^{(n)}) | \mathbf{R}_i^{(n)}]$$

$$= E^*[a^{(n)}(\mathbf{R}_{p+2}^{(n)}, \ldots, \mathbf{R}_{p+i}^{(n)}, \mathbf{R}_i^{(n)}, \mathbf{R}_{p+i+2}^{(n)}, \ldots, \mathbf{R}_{p+1}^{(n)}) | \mathbf{R}_i^{(n)}] + o_p(1).$$
so that

\[(2) - \sum_{i=1}^{p+1} E^* \{ E^*[a^{(n)}(R_i^{(n)}, ..., R_i^{(n)} R_{i+1}^{(n)}) | R_i^{(n)}] \times E^*[a^{(n)}(R_{p+2}^{(n)}, ..., R_{p+i}^{(n)}, R_{p+2+i}^{(n)}, ...) \times R_{2p+2}^{(n)} | R_1^{(n)}, R_1^{(n)}, ..., R_{p+1}^{(n)}] \} + o_p(1)\]

\[= \sum_{i=1}^{p+1} E^* \{ E^*[a^{(n)}(R_i^{(n)}, ..., R_i^{(n)} a^{(n)}(R_{p+2}^{(n)}, ..., R_{i+1}^{(n)})) + o_p(1) \times R_{2p+2}^{(n)} | R_1^{(n)}, R_1^{(n)}, ..., R_{p+1}^{(n)}] \} + o_p(1)\]

Since the random variable in this latter expectation converges, under the assumptions, in quadratic mean, to \(J(F(X_1^{(n)}), ..., F(X_{p+2}^{(n)})) J(F(X_{p+2}^{(n)}), ..., F(X_{p+2}^{(n)}), F(X_{p+2}^{(n)}), ..., F(X_{2p+2}^{(n)}))\), we obtain that (2) converges in probability to

\[\sum_{i=1}^{p+1} \int J(F(x_1), ..., F(x_{p+1})) dF(x_1) \cdots dF(x_{p+1}) \times \int J(F(x_{p+2}), ..., F(x_{p+i}), F(x_i)) F(x_{p+i+2}) \cdots F(x_{2p+2}) dF(x_{p+1}) \cdots dF(x_{p+1}) \]

(4) \(\rightarrow^p \sum_{i=1}^{p+1} F \left\{ J(F(X_i^{(n)}), ..., F(X_{p+1}^{(n)})) \times \int J(F(x_1), ..., F(X_i^{(n)}), ..., F(X_{p+1}) dF(x_1) \cdots dF(x_{p+1}) \right\}.

Rearranging these terms finally yields
\[ E \left\{ \left[ J(X_1^{(n)}, \ldots, X_p^{(n)}) \right. \right. \\
+ \left. \left. \sum_{i=1}^{p+1} J(F(x_1), \ldots, F(X_i^{(n)}), \ldots, F(x_{p+1})) \ dF(x_1) \cdots dF(x_{p+1}) \right] \right \}^2 \right \} = \left\{ J^*(F(X_1^{(n)}), \ldots, F(X_{p+1}^{(n)})) \right\}^2, \]
which completes the proof.

**Proof of Lemma 2.9.** The proof follows proceeding as in Lemma 2.8 and is left to the reader.

### 2.6. Permutationally Distribution-free Multirank Tests for Randomness

The convergence of permutation moments in Proposition 2.4 allows us to build permutationally distribution-free, asymptotically distribution-free test statistics for testing \( H^{(n)} \) (randomness with unspecified underlying density).

**Proposition 2.5.** Let \( S_1^{(n)} \) be some linear serial multirank statistic, and let the null hypothesis \( H^{(n)} \) be restricted to those white noises with densities \( f(\cdot) \) such that the assumptions of Proposition 2.4 hold. Then

(i) the test procedure:

\[
\text{reject } H^{(n)} \text{ if } S_1^{(n)}(X^{(n)}) - E^*(S_1^{(n)}) > k_{1-\alpha} \sqrt{\text{Var}^*(S_1^{(n)})}, \]

where \( k_{1-\alpha} \) is the \((1-\alpha)\)-quantile of the standard normal distribution and \( E^*(S_1^{(n)}) \) is given in Lemma 2.1(i), has asymptotic level \( \alpha \) (for the sequence of null hypotheses \( H^{(n)} \)); its asymptotic power is

\[ 1 - \Phi \left( k_{1-\alpha} - \sum_{i=1}^{p} \frac{\text{tr}[C_{(1),i}D_i]}{V_1} \right) \]

against \( K_{A,B,f}^{(n)} \) (where \( \Phi \) stands for the standard normal distribution function and \( V_1 \) and \( C_{(1),i} \) are computed as in Proposition 2.2).

(ii) denoting by \( S_2^{(n)} \) some other statistic of the same type and restricting the null hypothesis \( H^{(n)} \) so that the assumptions of Proposition 2.4 hold for both \( S_1^{(n)} \) and \( S_2^{(n)} \), the A.R.E. (asymptotic relative efficiency) of (2.15) with respect to the test constructed in a similar way from \( S_2^{(n)} \) is

\[ V_2 \sum_{i=1}^{p} \frac{\text{tr}[C_{(1),i}D_i]}{V_1} \sum_{i=1}^{p} \frac{\text{tr}[C_{(2),i}D_i]}{V_2} \]

\((V_2 \text{ and } C_{(2),i} \text{ defined as in Proposition 2.2}).

