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On some non-Archimedean spaces of Alexandroff and Urysohn

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Abstract

Classical characterizations of four separable metrizable spaces are recalled, and generalized to classes of spaces which admit a uniformity with a totally ordered base. The Alexandroff–Urysohn characterization of the irrationals finds its closest analogues for strongly inaccessible cardinals, while the other three spaces, including the Cantor set, find their most natural analogues for weakly compact cardinals. In addition, A.H. Stone's characterization of Baire's zero-dimensional spaces is extended to give internal characterizations of all spaces ${}^{\gamma}\lambda \times D$, where D is discrete and ${}^{\gamma}\lambda$ has the initial agreement topology. The historical background for the Alexandroff–Urysohn result is briefly surveyed. © 1999 Elsevier Science B.V. All rights reserved.

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In the wake of the untimely tragic death of Paul Urysohn, his close friend and collaborator, Paul Alexandroff, published several papers posthumously that either had Urysohn as author or co-author, or were about Urysohn's concepts. One of the co-authored papers appeared in 1928 [2] and had to do with internal characterizations of three zero-dimensional homogeneous separable metrizable spaces: the space of irrationals, the countable direct sum of copies of the Cantor set, and a space they designated Ψ, which is homeomorphic to the subspace $\bigcup\{{}^{\omega}n\colon\ n \in \omega\}$ of ${}^{\omega}\omega$.

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The characterizations were reminiscent of Brouwer's famous characterization of the Cantor set [5] as the only totally disconnected, compact, metrizable, dense-in-itself space. Of course, if one substitutes "separable, locally compact noncompact" for "compact", one obtains the countable direct sum of copies of the Cantor set, and this is one thing Alexandroff and Urysohn did. They also showed that there is only one zero-dimensional, nowhere locally compact, nowhere locally countable, metrizable absolute F_σ-set, and thereby also showed Ψ and another space they designated Ψ_0 to be homeomorphic.

Alexandroff and Urysohn also characterized the space of irrationals as being the only zero-dimensional, nowhere locally compact, separable, completely metrizable space, using the "absolute G_δ" characterization of complete metrizability in a metrizable space to state their result. In Section 1, we treat historical precursors of this result, along with a "rediscovery" and some generalizations. The treatment is far from definitive and there is much room for additional historical delving, particularly into the theory of ordered fields and valuations. In Section 2, we recall a theorem of [2] which seemingly anticipated a highly general modern theorem on non-Archimedean spaces, whose full proof may actually be published here for the first time although it has long been a part of the folklore. In Section 3, we will use the results of Section 2 to help us in the proof of a natural and in some ways definitive extension to higher cardinals of the characterization of the space of irrationals, and also of the Cantor set and of the countable direct sum of copies of the Cantor set. We also give some characterizations of the analogues of the space Ψ. Some interesting results on weakly compact cardinals will be used in distinguishing between analogues of the Cantor set and analogues of the space of irrationals. Our technique will even have some basic but not immediately obvious facts of cardinal arithmetic as corollaries.

1. Basic concepts and historical comments

In their paper [2], Alexandroff and Urysohn acknowledge in a footnote, 16a, that their characterization of the irrationals is already "Im wesentlichen" (in essence) in a 1917 paper of Brouwer [6] "für a priori als linear vorausgesetzter Mengen" (for sets presented a priori as linearly ordered). It also had even earlier precursors, such as a 1909 article of Baire [3, p. 103] who essentially showed that the irrationals are homeomorphic to what we now call ${}^{\mathbb{N}}\mathbb{N}$ with the product topology.

Baire's proof, like the proof of Alexandroff and Urysohn, used the fact that the irrationals can be represented as the collection of all infinite continued fractions, a theorem that goes well back into the 19th century. A lot of the ingredients of Baire's proof were presented in an earlier 1906 article [3] which actually formed the first half of a two-part paper of which the 1909 article [4] was the second part.

It is in recognition of Baire's pioneering work that the countable product of discrete spaces of cardinality κ is known, for infinite κ, as *Baire's zero-dimensional space of weight* κ. The case $\kappa = \omega$ was treated by Hausdorff in a 1937 paper [10] where he called this space simply "der Baerische Nullraum" or " der Nullraum N". He rediscovered the Alexandroff–Urysohn theorem by showing that N is the only zero-dimensional, completely metrizable, separable metric space with no compact open subsets.

Hausdorff's proof made no allusion to the irrationals, instead using a very simple straightforward method which will be used, with one minor generalization, in Section 3. This same method was used by A.H. Stone [18] in 1962 to characterize Baire's zero-dimensional space of weight κ as the only strongly zero-dimensional, completely metrizable space in which every nonempty open set is of weight κ. At the heart of the method are two facts: (1) Every strongly zero-dimensional metrizable space (hence every zero-dimensional separable metrizable space) has the property that every open set can be written as the disjoint union of clopen sets of diameter $< 1/n$ for each positive integer n; and (2) a noncompact open set G in such a space can be written as the disjoint union of κ open sets, where κ is the weight of G.

These facts generalize in a simple and straightforward way to the class of all ω_μ-metrizable spaces for $\mu > 0$:

Definition 1.1. Let ω_μ be a regular infinite cardinal. An *ω_μ-metrizable space* is a Hausdorff space which is either discrete or admits a compatible uniformity with a totally ordered base of cofinality ω_μ.

Of course, the class of ω_0-metrizable spaces is simply the class of metrizable spaces. The term "ω_μ-metrizable" is due to Roman Sikorski [17] while the wording of Definition 1.1 is essentially that of an earlier paper by Cohen and Goffman [7]. Sikorski used a definition like that of Definition 1.4 below, but with values in an ordered group instead of an ordered field. One of the most basic results about ω_μ-metrizable spaces is:

Theorem 1.2 [7]. *Every totally ordered uniformity with a totally ordered base of uncountable cofinality has a base of equivalence relations.*

Proof. Given any entourage U_0, let basic entourages U_n for $n > 0$ be defined to satisfy $U_n \circ U_n^{-1} \subset U_{n-1}$. The intersection of the U_n is in the uniformity (by uncountable cofinality of the base) and is easily seen to be an equivalence relation. □

Equivalence relations in a uniformity partition the space into clopen sets, and these partitions can be used to define metrics and generalizations and analogues of metrics in various ways. To keep things simple in this article, the following analogue will be used in the theorems and proofs.

Definition 1.3. Let γ be an ordinal. An *inverse γ-metric on a set X is a function* $\mu : X^2 \to \gamma + 1$ such that

(1) $\mu(x, x) = \gamma$ for all $x \in X$.
(2) $\mu(x, y) < \gamma$ whenever $x \neq y$.
(3) $\mu(x, y) = \mu(y, x)$ for all $x, y \in X$.
(4) $\mu(x, z) \geqslant \min\{\mu(x, y), \mu(y, z)\}$ for all $x, y, z \in X$.

Given $\xi < \gamma$, a *closed ball of inverse radius* ξ is a set of the form $B(x,\xi) = \{y \in X\colon \mu(x,y) \geqslant \xi\}$. The topology induced by an inverse γ-metric μ is the one whose base is the set of closed balls of inverse radius $< \gamma$. Of course, if γ has a greatest element, then X is discrete, but X could also be discrete in other ways, such as there being $\delta < \gamma$ such that $\mu(x,y) \geqslant \delta$ implies $x = y$.

Because of (4), we have $\mu(z,w) \geqslant \xi$ for all $z, w \in B(x,\xi)$. Consequently, a ball of inverse radius $\geqslant \xi$ can also be said to be of inverse diameter $\geqslant \xi$. Also because of (4), two closed balls are either disjoint, or one is contained in the other. Specifically, if two balls meet, and one is of greater inverse radius than the other, then the one of smaller inverse radius will contain the one of larger inverse radius; while if the inverse radii are equal, then the balls coincide.

In this way, each inverse γ-metric defines a well-ordered (by refinement) family of partitions of X into clopen sets. This gives a covering uniformity with a totally ordered base in a natural way, and of course each partition is associated with an equivalence relation, immediately giving us a uniformity with a totally ordered base of neighborhoods of the diagonal.

The concept of an inverse γ-metric is a very slight modification of a tool of ring and field theory that has been used by algebraists for most of this century. Algebraists have given the name *additive valuation* to certain functions ν that take on their values in an ordered abelian group A, often the integers or the reals, with an ideal point ∞ greater than any member. The domain is an algebraic structure X which is an abelian group under one of its operations, and ν satisfies axioms which yield essentially (1) through (4) of Definition 1.2 when applied to the function $\mu : X^2 \to A \cup \{\infty\}$ defined by

$$\mu(x,y) = \nu(x-y).$$

These axioms, which obviously imply the corresponding ones of Definition 1.3 when ∞ is substituted for γ, are:

(1) $\nu(0) = \infty$.

(2) $\nu(x) \in A$ if $x \neq 0$.

(3) $\nu(-x) = \nu(x)$ for all $x \in X$.

(4) $\nu(x+y) \geqslant \min\{\nu(x), \nu(y)\}$ for all $x, y \in X$.

If $\langle X, +, \times\rangle$ is a field, (3) is replaced by the more demanding axiom:

(3+): $\nu(xy) = \nu(x) + \nu(y)$ for all $x, y \in X$.

Closed balls of radii $a \in A$ are defined as in Definition 1.2, and the induced topology is defined in the same way.

Any ordinal can be embedded in an ordered abelian group. One easy method is to take the direct sum of copies of the additive group of integers, indexed by the limit ordinals of γ, identify γ with the set of characteristic functions of the singletons $\{\xi\} \subset \gamma$, and give the group the reverse lexicographical order. That is, the elements of the group are the functions with finite support, and the last nonzero coordinate of the union of two functions determines which comes first; if there is a tie, we go backwards until the functions differ.

Closely related to the additive valuations are multiplicative valuations that take on their values in an ordered field. When an additive valuation takes on its values in $\mathbb{R}$, one can use an exponential function to produce something that behaves like absolute value on the reals; in fact the absolute value on $\mathbb{R}$ is a special case of:

Definition 1.4. Let F be a field and let K be a totally ordered field. A *multiplicative K-valuation* is a function which assigns to each $x \in F$ an element $|x| \geqslant 0$ in K such that:

(1) $|x| = 0$ if and only if $x = 0$.
(2) $|xy| = |x||y|$ for all $x, y \in F$.
(3) $|x + y| \leqslant |x| + |y|$ for all $x, y \in F$.

If one begins with an additive valuation ν with values in $\mathbb{R}$ as described, and lets $|x| = 1/2^{\nu(x)}$ then one gets a valuation that even satisfies the strong triangle inequality,

$$|x + y| \leqslant \max\{|x|, |y|\}$$

and hence the associated metric satisfies the strong triangle inequality for metrics,

$$d(x, z) \leqslant \max\{d(x, y), d(y, z)\}.$$

One can convert any additive valuation into a multiplicative one by a formally similar process. Instead of using $\mathbb{R}$ as the ordered field, one can use the field $S(C, \Gamma, 1)$ of formal power series as defined in [16, p. 23] as long as C is an ordered field. The construction of $S(C, \Gamma, 1)$ goes back to a 1907 paper of Hahn [9]. The ordered abelian group Γ becomes the family of exponents in the formal powers of an indeterminate t, and the formal sums in turn are formal summations over an ordinal as the indexing set, with powers of t increasing monotonically in Γ. When C is ordered, the order on $S(C, \Gamma, 1)$ is determined by having the set of all positive elements be those formal sums whose first nonzero coefficient is a positive element of C. This makes the embedding $\Gamma \to S(C, \Gamma, 1)$ that sends α to t^α order-reversing. Letting $\Gamma = A$ when $A \cup \infty$ is the range of an additive valuation ν, one can then let $|x| = t^{\nu(x)}$ if $x \neq 0$ and let $|0| = 0$.