**Proof.** The proof follows by applying Propositions 2.2, 2.3, and 2.4.
3. Asymptotically Most Powerful Multirank Tests Against Local Alternatives

3.1. Rank Autocovariance Matrices Associated with a Density \( f \).

We have shown in Hallin et al. [7] and Hallin and Puri [9] that in the univariate version of the problem of testing for randomness, a key role is played by serial rank statistics of the form

\[
\phi(F^{-1}(R_{i}^{(n)}/(n + 1))) - \phi(F^{-1}(R_{i}^{(n)}/(n + 1))) - m^{(n)} \big/ s^{(n)},
\]

where \( m^{(n)} = \) standardizing constants under \( H^{(n)} \). Because of the close similarity of the asymptotic behaviour of these statistics to that of usual sample autocorrelations under Gaussian assumptions (cf. also [8]), we termed \( r_{i,j}^{(n)} \) the rank autocorrelation coefficient of order \( i \) associated with \( f \)—although, unless \( f \) is normal, it does not take the form of an autocorrelation coefficient. We introduce here a multivariate extension of these univariate coefficients; because, however, of notational and some technical problems inherent to their multivariate nature, we define here rank autocovariances instead of rank autocorrelations.

**Definition 3.1.** The rank autocovariance matrix of order \( i \) associated with the multivariate density \( f \) (shortly, the \( f \)-rank autocovariance matrix of order \( i \)) is the matrix (with elements \( \gamma_{i,f,k,l}^{(n)} \))

\[
\Gamma_{i,f}^{(n)} = (n - i)^{-1} \sum_{t = i + 1}^{n} \phi \left( F^{-1} \left( \frac{R_{i}^{(n)}}{n + 1} \right) \right) \left( F^{-1} \left( \frac{R_{i}^{(n)}}{n + 1} \right) \right) - m_{i,f}^{(n)},
\]

where

\[
m_{i,f}^{(n)} = (n(n - 1))^{-1} \sum_{1 \leq i_{1} \neq i_{2} \leq n} \phi \left( F^{-1} \left( \frac{R_{i_{1}}^{(n)}}{n + 1} \right) \right) \left( F^{-1} \left( \frac{R_{i_{2}}^{(n)}}{n + 1} \right) \right),
\]

\( \gamma_{i,f,k,l}^{(n)} \) is a cross-covariance whenever \( k \neq l \), an autocovariance if \( k = l \); note that it is of the form \( S^{(n)} - E^{*}(S^{(n)}) \), with score function (for \( S^{(n)} \))

\[
a^{(n)}(R_{1}^{(n)}, ..., R_{i}^{(n)}) = \phi \left( F^{-1} \left( \frac{R_{i}^{(n)}}{n + 1} \right) \right) F_{i}^{-1} \left( \frac{R_{i}^{(n)}}{n + 1} \right).
\]

3.2. Asymptotic Multinormality of Rank Autocovariance Matrices

If we are to use Propositions 2.2 and 2.5 in order to obtain the asymptotic distribution of rank autocovariance matrices, we first have to ensure that condition (2.3) and the various assumptions in Proposition 2.4
are satisfied. We therefore first establish the following convergence property.

**Proposition 3.1.** Let the function $J(u_1, \ldots, u_{p+1})$, $u_i \in (0, 1)^m$, $i = 1, \ldots, p + 1$ be expressible as a finite sum $\sum_{k=1}^{\infty} \prod_{i=1}^{p+1} J_k(u_i)$, where $J_{k(i)}$ is monotone (with respect to all the components $u_{i,1}, \ldots, u_{i,m}$ of $u_i$), and $E[J_{k(i)}(F(X_i^{(n)}))] < \infty$, $i = 1, \ldots, p + 1, k = 1, \ldots, K$. Then

$$\lim_{n \to \infty} E \left\{ J \left( \frac{R_i^{(n)}}{n+1}, \frac{R_{i-1}^{(n)}}{n+1}, \ldots, \frac{R_{1-p}^{(n)}}{n+1} \right) - J(F(X_i^{(n)}), \ldots, F(X_{i-p}^{(n)})) \right\}^2 = 0.$$  

(3.5)

**Corollary 3.1.** Let $f(x)$ be a strongly unimodal density (i.e., let $\phi_k(x)$, $k = 1, \ldots, m$ be non-decreasing with respect to all its argument $x_1, \ldots, x_m$). Then condition (2.3) and the assumptions in Propositions 2.4 are satisfied by the score functions (of the form $\phi_k(F^{-1}(R_i^{(n)})/(n + 1))) F^{-1}(R_{i-1}^{(n)}/(n + 1))$ appearing in Definition 3.1.

**Proof.** We may restrict ourselves, without any loss of generality, to the case $J(u, v) = J_{(1)}(u) J_{(2)}(v)$, where $J_{(1)}$ and $J_{(2)}$ are non-decreasing on $(0, 1)^m$, thus a.e. continuous. Then, $J(R_i^{(n)}/(n + 1), R_{i-1}^{(n)}/(n + 1)) - J(F(X_i^{(n)}), F(X_{i-1}^{(n)}))$ converges a.s. to zero. Now consider the (random) hypercubes (associated with elements $R_i^{(n)}$ of $R_i^{(n)}$) defined by

$$\mathcal{D}_{\star(i)}^{(n)} = \left\{ \frac{i}{n+1}, \frac{i+1}{n+1}, \ldots, \frac{n+1}{n+1} \right\} \times \left\{ \frac{R_{2,i}^{(n)}}{n+1}, \frac{R_{2,i+1}^{(n)}}{n+1}, \ldots, \frac{R_{m,i}^{(n)}}{n+1}, \frac{R_{m,i+1}^{(n)}}{n+1} \right\} \times \cdots \times \left\{ \frac{R_{m,i}^{(n)}}{n+1}, \frac{R_{m,i+1}^{(n)}}{n+1} \right\}.$$  

Let $\mathcal{D}_{\star(i)} = \bigcup_{i=1}^{n} \mathcal{D}_{\star(i)}^{(n)}$ and $F^{(n)}(u) = P[F(X_i^{(n)}) \leq u \mid F(X_i^{(n)}) \in \mathcal{D}_{\star(i)}^{(n)}]$, $u \in [0, 1]^m$. Clearly,

$$\int_{\mathcal{D}_{\star(i)}^{(n)}} dF^{(n)}(u) = 1/(n + 1), \quad i = 1, \ldots, n.$$  

Monotonicity then implies

$$\frac{1}{(n+1)^2} \left[ J_{(1)} \left( \frac{R_i^{(n)}}{n+1} \right) J_{(2)} \left( \frac{R_{i-1}^{(n)}}{n+1} \right) \right]^2 \leq \int_{\mathcal{D}_{\star(i)}^{(n)}} J_{(1)}^2(u) dF^{(n)}(u) \cdot \int_{\mathcal{D}_{\star(i)}^{(n)}} J_{(2)}^2(u) F^{(n)}(u).$$  

Hence
RANK TESTS FOR MULTIVARIATE RANDOMNESS

\[ E^* \left[ J^2 \left( \frac{R_{i}^{(n)}}{n+1}, \frac{R_{i-1}^{(n)}}{n+1} \right) \right] = \left[ n(n-1) \right]^{-1} \sum_{1 \leq i_1 \neq i_2 \leq n} \sum_{j=1}^{n} J^2_{i_1}(\frac{R_{j}^{(n)}}{n+1}) J^2_{i_2}(\frac{R_{j}^{(n)}}{n+1}) \]

\[ \leq \frac{(n+1)^2}{n(n-1)} \int_{\Theta_{n}} J^2_{i_1}(u) d\Phi^{(n)}(u) \int_{\Theta_{n}} J^2_{i_2}(u) d\Phi^{(n)}(u) \]

\[ = \frac{(n+1)^2}{n(n-1)} E^*_{\Phi} \left[ J^2(\Phi(X_i^{(n)}), \Phi(X_{i-1}^{(n)})) \right] \]

Taking expectations on both sides yields

\[ E \left[ J^2 \left( \frac{R_{i}^{(n)}}{n+1}, \frac{R_{i-1}^{(n)}}{n+1} \right) \right] \leq \frac{(n+1)^2}{n(n-1)} E[J^2(\Phi(X_i^{(n)}), \Phi(X_{i-1}^{(n)}))] \]

and thus, letting \( n \to \infty \),

\[ \limsup_{n \to \infty} E \left[ J^2 \left( \frac{R_{i}^{(n)}}{n+1}, \frac{R_{i-1}^{(n)}}{n+1} \right) \right] \leq E[J^2(\Phi(X_i^{(n)}), \Phi(X_{i-1}^{(n)}))] ; \]

(3.5) then follows from Theorem 1.3 on page 154 of Hájek and Šidák [3].

Corollary 3.1 is a straightforward consequence of Proposition 3.1. The score-generating functions associated with \( \gamma_{i,j,k}^{(n)} \) are

\[ J(u, v) = \phi_k(F^{-1}(u)) F_i^{-1}(v_i), \quad (3.6) \]

and, referring to (2.13), the “centered” (under \( H_{g}^{(n)} \), where \( g(x) \) denotes a density with distribution function \( G(x) \), marginal densities \( g_k \) and marginal distribution functions \( G_k \), \( k = 1, ..., m \) ) version of \( J \) is

\[ J^*(u, v) = \phi_k(F^{-1}(u)) F_i^{-1}(v_i) - \int \phi_k(F^{-1}(G(x))) g(x) dx F_i^{-1}(v_i) \]

\[ - \phi_k(F^{-1}(u)) \int F_i^{-1}(G(y)) g_i(y) dy \]

\[ + 2 \int \phi_k(F^{-1}(G(x))) F_i^{-1}(G(y)) g(x) g_i(y) dx dy \]

\[ = \left[ \phi_k(F^{-1}(u)) - \int \phi_k(F^{-1}(G(x))) g(x) dx \right] F_i^{-1}(v_i). \quad (3.7) \]

We now may derive the joint asymptotic distribution of the cross- and autocovariances \( \gamma_{i,j,k}^{(n)} \). Let us therefore introduce some additional
notation. Let \( f(x) \) and \( g(x) \) be two densities satisfying the technical assumptions of Section 1.3. Using the notation \( G(x) \) in an obvious way, let

\[
\phi_f(x) = -\text{grad} \log f(x) \quad \text{and} \quad \phi_g(x) = -\text{grad} \log g(x),
\]

and define

\[
\mathcal{I}(f, g) = \int \phi_f(F^{-1}(G(x))) \phi'_g(x) g(x) \, dx, \tag{3.8}
\]

\[
\mathcal{I}(f \mid g) = \int \left( \int \phi_f(F^{-1}(G(x))) g(x) \, dx \right) \phi'_g(F^{-1}(G(x))) g(x) \, dx
\]

\[
\times \left[ \int \phi_f(F^{-1}(G(x))) g(x) \, dx \right]'
\]

\[
\mathcal{I}(f \mid g) = \int F^{-1}(G(x)) x' g(x) \, dx \tag{3.9}
\]

\[
\mathcal{I}(f \mid g) = \int F^{-1}(G(x))(F^{-1}(G(x)))' g(x) \, dx; \tag{3.10}
\]

\[
\mathcal{I}(f \mid g) = \int F^{-1}(G(x))(F^{-1}(G(x)))' g(x) \, dx; \tag{3.11}
\]

clearly, \( \mathcal{I}(f, f) = \mathcal{I}(f \mid f) = \mathcal{I}(f) \) and \( \mathcal{I}(f, f) = \mathcal{I}(f \mid f) = \mathcal{I} \). Also, we use the classical notation \( \text{vec}(A) \), where \( A \) is a \( q \times r \) matrix, for the \( qr \times 1 \) vector obtained by stacking the columns of \( A \) on top of each other; \( \otimes \) denotes, as usual, the Kronecker product.