In any space that admits a metric with the strong triangle inequality, the open balls of radius ε partition the space into clopen sets, as do the closed balls of radius ε; of course, the latter need not be (the closures of) the former.

Valuations and metrics satisfying the strong triangle equality are called *non-Archimedean*, and the metrics are also referred to as *ultrametrics*. This terminology, together with the partition result just mentioned, inspired A.F. Monna [13] to make the following definition:

Definition 1.5. A space X is *non-Archimedean* if it is Hausdorff and has a base $\mathcal{B}$ such that if $B_i \in \mathcal{B}$ for $i \in \{0, 1\}$, then either (a) $B_0 \subset B_1$ or (b) $B_1 \subset B_0$ or (c) $B_0 \cap B_1 = \emptyset$.

The trichotomy in Definition 1.5 is expressed by saying $\mathcal{B}$ is *of rank* 1, a term due to Nagata [14], who also introduced the concept of a rank n base for all n. However, the concept of a rank 1 base goes back at least to the 1928 paper of Alexandroff and

Urysohn, where we find a theorem [2, p. 90] which, more or less literally translated, reads:

Theorem 1.6. *For every infinite zero-dimensional separable metrizable space M there is (when one views M as a topological space) a countable base $\mathfrak{B}$ satisfying the following properties:*

(1) *Of any two non-disjoint, otherwise arbitrary regions of the base $\mathfrak{B}$, one is a subset of the other.*

(2) *There is no infinite ascending sequence of regions in $\mathfrak{B}$ (that is, every increasing sequence*

$$G_1 \subset G_2 \subset \cdots, \quad G_n \neq G_{n+1},$$

of regions of $\mathfrak{B}$ breaks off after finitely many terms).

In other words, every separable zero-dimensional metrizable space has a rank 1 Noetherian base. Alexandroff and Urysohn were probably unaware that every Hausdorff or even T_1 space with a base satisfying (1) in Theorem 1.6 had a base satisfying both (1) and (2) simultaneously, so it is interesting that they listed these two properties in the same theorem. This is indeed one of the most useful facts about non-Archimedean spaces, and has been often referred to in papers although I am unaware of any place where a detailed proof has appeared. Section 2 gives an efficient proof of this fairly nontrivial fact, giving several basic properties of these spaces along the way.

It is clear from what was said after Definition 1.3 that every ω_μ-metrizable space with ω_μ regular uncountable, is also non-Archimedean. As for metrizable spaces, it is clear from a theorem of de Groot [8] that a metrizable space is non-Archimedean if, and only if, it has covering dimension zero. These two kinds of non-Archimedean spaces are precisely the spaces admitting a uniformity with a well-ordered base of equivalence relations. They are also the natural setting for generalizing the Alexandroff–Urysohn characterization of the irrationals. We have already seen it in the metrizable case, and the nonmetrizable case will be dealt with in Section 3.

For the generalization, it turns out that we need more than the analogue of completeness in metric spaces. We need a more demanding concept familiar to those working in normed vector spaces over fields with non-Archimedean valuations:

Definition 1.7. An ω_μ-metric space X is *spherically complete* if every nested family of closed balls $B(x, \xi)$ has nonempty intersection.

Example 1.8. Let X be the space of all binary ω_1-sequences in which all but finitely many terms are 0, with the inverse ω_1-metric defined by letting $\mu(x, y) = \min\{\alpha\colon x(\alpha) \neq y(\alpha)\}$. Then every Cauchy filter in the induced uniformity converges, but if we let x_n be the sequence whose first n terms are 1 while all others are zero, the nest of spheres $B(x_n, n)$ has empty intersection.

The concept of spherical completeness coincides with the property of every well-ordered pseudo-Cauchy net having a pseudo-limit.

Definition 1.9. A well-ordered net $\langle x_\alpha\colon\ \alpha < \delta\rangle$ in a space with inverse γ-metric μ is *pseudo-Cauchy* if δ is a limit ordinal and $\mu(x_\xi, x_\eta) < \mu(x_\sigma, x_\xi)$ whenever $\eta < \xi < \sigma$. The net $\langle x_\alpha\colon\ \alpha < \delta\rangle$ is *pseudo-convergent to* x if, for all ξ and η sufficiently large, $\eta < \xi$ implies $\mu(x, x_\eta) < \mu(x, x_\xi)$. In this case, we say x is the *pseudo-limit* of $\langle x_\alpha\rangle$.

The modification of this definition for valuations of both kinds is easy to guess at and can be found in [16] and in [15], respectively, and [15] shows the equivalence of spherical completeness with every pseudo-Cauchy sequence having a pseudo-limit in the metric case, in a way that is easily adaptable to the other settings we have looked at.

There is also an essentially algebraic concept called *maximal completeness* which is equivalent to spherical completeness in a valuated field. See [16] for a definition and proofs of two fundamental facts, done in the setting of arbitrary additive valuations: (1) every valuated field can be embedded in a maximally complete field and (2) a valuated field is maximally complete if, and only if, every well-ordered pseudo-Cauchy net has a pseudo-limit. Both results are highly nontrivial. The first is due to Krull [11] and its proof is only sketched in [16]; the second is apparently credited to Kaplansky's 1941 Harvard thesis by Schilling. Unfortunately, there seems to be no easy alternative way of showing that every valuated field can be embedded in a spherically complete field.

2. Non-Archimedean spaces and trees

We begin by recalling some definitions from the theory of partially ordered sets.

Definition 2.1. Given two elements $x < y$ of a poset, we say x is a *predecessor* of y and y is a *successor* of x. A *tree* is a partially ordered set in which the predecessors of any element are well-ordered. A totally ordered subset of a poset is called a *chain* and a maximal chain in a tree T is called a *branch* of T.