**Proposition 3.2.** Let \( f(x) \) be strongly unimodal. Then

\[
\left( \begin{array}{c}
(n - i)^{1/2} \text{vec}(\Gamma_i^{[n]}_f) \\
(n - i')^{1/2} \text{vec}(\Gamma_i^{[n]}_{i'})
\end{array} \right), \quad i \neq i',
\]

is asymptotically normal with mean \((0, 0)'\) under \( H_{1, g}^{[n]} \), and mean

\[
\left( \begin{array}{c}
\text{vec}(\mathcal{I}(f, g) D_1 \Sigma(f, g)) \\
\text{vec}(\mathcal{I}(f, g) D_r \Sigma(f, g))
\end{array} \right) = \left( \begin{array}{c}
\Sigma(f, g) \otimes \mathcal{I}(f, g) \vec{D}_1 \\
\Sigma(f, g) \otimes \mathcal{I}(f, g) \vec{D}_r
\end{array} \right)
\]

(with the convention \( D_j = 0, \quad j > p \)) under \( K_{A, B, g}^{[n]} \), and with covariance matrix

\[
\left( \begin{array}{cc}
\Sigma(f \mid g) \otimes \mathcal{I}(f \mid g) & 0 \\
0 & \Sigma(f \mid g) \otimes \mathcal{I}(f \mid g)
\end{array} \right)
\]

under both.

Note that the asymptotic covariance matrix, which is the same under the null hypothesis of randomness and under the alternative, is not dis-
distribution-free. Under $K_{A,B,f}^{(i)}$, the asymptotic mean and covariance matrices take the simpler forms
\[
\begin{pmatrix}
\Sigma \otimes \mathcal{J}(f) & \text{vec } \mathbf{D}_i \\
\Sigma \otimes \mathcal{J}(f) & \text{vec } \mathbf{D}_i
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\Sigma \otimes \mathcal{J}(f) & 0 \\
0 & \Sigma \otimes \mathcal{J}(f)
\end{pmatrix}
\]

**Proof of Proposition 3.2.** It follows from Corollary 3.1 and the strong unimodality of $f(x)$ that Proposition 2.2 holds here. Taking (3.7) into account, the matrices (2.11) associated with $\gamma_{i;j;f,k}^{(n)}$ are all zero, except (provided $i \leq p$) for
\[
\mathbf{C}_{i;j,k}^{(n)} = \int \phi_{f,j}(\mathbf{F}^{-1}(\mathbf{G}(x))) \phi_{g}(x) \ g(x) \ dx
\]
\[
\times \int \mathbf{F}_k^{-1}(G_k(x)) \ x' g(x) \ dx
\]
\[
= \{ [\mathbf{\Sigma}(f,g)]_j \mathcal{J}(f,g)_k \}_i.'
\]
Accordingly, the asymptotic mean under $K_{A,B,g}^{(i)}$ of $(n-i)^{1/2} \gamma_{i;j,f,k}^{(n)}$ is
\[
\text{tr}[\mathbf{C}_{j,k}^{(n)} \mathbf{D}_i] = \{ \text{vec}[\mathcal{J}(f,g)_j] \}' \text{vec}[\mathbf{\Sigma}(f,g)_k \mathbf{D}_i]
\]
\[
= [\mathcal{J}(f,g)_j \mathbf{D}_i \mathbf{\Sigma}(f,g)_k],
\]
and the asymptotic mean of $(n-i)^{1/2} \Gamma_i^{(n)}$ takes the form $\mathcal{J}(f,g) \mathbf{D}_i \mathbf{\Sigma}(f,g)$. The identity $\text{vec}[\mathcal{J}(f,g) \mathbf{D}_i \mathbf{\Sigma}(f,g)] = \mathbf{\Sigma}(f,g) \otimes \mathcal{J}(f,g) \text{vec}(\mathbf{D}_i)$ is a classical property of the Kronecker product (see, e.g., Mardia et al. [21]). As for the asymptotic covariance matrix, it is easy to check that the off-diagonal blocks are zero, since, for $i \neq i'$, (2.14) yields terms of the form
\[
\int \left[ \phi_{f,j}(\mathbf{F}^{-1}(\mathbf{G}(x))) - \int \phi_{f,j}(\mathbf{F}^{-1}(\mathbf{G}(x))) \ g(x) \ dx \right]
\times \left[ \phi_{f,j}(\mathbf{F}^{-1}(\mathbf{G}(x))) - \int \phi_{f,j}(\mathbf{F}^{-1}(\mathbf{G}(x))) \ g(x) \ dx \right] g(x) \ dx
\]
\[
\times \int \mathbf{F}_k^{-1}(G_k(y_1)) \mathbf{F}_k^{-1}(G_k(y_2)) \ g_k(y_1) \ g_k(y_2) \ dy_1 \ dy_2
\]
\[
- \int \left[ \phi_{f,j}(...) - \int \phi_{f,j}(\cdot \cdot) \ g(x) \ dx \right]
\times \left[ \phi_{f,j}(...) - \int \phi_{f,j}(\cdot \cdot) \ g(x) \ dx \right] g(x) \ dx
\]
\[
\times \int \mathbf{F}_k^{-1}(u) \ du \cdot \int \mathbf{F}_k^{-1}(u) \ du = 0, \quad \forall j, j', k, k'
\]
or
\[
\int \left[ \phi_{f,j}(F^{-1}(G(x))) - \int \phi_{f,j}(\cdots) g(x) \, dx \right] g(x) \, dx \\
\times \int \left[ \phi_{f,j}(F^{-1}(G(x))) - \int \phi_{f,j}(\cdots) g(x) \, dx \right] g(x) \, dx \\
\times \int F_k^{-1}(G_k(x_k)) F_k^{-1}(G_k(x_k)) g(y) \, dy = 0, \quad \forall j, j', k, k'.
\]