It is easy to show that a base $\mathcal{B}$ for a Hausdorff space X is a tree by reverse inclusion if, and only if, it is rank 1 and Noetherian—the conditions in Alexandroff and Urysohn's Theorem 1.6. Having a base like this makes proofs of many results very easy. For example, to show that a space with a tree base is ultraparacompact, one associates to each open cover $\mathcal{U}$ the family of all members of the tree base which are $\subset$-maximal with respect to being contained in members of $\mathcal{U}$. This family is then easily seen to be a partition of X into clopen sets refining $\mathcal{U}$. If instead one is given a rank 1 base which is not Noetherian, then the proof of ultraparacompactness is not nearly this easy. So it is handy to be able to produce a tree base for any non-Archimedean space. Before proving that it is always possible to do this, we take a quick look at the converse operation: producing a non-Archimedean space from a tree.

Definition 2.2. Given a tree T, the *branch space* of T is the space $\mathcal{B}(T)$ whose points are the branches of T, with a base for the open sets of $\mathcal{B}(T)$ consisting of the sets of the form $U_t = \{B\colon\ B \text{ is a branch of } T \text{ and } t \in B\}$.

If every nonmaximal member of T has at least two immediate successors, then the correspondence between t and U_t is an order-preserving bijection if the order of reverse inclusion is put on the family of U_t's. In any event, $\{U_t\colon\ t \in T\}$ is itself a tree and so $\mathcal{B}(T)$ has a tree base.

Definition 2.2 is reminiscent of what is done in Stone duality: in fact, branches of T can be viewed as maximal ideals of T, or as ultrafilters of the poset obtained by turning T upside down, as logicians are wont to do. The definition of the basic open sets (which are easily shown to be closed as well) is then the same in both cases, and we get a Hausdorff, indeed ultraparacompact topology in either case. For $\mathcal{B}(T)$ this is clear from the remarks preceding and immediately following Definition 2.2. In particular, $\mathcal{B}(T)$ is non-Archimedean.

There are, of course, differences between the spaces we get as a result. The branch space of a tree T is compact iff every antichain of T is finite, and one almost never has a unique tree-base for a space even if it is the branch space of some tree; on the other hand, every clopen set is a basic clopen set in the Stone space of a Boolean algebra. Also, while the Stone spaces of Boolean algebras are precisely the compact zero-dimensional spaces, there seems to be no convenient topological characterization of the branch spaces of trees.

Still, there are many equivalences between order properties of T and topological properties of $\mathcal{B}(T)$, just as one can find many in the Stone duality. For instance, $\mathcal{B}(T)$ is second countable iff it is separable iff T is countable, and $\mathcal{B}(T)$ is nonseparable and has the countable chain condition iff T is a Souslin tree.

More closely related to our aims in this paper is a basic embedding theorem: just as zero-dimensional spaces can be characterized as the subspaces of Stone spaces of Boolean algebras, so the non-Archimedean spaces can be characterized as the subspaces of branch spaces of trees. To show this, however, it is obviously necessary to show that every non-Archimedean space has a tree base; in other words, to show that the second clause in the Alexandroff–Urysohn Theorem 1.6 follows from the first for arbitrary Hausdorff spaces. (One could substitute "T_1" for "Hausdorff" but there is no gain in generality: every T_1 space with a rank 1 base is Hausdorff.) We head for this goal with:

Theorem 2.3. *Every rank* 1 *base for a Hausdorff space is an ortho-base; that is, every subcollection either has open intersection, or else it intersects in a nonisolated point for which the subcollection is a local base.*

Proof. Let $\mathcal{B}$ be a rank 1 base and let $\mathcal{B}' \subset \mathcal{B}$. If $\bigcap \mathcal{B}'$ is open there is nothing to prove, so suppose not. If $\bigcap \mathcal{B}'$ contains more than one point, say x and p are both in $\bigcap \mathcal{B}'$, then there exists $B_x \in \mathcal{B}$ such that $x \in B_x$ and $p \notin B_x$, but then by the rank 1 property, $B_x \subset B$ for all $B \in \mathcal{B}'$, whence $\bigcap \mathcal{B}'$ is a neighborhood of x, and similarly $\bigcap \mathcal{B}'$ is a neighborhood of each of its points, as desired.

If $\bigcap \mathcal{B}'$ is a singleton $\{p\}$, let B be an open nbhd of p. If $B = \{p\}$ there is nothing to prove; otherwise, let $x \in U \setminus \{p\}$ and let B' be a member of $\mathcal{B}'$ containing p and missing x. Then the rank 1 property insures that $B' \subset B$ and it follows that $\mathcal{B}'$ is a local base at p. $\square$

Notation 2.4. Given a rank 1 collection $\mathcal{B}$ of sets, we let $\overline{\mathcal{B}} = \{\bigcup \mathcal{C}\colon \mathcal{C}$ is a chain in $\mathcal{B}\}$.

Of course, we are including one-element chains and empty chains, so $\mathcal{B} \subset \overline{\mathcal{B}}$ and $\emptyset \in \overline{\mathcal{B}}$. So if $\mathcal{B}$ is a base for a space X then $\overline{\mathcal{B}}$ is one also; furthermore:

Lemma 2.5. *$\overline{\mathcal{B}}$ is a rank* 1 *collection.*

Proof. The key is to observe that if $D \in \mathcal{B}$ and $\mathcal{C}$ is a chain in $\mathcal{B}$, then either (a) $\bigcup \mathcal{C} \cap D = \emptyset$ or (b) D is a subset of some member of $\mathcal{C}$ or (c) $\bigcup \mathcal{C} \subset D$.

Indeed, if alternative (a) fails, let $C \in \mathcal{C}$ satisfy $C \cap D \neq \emptyset$. Then if alternative (b) fails, we must have $C \subset D$ and hence (c) holds.

So now, let B_1 and B_2 be any members of $\overline{\mathcal{B}}$. Let $B_1 = \bigcup \mathcal{C}$ and let $B_2 = \bigcup \mathcal{D}$ where $\mathcal{C}$ and $\mathcal{D}$ are chains in $\mathcal{B}$. If $B_1 \cap B_2 \neq \emptyset$, then either $B_1 \subset D$ for some $D \in \mathcal{D}$ (whence $B_1 \subset B_2$) or else every D in $\mathcal{D}$ is a subset of some member of $\mathcal{C}$, whence $B_2 \subset B_1$. $\square$

Lemma 2.6. *Every non-Archimedean space has a rank* 1 *base which is closed under the union of chains.*

Proof. We will be done as soon as we show $\overline{\mathcal{B}}$ is closed under the union of chains, for a rank 1 base $\mathcal{B}$.

Let $\mathcal{D}$ be a chain in $\overline{\mathcal{B}}$. Without loss of generality, we may assume $\mathcal{D}$ is well-ordered by inclusion with no maximal member, $\mathcal{D} = \bigcup\{D_\xi\colon \xi < \gamma\}$. Given $\xi < \gamma$, let $D_{\xi+1} = \bigcup \mathcal{C}_\xi$ for some chain $\mathcal{C}_\xi \subset \mathcal{B}$. Then by the key to Lemma 2.5, there is a member $C(\xi)$ of $\mathcal{C}_\xi$ such that $D_\xi \subset C(\xi)$. Clearly $\bigcup \mathcal{D} = \bigcup\{C(\xi)\colon \xi < \gamma\}$, and so $\bigcup \mathcal{D} \in \overline{\mathcal{B}}$, as desired. $\square$

The following result is very useful in the theory of non-Archimedean spaces.

Theorem 2.7. *Let X be a non-Archimedean space and let $\mathcal{B}$ be a* rank 1 *base for X that is closed under the union of chains. Every open subset of X is the union of a disjoint collection of members of $\mathcal{B}$.*

Proof. Let U be an open subset of X and let $\mathcal{V} = \{B \in \mathcal{B}\colon B \subset U\}$. By Zorn's Lemma, every member of $\mathcal{V}$ is a subset of a maximal member of $\mathcal{V}$, and the maximal members of $\mathcal{V}$ are obviously disjoint and cover U. $\square$

For the next result, we recall the usual notation for levels $T(\alpha)$ of a tree T:

Notation 2.8. If T is a tree, then $T(0)$ is its set of minimal members. Given an ordinal α, if $T(\beta)$ has been defined for all $\beta < \alpha$, then $T \restriction \alpha = \bigcup\{T(\beta)\colon \beta < \alpha\}$, while $T(\alpha)$ is the set of minimal members of $T \setminus T \restriction \alpha$.

Now we are ready to prove the main theorem of this section.

Theorem 2.9. *Every non-Archimedean space has a base which is a tree by reverse inclusion.*

Proof. Let X be a non-Archimedean space and let $\mathcal{B}$ be a rank 1 base for X that is closed under the union of chains.

We define the tree $\mathcal{T}$ by induction. Using Theorem 2.7, let $\mathcal{T}(0)$ be a partition of X into members of $\mathcal{B}$. If β is an ordinal and the collection of disjoint base members $\mathcal{T}(\beta)$ has been defined, partition any nonsingleton member B of $\mathcal{T}(\beta)$ into at least two members by first letting $B' \in \mathcal{B}$ be a proper subset of B and then partitioning $B \setminus B'$ into members of $\mathcal{B}$. Doing this for every nonsingleton member of $\mathcal{T}(\beta)$ gives $\mathcal{T}(\beta+1)$.

If β is a limit ordinal, and $\mathcal{T}(\alpha)$ has been defined for all $\alpha < \beta$, let $\mathcal{P}$ be the set of all intersections of chains in $\mathcal{T} \restriction \beta$ which meet every $\mathcal{T}(\alpha)$ such that $\alpha < \beta$. It is easy to see that any two distinct members of $\mathcal{P}$ are disjoint. By Theorem 2.3, every $P \in \mathcal{P}$ is either clopen or else is a nonisolated singleton $\{p\}$ for which the members of $\mathcal{T} \restriction \beta$ containing p constitute a local base at p. We let $\mathcal{T}(\beta)$ be the collection of all clopen members of $\mathcal{P}$.

We continue the induction until we arrive at a γ such that $\mathcal{T}(\gamma)$ is empty. This could either occur at a limit stage in which no member of $\mathcal{P}$ is clopen, or at a successor stage $\gamma = \beta + 1$ in which every member of $\mathcal{T}(\beta)$ is a singleton. In either case, $\mathcal{T} = \bigcup\{\mathcal{T}(\beta)\colon \beta < \gamma\}$.

To see that $\mathcal{T}$ is a base for X, let $p \in X$. Let α be the least ordinal such that $p \notin \bigcup \mathcal{T}(\alpha)$. If α is a limit ordinal, then $\mathcal{T} \restriction \alpha$ contains a local base at p, while if $\alpha = \beta + 1$ is a successor then p is isolated and $\{p\} \in \mathcal{T}(\beta)$. $\square$

Now for the basic embedding theorem.

Theorem 2.10. *A space is non-Archimedean if, and only if, it can be embedded into the branch space of a tree.*

Proof. Sufficiency follows from the fact that the branch space of any tree is non-Archimedean, and the fact that every subspace of a non-Archimedean space is non-Archimedean.

To show necessity, let T be a tree base for the non-Archimedean space X, and define a map $e : X \to \mathcal{B}(T)$ by letting $e(x)$ be the branch $B_x = \{t \in T\colon x \in t\}$. So $e(x) = \bigcap\{U_t\colon x \in t\}$ where, as before, $U_t = \{B\colon B \text{ is a branch of } T \text{ and } t \in B\}$. This map e is injective because X is Hausdorff, and we have $t = e^{\leftarrow}[U_t]$ since if $x \in t$ then $e(x) \in U_t$ and conversely. Thus e is an embedding. $\square$

3. Characterization theorems and applications

One theorem of this section is an embedding characterization of ω_μ-metrizable spaces similar to Theorem 2.10, involving a nice subclass of the following kinds of trees.