The diagonal blocks, still on account on (2.14), have entries of the form
\[
\int \left[ \phi_{f,j}(F^{-1}(G(x))) - \int \phi_{f,j}(F^{-1}(G(x))) g(x) \, dx \right] \\
\times \left[ \phi_{f,j}(F^{-1}(G(x))) - \int \phi_{f,j}(F^{-1}(G(x))) g(x) \, dx \right] g(x) \, dx \\
\times \int F_k^{-1}(G_k(x_k)) F_k^{-1}(G_k(x_k)) g(x) \, dx,
\]

which completes the proof.

### 3.3. Locally Asymptotically Most Powerful Test for \(H^{(n)}\) against Specified ARMA Alternatives

Consider the multirank serial statistic
\[
T^{(n)}_\star = (n - p)^{-1/2} \sum_{i=1}^{p} (n - i)^{1/2} \text{tr}[\Gamma^{(n)}_{ij} D_i]/\sigma^{(n)},
\]

where \((\sigma^{(n)}_\star)^2\) denotes the permutational variance \(\text{Var}_\star(\sum_{i=1}^{p} (n - i)^{1/2} \text{tr}[\Gamma^{(n)}_{ij} D_i])\). Then \((n - p)^{1/2} T^{(n)}_\star\) is a permutationally distribution-free statistic, with (conditional and unconditional) mean zero and variance one under \(H^{(n)}\); consequently, it can be used to build permutationally distribution-free tests, resulting in (unconditionally) similar, asymptotically distribution-free tests for \(H^{(n)}\) (see Proposition 3.3 below).

An exact computation of the permutation variance \(\sigma^{(n)}_\star\), although possible, appears to be rather unpleasant, even for moderate values of the series length \(n\). On the other hand, the corresponding asymptotic value
\[
\lim_{n \to \infty} (\sigma^{(n)}_\star)^2 = \sum_{i=1}^{p} (\text{vec } D_i)' \Sigma(f \mid g) \otimes \mathcal{F}(f \mid g)(\text{vec } D_i)
\]

under \(H^{(n)}\) explicitly depends on the "true" underlying density \(g(\cdot)\), and thus cannot be substituted for the exact value of \((\sigma^{(n)}_\star)^2\) in (3.15). An
approximate value, however, can be used which is easier to compute and takes advantage of the asymptotic independence (Proposition 3.2) of the autocovariances $\Gamma_{i;j}^{(n)}$ and $\Gamma_{i';j'}^{(n)}$ for distinct lags $i$ and $i'$.

**Lemma 3.1.** Define

$$T_{n}^{(n)} = (n - p)^{-1/2} \sum_{i = 1}^{p} (n - i)^{1/2} \text{tr}[\Gamma_{i;j}^{(n)}D_i']/\tilde{\sigma}_{n}^{(n)}, (3.17)$$

where

$$\tilde{\sigma}_{n}^{(n)} = \sum_{i = 1}^{p} (\text{vec } D_i)' [W_{i}^{(n)}]^{2} (\text{vec } D_i), (3.18)$$

and

$$[W_{i}^{(n)}]^{2} =\sum_{i = 1}^{p} \sum_{1 \leq i \neq i' \leq n} \left[ F^{-1}\left( \frac{R_{i}^{(n)}}{n + 1} \right) F^{-1}\left( \frac{R_{i'}^{(n)}}{n + 1} \right) \right]$$

$$\otimes \left[ \phi\left( F^{-1}\left( \frac{R_{i}^{(n)}}{n + 1} \right) \right) \phi'\left( F^{-1}\left( \frac{R_{i'}^{(n)}}{n + 1} \right) \right) \right]$$

$$- [\text{vec}(m_{i}^{(n)})][\text{vec}(m_{i'}^{(n)})]' .$$

Then $(n - p)^{1/2} (T_{n}^{(n)} - \tilde{T}_{n}^{(n)})$ is $o_{p}(1)$ under $H^{(n)}$ as well as under any alternative of the form $K_{n}^{(n)}$. 

**Proof.** The permutation covariance matrix of $(n - i)^{1/2} \text{vec}(\Gamma_{i;j}^{(n)})$ takes (cf. part (ii) of Lemma 2.1) the form

$$\frac{1}{n(n - 1)} \sum_{i \leq i \neq i' \leq n} \left[ F^{-1}\left( \frac{R_{i}^{(n)}}{n + 1} \right) F^{-1}\left( \frac{R_{i'}^{(n)}}{n + 1} \right) \right]$$

$$\otimes \left[ \phi\left( F^{-1}\left( \frac{R_{i}^{(n)}}{n + 1} \right) \right) \phi'\left( F^{-1}\left( \frac{R_{i'}^{(n)}}{n + 1} \right) \right) \right]$$

$$+ \frac{n - 2i}{n - i} \cdot \sum_{1 \leq i \neq i' \leq n} \left[ F^{-1}\left( \frac{R_{i}^{(n)}}{n + 1} \right) F^{-1}\left( \frac{R_{i'}^{(n)}}{n + 1} \right) \right]$$

$$\otimes \left[ \phi\left( F^{-1}\left( \frac{R_{i}^{(n)}}{n + 1} \right) \right) \phi'\left( F^{-1}\left( \frac{R_{i'}^{(n)}}{n + 1} \right) \right) \right]$$
On account of Proposition 3.2, the limiting (in probability, under $H^{(n)}_g$) value of (3.19) is $\Sigma(f \mid g) \otimes \mathcal{J}(f \mid g)$. Since the first term in (3.19) is the permutational expectation (under $H^{(n)}_g$) of its limiting value (in probability, under $E^{(n)}_g$) is also (Proposition 3.2) $\Sigma(f \mid g) \otimes \mathcal{J}(f \mid g)$. Also, since $\int F^{-1}(G_k(x)) g_k(x) dx = \int F^{-1}(u) du = 0$, the limiting value (still in probability, under $H^{(n)}_g$) of $m^{(n)}_*$ is zero. (3.19) thus reduces (up to $o_p(1)$ terms, under $H^{(n)}_g$) to its first term, which in turn is asymptotically equal to $[W^{(n)}_*]^2$. Now since

$$\sum_{i=1}^p (n-i)^{1/2} \text{tr}[\Gamma^{(n)}_{i:f} D_i] = \sum_{i=1}^p (\text{vec } D_i)' (\text{vec}(n-i)^{1/2} \Gamma^{(n)}_{i:f}),$$

and since $(n-i)^{1/2} \Gamma^{(n)}_{i:f}$ and $(n-i')^{1/2} \Gamma^{(n)}_{i':f}$ are asymptotically independent, $\sigma^{(n)}_* - \bar{\sigma}^{(n)}_*$ is $o_p(1)$, both under $H^{(n)}$ and under $K^{(n)}_{A,B:f}$, $\forall A, B, g$.

We now establish that the multirank statistic $T^{(n)}_*$ (or, equivalently, $\tilde{T}^{(n)}_*$) provides a locally asymptotically most powerful test for $H^{(n)}$ against $H^{(n)}_{A,B:f}$ (not only within the class of rank-based tests, but within the whole class of tests having the same probability level).

**Proposition 3.3.** The permutationally distribution-free test procedure

reject $H^{(n)}$ if $T^{(n)}_* > (n-p)^{-1/2} k_{1-\alpha}$, \hspace{1cm} (3.20)

and the asymptotically distribution-free test procedure

reject $H^{(n)}$ if $\tilde{T}^{(n)}_* > (n-p)^{-1/2} k_{1-\alpha}$, \hspace{1cm} (3.21)

where $k_{1-\alpha}$ denotes the $(1-\alpha)$-fractile of the standard normal distribution.
(i) have asymptotic level $\alpha$ (under $H^{(n)}$)
(ii) have asymptotic power

\[
1 - \Phi \left( k_{1 - \alpha} - \sum_{i=1}^{p} (\text{vec } D_i)' \Sigma(f, g) \otimes \mathcal{I}(f, g)(\text{vec } E_i) \right) \right)^{1/2} \tag{3.22}
\]

against $K^{(n)}_{E, g} = U\{ K^{(n)}_{A_i, B_i, g} | A_i + B_i = E_i, i = 1, \ldots, p \}$, where $\Phi(\cdot)$ denotes the standard normal distribution function, and where $E_1, \ldots, E_q$ is any $q$-tuple of $m \times m$ matrices such that $|E_q| \neq 0$, and with the convention that $E_{q+1} = \cdots = E_p = 0$ if $q < p$.

(iii) are asymptotically most powerful against

\[
K^{(n)}_{D_i, f} = U\{ K^{(n)}_{A_i, B_i, f} | A_i + B_i = D_i, i = 1, \ldots, p \}.
\]

Consequently, the limit, as $n \to \infty$, of the envelope power function for $H^{(n)}$ against $K^{(n)}_{A_i, B_i, f}$ exists, and takes the value

\[
\lim_{n \to \infty} \beta(\alpha; H^{(n)}, K^{(n)}_{A_i, B_i, f})
= 1 - \Phi \left( k_{1 - \alpha} - \left\{ \sum_{i=1}^{p} (\text{vec } D_i)' \Sigma(f, g) \otimes \mathcal{I}(f, g)(\text{vec } D_i) \right\}^{1/2} \right). \tag{3.23}
\]

The asymptotic power values in (3.22) and (3.23) are consistent with the corresponding univariate values (cf. Hallin et al. [5]), where the limit of the envelope power function was shown to be of the form $1 - \Phi(k_{1 - \alpha} - (\sum_{i=1}^{p} d_i^2 \sigma_i^2)^{1/2})$.

Proof of Proposition 3.3. It is an immediate consequence of Proposition 3.2 that $(n - p)^{1/2} T_{*}$ is asymptotically normal with mean zero and variance one under $H^{(n)}$, with mean $\sum_{i=1}^{p} (\text{vec } D_i)' \Sigma(f, g) \otimes \mathcal{I}(f, g)(\text{vec } E_i)/\lim_{n \to \infty} \sigma_i^{(n)}$ and variance one under $K^{(n)}_{E, g}$.