Definition 3.1. Let γ be an ordinal and let λ be a cardinal. The *full λ-ary tree of height γ* is the set ${}^{<\gamma}\lambda$ of all transfinite sequences $f : \alpha \to \lambda$ such that $\alpha < \gamma$, with the order on the tree given by end extension: $f \leqslant g$ iff $\operatorname{dom} f \subset \operatorname{dom} g$ and $g \restriction \operatorname{dom} f = f$.

As usual, Definition 3.1 interprets an ordinal as the set of all smaller ordinals, starting with $0 = \emptyset$. There is no gain in generality if we replace λ by an ordinal δ since ${}^{<\gamma}\delta$ is obviously order-isomorphic to ${}^{<\gamma}\lambda$ where $\lambda = |\delta|$. Of course, the branch spaces are homeomorphic as well, but we will also see homeomorphisms between branch spaces of nonisomorphic trees of the form ${}^{<\gamma}\lambda$. The following concepts will be used in clarifying the picture.

Definition 3.2. A nonempty subset A of an ordinal γ is a *tail* if $\alpha \in A$ and $\xi \in \gamma$, $\xi > \alpha$ together imply $\xi \in A$. An ordinal is *uniform* if it is order-isomorphic to every tail of itself. Given an ordinal γ, the *eventual tail* of γ is the order type of those tails of γ which are order-isomorphic to any of their sub-tails.

Every ordinal has an eventual tail, because when we take successively smaller tails, the order type can never increase, hence becomes constant after finitely many reductions of order type.

The branches of ${}^{<\gamma}\lambda$ are in a natural one-to-one correspondence with the elements of ${}^{\gamma}\lambda = \{f\colon\ f \text{ is a function from } \gamma \text{ into } \lambda\}$. When we identify the branch space $\mathcal{B}({}^{<\gamma}\lambda)$ with ${}^{\gamma}\lambda$ in this way, the basic clopen set U_σ of $\mathcal{B}({}^{<\gamma}\lambda)$ gets identified with

$$B(f, \operatorname{dom}\sigma) = \{f \in {}^{\gamma}\lambda\colon\ f \restriction \operatorname{dom}\sigma = \sigma\}.$$

The notation $B(f, \operatorname{dom}\sigma)$ harks back to Definition 1.3 and associates an inverse γ-metric with ${}^{\gamma}\lambda$: $\mu(f, f) = \gamma$ for all $f \in {}^{\gamma}\lambda$ and if $f \neq g$, then $\mu(f, g) = \min\{\xi\colon\ f(\xi) \neq g(\xi)\}$. In other words, $\mu(f, g) \geqslant \xi$ iff $f \restriction \xi = g \restriction \xi$. It seems appropriate, then, to call the resulting topology on ${}^{\gamma}\lambda$ the *initial agreement topology* and to call μ the *canonical inverse γ-metric* on ${}^{\gamma}\lambda$.

Any nest of spheres in ${}^{\gamma}\lambda$ corresponds to a nested family of sets of the form U_σ, in other words, a family where the σ are totally ordered. Therefore, they are in a branch and have nonempty intersection; so ${}^{\gamma}\lambda$ is spherically complete.

To simplify notation, we will call a space *γ-metrizable* if it has an inverse γ-metric inducing the topology. Analogously to Theorem 2.10, we have:

Theorem 3.3. *Let X be a topological space. The following are equivalent.*

(1) *X is γ-metrizable for some ordinal γ.*

(2) *X is either κ-metrizable for some uncountable regular κ or X is metrizable and non-Archimedean.*

(3) *X can be embedded in the branch space of a tree of the form ${}^{<\kappa}\lambda$ where κ and λ are regular cardinals.*

(4) *X is homeomorphic to a subspace of some ${}^{\kappa}\lambda$ with the initial agreement topology, where κ and λ are regular cardinals.*

Proof. (1) $\Rightarrow$ (2) If γ is a successor ordinal, then X is discrete. Otherwise, let κ be the cofinality of γ, and pick a cofinal increasing sequence $\langle \xi_\alpha\colon \alpha < \kappa\rangle$ in γ. Define $\mu'(x,y) = \min\{\xi_\alpha\colon \xi_\alpha \geqslant \mu(x,y)\}$ where μ is the inverse g-metric. It is easy to see that μ' is an inverse κ-metric uniformly equivalent to μ.

(2) $\Rightarrow$ (3) The balls $B(x,\xi)$ form a tree base as x ranges over X and ξ ranges over k, ordered by reverse inclusion as usual. If $B(x,\xi) = \{x\}$ for some $\xi < \kappa$ then we put in copies of $B(x,\xi)$ to make all branches of the tree of length κ. Letting λ be the supremum of the cardinalities of sets of immediate successors (by reverse inclusion) of members of the resulting tree, one easily embeds this tree into ${}^{<\kappa}\lambda$, and X embeds in the branch space as in the proof of Theorem 2.10.

(3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) were shown prior to the statement of the theorem. $\square$

If κ is a regular cardinal and λ is given the discrete topology, the initial agreement topology on ${}^{\kappa}\lambda$ is the coarsest topology which is finer than the product topology and has the property that the intersection of fewer than κ open sets is open. In some ways it is a more well-behaved topology than either the product topology or the box topology, and provides the natural setting for the generalization of the Alexandroff–Urysohn characterization theorem to higher cardinals. The following well-known example illustrates how ω is actually an unusual cardinal number in this context.

Example 3.4. Let $\kappa = \omega_1$. Let ${}^{\kappa}2$ be given the initial agreement topology. Each nonempty clopen set can be partitioned into $\mathfrak{c}$ clopen sets. Indeed, if $g \in B(f,\alpha)$ then $B(g,\alpha+\omega) = B(f,\alpha+\omega)$ if, and only if, $g \restriction \{\xi\colon \alpha \leqslant \xi < \alpha+\omega\} = f \restriction \{\xi\colon \alpha \leqslant \xi < \alpha+\omega\}$, and there are $\mathfrak{c}$ ways of choosing the coordinates in $\{\xi\colon \alpha \leqslant \xi < \alpha+\omega\}$.