This establishes parts (i) and (ii) of the proposition. Now, note that against $K^{(n)}_{D_i, f}$ (3.22) reduces to

\[
1 - \Phi \left( k_{1 - \alpha} - \left\{ \sum_{i=1}^{p} \text{tr}[\mathcal{I}(f, g) D_i \Sigma D_i'] \right\}^{1/2} \right) = 1 - \Phi \left( k_{1 - \alpha} - \left\{ \sum_{i=1}^{p} \text{tr}[\mathcal{I}(f, g) D_i \Sigma D_i'] \right\}^{1/2} \right). \tag{3.24}
\]

In order to prove that (3.20) and (3.21) are asymptotically most powerful against $K^{(n)}_{A_i, B_i, f}$, hence against $K^{(n)}_{D_i, f}$, it is therefore sufficient to show that (3.24) is the limit, as $n \to \infty$, of the envelope power function $\beta(\alpha; H^{(n)}, K^{(n)}_{A_i, B_i, f}), \alpha \in (0, 1)$. Consider the simple null hypothesis $H^{(n)}_j$, and the simple alternative $K^{(n)}_{A_i, B_i, f}$: the most powerful test is then the
Neyman test based on $\mathcal{L}_{\text{A,B};f}^{(n)}$, which, on account of Proposition 1.1 and LeCam's third lemma, is asymptotically normal, with mean $-\frac{1}{2} \sum_{i=1}^{p} \text{tr}[\mathcal{I}(f) D_i \Sigma D_i']$ under $H_f^{(n)}$, mean $\frac{1}{2} \sum_{i=1}^{p} \text{tr}[\mathcal{I}(f) D_i \Sigma D_i']$ under $K_{\text{A,B};f}^{(n)}$, and variance $\sum_{i=1}^{p} \text{tr}[\mathcal{I}(f) D_i \Sigma D_i']$ under both. It follows that the asymptotic power of the Neyman test, hence the limit as $n \to \infty$ of the envelope power function $\beta(\alpha; H_f^{(n)}, K_{\text{A,B};f}^{(n)})$, is (3.24). Now $\beta(\alpha; H_f^{(n)}, K_{\text{A,B};f}^{(n)}) \geq \beta(\alpha, H_f^{(n)}, K_{\text{A,B};f}^{(n)})$ for every $n$; on the other hand, it follows from (3.22) that

$$\limsup_{n \to \infty} \beta(\alpha; H_f^{(n)}, K_{\text{A,B};f}^{(n)}) \geq \lim_{n \to \infty} \beta(\alpha; H_f^{(n)}, K_{\text{A,B};f}^{(n)}).$$

Since this latter limit exists (and is given in (3.24)), (3.23) holds for any $\alpha \in (0, 1)$, which completes the proof.

### 4. Normal Theory Procedure

#### 4.1. Asymptotically Locally Most Powerful Parametric Test (Gaussian Case)

Assume that the density $f$ is normal with full-rank covariance matrix $\Sigma$. Then $\Phi(x) = \Sigma^{-1} x$, and the first-order term $\mathcal{L}_0^{(n)}$ in the expansion (1.9) of the log-likelihood $\mathcal{L}_0^{(n)}$ takes the form

$$\mathcal{L}_0^{(n)} = n^{-1/2} \sum_{i=p+1}^{p} \sum_{i=1}^{p} \text{tr}[\Sigma^{-1} \mathbf{X}_i^{(n)}(\mathbf{X}_i^{(n)})' D_i']$$

$$= n^{-1/2} \sum_{i=1}^{p} (n - i) \text{tr}[\Sigma^{-1} \Gamma_i^{(n)} D_i'], \quad (4.1)$$

where $\Gamma_i^{(n)}$ stands for the usual sample autocovariance matrix

$$\Gamma_i^{(n)} = (n - i)^{-1} \sum_{t=i+1}^{n} \mathbf{X}_i^{(n)}(\mathbf{X}_i^{(n)})'. \quad (4.2)$$

Since $(n - i)^{1/2} \text{vec}(\Gamma_i^{(n)})$ is asymptotically normal (cf., e.g., [2]), with mean $0$ (under $H_f^{(n)}$) and covariance matrix $\Sigma \otimes \Sigma$ (under $H_f^{(n)}$ as well as under contiguous alternatives), and since $\Gamma_i^{(n)}$ and $\Gamma_i^{(n)}$, $i \neq i'$ are uncorrelated, the test procedure rejecting $H_f^{(n)}$ when

$$\sum_{i=1}^{p} (n - i)^{1/2} \text{tr}[\Gamma_i^{(n)} D_i' \Sigma^{-1}] \left( \sum_{i=1}^{p} (\text{vec} D_i)' \Sigma \otimes \Sigma^{-1}(\text{vec} D_i) \right)^{1/2} > k_{1-\alpha} \quad (4.3)$$

has asymptotical level $\alpha$, $\alpha \in (0, 1)$. It is easy to gauge from (4.1) that (4.3)
also provides an asymptotically most powerful test for $H^{(\alpha)}$ against gaussian alternatives of the form $K_{\mathbf{d}_i}^{(n)}$ (see Proposition 3.3(ii) for a precise definition). Actually, the (parametric) test statistic in (4.3) is asymptotically equivalent (under normality) to the optimal rank-based statistic (3.15) (here of the van der Waerden type to be described in Section 4.2). This easily follows from the asymptotic equivalence of (3.15) with a $U$-statistic (Section 2.3). The asymptotic power of (4.3) against $K_{\mathbf{d}_i}^{(n)}$, where $\tilde{f}$ denotes a gaussian density with correlation matrix $\tilde{\mathbf{P}}$ and covariance matrix $\Sigma = \text{diag}(\tilde{\sigma}_i) \mathbf{P} \text{diag}(\tilde{\sigma}_i)$ is

$$1 - \Phi \left( k_{1-\alpha} - \frac{\sum_{i=1}^{\rho} (\text{vec} \mathbf{D}_i)' \text{diag}(\sigma_i) \tilde{\mathbf{P}} \text{diag}(\sigma_i) \otimes \Sigma^{-1} (\text{vec} \mathbf{D}_i)}{\left( \sum_{i=1}^{\rho} (\text{vec} \mathbf{D}_i)' \text{diag}(\sigma_i) \tilde{\mathbf{P}} \text{diag}(\sigma_i) \otimes \Sigma^{-1} (\text{vec} \mathbf{D}_i) \right)^{1/2}} \right)$$

and reduces to

$$1 - \Phi \left( k_{1-\alpha} - \left\{ \sum_{i=1}^{\rho} (\text{vec} \mathbf{D}_i)' \Sigma \otimes \Sigma^{-1} (\text{vec} \mathbf{D}_i) \right\}^{1/2} \right)$$

against $K_{\mathbf{d}_i}^{(n)}$.

4.2. A van der Waerden Type Test

Under the same assumptions as in Section 4.1, the test statistic $T_{\text{v.d.w.}}^{(\alpha)}$, which is optimal against $K_{\mathbf{d}_i}^{(n)}$, is obtained from (3.15) by considering the van der Waerden rank autocovariance matrices, defined as

$$\Gamma_{\text{v.d.w.}}^{(\alpha)} = (n-i)^{-1} \Sigma^{-1} \sum_{i=1}^{n} \text{diag}(\sigma_j)$$

$$\times \phi^{-1} \left( \frac{R_{j}^{(n)}}{n+1} \right) \left( \phi^{-1} \left( \frac{R_{j-i}^{(n)}}{n+1} \right) \right)' \text{diag}(\sigma_j) - \mathbf{m}_{\text{v.d.w.}}^{(\alpha)}$$

$$(\phi^{-1}(R_{j}^{(n)}/(n+1)), \ldots, \phi^{-1}(R_{j,m}^{(n)}/(n+1)))$$ denotes the vector whose components are $\phi^{-1}(R_{j}^{(n)}/(n+1)), \ldots, \phi^{-1}(R_{j,m}^{(n)}/(n+1))$. Since $\Sigma \Gamma_{\text{v.d.w.}}^{(\alpha)} - \Gamma_{\text{v.d.w.}}^{(n)}$ is $O_p(n^{-1/2})$ under $H^{(\alpha)}$, it follows that the ARE (asymptotic relative efficiency) of the van der Waerden test (based on (4.6)) with respect to its parametric counterpart (4.3) is one. The van der Waerden test, however, has over the parametric test, the many advantages of rank-based procedures: permutational distribution-freeness, robustness, and low sensitivity to gross error—which should make it a rather attractive testing procedure, especially for relatively short series, when the underlying noise is suspected to be nongaussian, and against moving average dependencies rather than autoregressive ones (see [14]).
Now, the definition (4.6) of van der Waerden autocovariances, as well as that (4.3) of the optimal parametric test statistic explicitly involves the covariance matrix $\Sigma$. Since $\Sigma$ remains unspecified in most applications, it usually has to be replaced with some consistent estimator $\Sigma^{(e)}$. Provided that $\Sigma^{(e)}$ is a symmetric function of the observations, such a substitution does not affect the exchangeability property of the observed series (see, e.g., [18], pp. 356–357). Neither does it affect the asymptotic distribution of the test statistic. The resulting van der Waerden test accordingly remains genuinely permutationally distribution-free and asymptotically locally as powerful as its parametric counterpart.

5. CONCLUDING REMARKS

The contribution of this paper is twofold. Sections 1 and 2 mainly provide the asymptotic distribution theory of linear serial multirank statistics under randomness and under local alternatives of ARMA dependence. The results in these sections allow for constructing rank-based tests for multivariate white noise, evaluating their asymptotic power against local alternatives, and computing their AREs with respect to each other and with respect to related parametric procedures. Sections 3 and 4 show that asymptotic local optimality can be achieved by means of adequate linear multirank tests and suggest a procedure of the van der Waerden type, possessing all the distinctive features of rank-based methods (distribution-freeness, robustness against gross errors, etc.), yet asymptotically as powerful as its normal-theory counterpart. The optimality results in Section 3 also attest to the prominent role of a class of measures of dependence (closely depending on the underlying white noise density $f$) defined as the rank autocovariance matrices $\Gamma^{(e)}_{(m)}$ associated with $f$, which provide multivariate analogues of the rank autocorrelation coefficients previously considered for univariate series [7, 9].

This paper constitutes a first step towards a systematic and logically coherent nonparametric approach towards multiple time series problems, an approach that has not been explored so far. Many questions remain open (in this area as well as in the corresponding parametric approach). In an ongoing research, the authors propose to derive asymptotically maximin rank-based tests against unspecified ARMA alternatives. These tests rely on quadratic forms involving the rank autocovariance matrices $\Gamma^{(e)}_{(m)}$. Their asymptotic performances should be compared with those of their parametric, normal-theory competitors based on sample cross-correlations—such as Hosking's [11, 12] or Poskitt and Tremayne's [15] multivariate portmanteau tests.

On the other hand, the present paper also constitutes, to the best of our knowledge, the first attempt to obtain explicit local optimality results in a
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multiple time series context. In this respect, it could also serve as a basis for deriving locally asymptotically most powerful or maximin parametric (gaussian or nongaussian) procedures (e.g., substituting alternative local asymptotical maximin criteria for the gaussian Lagrange multiplier principle used in Hosking [12] in the definition of a multivariate portmanteau statistic).

REFERENCES