So ${}^{\omega_1}2$ is not the ω_1-analogue of the Cantor set, and it is not even the ω_1-analogue of the space of irrationals unless we assume CH. This fact generalizes to all ordinals γ other than strongly inaccessible cardinals. There indeed we have an analogue of the space of irrationals, at least; and, as will be seen, we have an analogue of the Cantor set if and only if γ is a weakly compact cardinal number.

If γ is a cardinal number, it is easy to see that ${}^{\gamma}\lambda$ with the initial agreement topology is homeomorphic to each basic clopen subset of itself. In fact, the tree of all nets in λ indexed by members of γ is order-isomorphic to the tree of all nets in λ indexed by members of $\{\eta\colon \xi < \eta < \gamma\}$, thanks to the order-isomorphism of γ with each tail of γ. This holds for all uniform limit ordinals, leading to:

Theorem 3.5. *Let γ be a uniform limit ordinal and let λ be a cardinal > 1. If $\gamma \leqslant \lambda$, or if γ is not strongly inaccessible, then the following are equivalent.*

(1) *X is homeomorphic to ${}^{\gamma}\lambda$ with the initial agreement topology.*

(2) *X is γ-metrizable by a spherically complete inverse metric, and every nonempty open set can be partitioned into $\lambda^{<\gamma}$ clopen sets, but no more.*

(3) *X is homeomorphic to ${}^{\mathrm{cf}\,\gamma}\overline{\lambda}$ with the initial agreement topology, where $\overline{\lambda} = \lambda^{<\gamma}$.*

Proof. (1) $\Rightarrow$ (2) We have already demonstrated the former clause. The latter clause follows from Theorem 2.7, from the fact that there are $\lambda^{<\gamma}$ basic clopen sets altogether, and from the fact that each basic clopen subset of ${}^{\gamma}\lambda$ is homeomorphic to ${}^{\gamma}\lambda$.

(2) $\Rightarrow$ (3) We first assume γ is regular. Let $\mathcal{B}_0 = \{X\}$. If $\alpha = \beta + 1$ and the clopen partition $\mathcal{B}_\beta$ has been defined, let $\mathcal{B}_\alpha$ be defined by splitting each member of $\mathcal{B}_\beta$ into $\lambda^{<\gamma}$ clopen sets and thence into $\lambda^{<\gamma}$ spheres of inverse diameter $\geqslant \alpha$. If $\alpha < \gamma$ is a limit ordinal, and $\mathcal{B}_\beta$ is a partition of $\lambda^{<\gamma}$ into spheres of inverse diameter $\geqslant \beta$ for all $\beta < \alpha$, then the common refinement of the partitions $\mathcal{B}_\beta$ is the partition whose members are the intersections of all maximal chains in $\bigcup\{\mathcal{B}_\beta\colon \beta < \alpha\}$. Each such intersection is nonempty by spherical completeness, and is itself a sphere, being the intersection of spheres, and open because fewer than γ spheres are being intersected. Now $\mathcal{B} = \bigcup\{\mathcal{B}_\alpha\colon \alpha < \gamma\}$ is a tree by reverse containment, and is clearly isomorphic to ${}^{<\gamma}\overline{\lambda}$, where $\overline{\lambda} = \lambda^{<\gamma}$. And X is homeomorphic to the branch space of $\mathcal{B}$ since each branch of $\mathcal{B}$ has a singleton intersection. This last fact follows from the spherical completeness of X and the fact that the set of inverse diameters of each maximal chain in $\mathcal{B}$ is unbounded. So X is homeomorphic to ${}^{\gamma}\overline{\lambda}$.

If γ is singular, let $\{\beta_\xi\colon \xi < \operatorname{cf}\gamma\}$ be a cofinal sequence in γ and define

$$\mathcal{B} = \bigcup\{\mathcal{B}_\alpha\colon \alpha < \operatorname{cf}\gamma\}$$

just like we did $\bigcup\{\mathcal{B}_\alpha\colon \alpha < \gamma\}$ above. The remainder of the proof is as before.

Applying this to $X = {}^{\gamma}\overline{\lambda}$ in particular shows that ${}^{\gamma}\overline{\lambda}$ is homeomorphic to ${}^{\operatorname{cf}\gamma}\overline{\lambda}$ for all uniform limit ordinals γ and all cardinals $\lambda > 1$. From this it is clear that (3) implies (1). $\square$

Note that from the foregoing theorem one can deduce such facts as ${}^{\omega_1}2$ being homeomorphic to ${}^{\omega_1}\mathfrak{c}$ and the fact that if $\kappa = \sup\{2^{\omega_n}\colon n \in \omega\}$, then ${}^{\kappa}2$ is homeomorphic to ${}^{\omega}\kappa$. In particular, we get the purely set-theoretic fact that $2^\kappa = \kappa^\omega$. One can get many other set-theoretic facts by similar uses of Theorem 3.5. For example, $\prod\{\aleph_n\colon n \in \omega\}$ is obviously of weight $\aleph_\omega$ as is every nonempty basic clopen set, and is easily partitionable into $\aleph_\omega$ clopen sets, and is spherically complete. So it is homeomorphic to ${}^{\omega}\lambda$ where $\lambda = \aleph_\omega^{<\omega} = \aleph_\omega$. And so we have the purely set-theoretic fact that

$$\prod\{\aleph_n\colon n \in \omega\} = (\aleph_\omega)^\omega.$$

Some of the spaces ${}^{\gamma}\lambda$ not covered by Theorem 3.5 can be quickly disposed of. If $\lambda = 0$ then $X = \emptyset$; if $\lambda = 1$ then X is a singleton. If γ is not a limit ordinal, X is discrete, of cardinality λ^γ. More generally, if γ is not a uniform limit ordinal, then X is homeomorphic to the direct sum of λ^γ copies of ${}^{\delta}\lambda$, where δ is the eventual tail of γ. Of course, if γ is a successor, this eventual tail is $\{0\}$. If γ is a nonuniform limit ordinal, the question of whether X is subsumed under Theorem 3.5 revolves around the question of whether every open set can be partitioned into λ^γ clopen sets, and this happens iff $\lambda^\gamma = \lambda^{<\delta}$ iff X is homeomorphic to each of its nonempty clopen (also each of its nonempty open) subsets. So all that is left is the case covered by:

Theorem 3.6. *If $\gamma > \lambda$ and γ is strongly inaccessible, then ${}^{\gamma}\lambda$ is homeomorphic to ${}^{\gamma}2$.*

Proof. As we have seen, ${}^{\gamma}2$ is homeomorphic to each of its basic clopen sets, and it can be partitioned into λ basic clopen sets as follows. For each $\xi < \lambda$ let $B_\xi = \{x \in {}^{\gamma}2\colon x(\eta) = 0$ for all $\eta < \xi,\ x(\xi) = 1\}$, and let $B_\lambda = \{x \in {}^{\gamma}2\colon x(\xi) = 0$ for all $\xi < \lambda\}$. Now a procedure like that in the regular case of (2) $\Rightarrow$ (3) shows that ${}^{\gamma}2$ is homeomorphic to ${}^{\gamma}\lambda$. □

For most cardinals λ, the homeomorphism of ${}^{\gamma}\lambda$ with ${}^{\gamma}2$ also holds if $\lambda = \gamma$. It all depends on whether ${}^{\lambda}2$ can be partitioned into $\lambda^{<\lambda}$ clopen sets, by Theorem 3.5, and it can be partitioned in this way if λ is not strongly inaccessible. When λ is strongly inaccessible, $\lambda^{<\lambda} = \lambda$; in fact,

$$^{<\lambda}\lambda = \bigcup\{{}^{\alpha}\beta\colon\ \alpha \leqslant \beta < \lambda\}.$$

In other words, when λ is strongly inaccessible, we are asking whether ${}^{\lambda}2$ is just an analogue of the irrationals, or whether there really is an analogue of the Cantor set.

This problem was solved in 1964 by Monk and Scott [12], with a purely set-theoretic characterization of the right λ.

Definition 3.7. Let κ be an infinite cardinal number. A topological space X is *κ-compact* if every open cover of X has a subcover of cardinality $< \kappa$. A *κ-Aronszajn tree* is a tree T of cardinality κ such that $|T(\alpha)| < \kappa$ for all $\alpha < \kappa$, and such that every branch of T is of cardinality $< \kappa$. An cardinal number κ is *weakly compact* if it is strongly inaccessible and there are no κ-Aronszajn trees.

Theorem 3.8. *Let κ be a cardinal number. The following are equivalent.*
(1) *κ is weakly compact.*
(2) *${}^{\kappa}2$ is κ-compact.*
(3) *${}^{\kappa}2$ cannot be partitioned into κ-many disjoint open sets.*

Proof. The equivalence of (1) and (2) was shown in [12]. (2) clearly implies (3); conversely, every open cover of ${}^{\kappa}2$ refines to a partition into members of the usual tree base for ${}^{\kappa}2$ (see the beginning of Section 2). □

Since ${}^{\kappa}2$ is not subsumed by Theorem 3.5 when κ is weakly compact, it is worth going into a little more detail about its topology. From Theorem 3.8 it follows that every open cover $\mathcal{U}$ is *uniform*; that is, there is a partition refining $\mathcal{U}$ right in the uniformity induced by the canonical inverse κ-metric on ${}^{\kappa}2$. Indeed, we can first refine $\mathcal{U}$ to a partition into basic clopen sets, take the supremum σ of the inverse radii involved, and use the partition into spheres of inverse radius τ for any $\tau \geqslant \sigma$.

There are a number of other pleasing parallels with the Cantor set. It is easy to see that there are two kinds of nonempty open sets up to homeomorphism, the clopen ones and the nonclosed ones, just like with the Cantor set, and that the latter ones are partitionable into κ clopen subsets of ${}^{\kappa}2$. It is also easy to show, just like with the Cantor set, that every κ-metrizable space of weight κ embeds in ${}^{\kappa}2$. Also, we have a generalization of the Brouwer 1910 characterization of the Cantor set, except that spherical completeness needs to be added:

Theorem 3.9. *A space is homeomorphic to* ${}^{\kappa}2$ *for some weakly compact cardinal* κ *if, and only if, it is* κ*-compact, totally disconnected, has no isolated points, and is* κ*-metrizable by a spherically complete inverse* κ*-metric.*

Proof. The case $\kappa = \aleph_0$ just gives the Brouwer Theorem. If κ is uncountable, then necessity is clear. As for sufficiency, the fact that X has no isolated points insures that every nonempty clopen set splits into infinitely many clopen sets, and spherical completeness gives us κ being strongly inaccessible by Theorem 3.5. We will define partitions $\mathcal{B}_\alpha$ similarly to what was done in (2) $\Rightarrow$ (3) in the proof of Theorem 3.5. As before, let $\mathcal{B}_0 = \{X\}$.

Suppose $\mathcal{B}_\alpha$ has been defined, to be a partition of X into $2^{|\alpha|}$ clopen sets, but $\mathcal{B}_{\alpha+1}$ has not yet been defined. Let $\mathcal{P}_\alpha$ be a partition of X which partitions each member of $\mathcal{B}_\alpha$ into infinitely many clopen balls of inverse radius $\geqslant \alpha$. Let $\lambda < \kappa$ be a cardinal such that no member of $\mathcal{P}_\alpha$ partitions any member of $\mathcal{B}_\alpha$ into more than 2^λ sets. Let $\mathcal{B}_{\alpha+\lambda}$ partition every member of $\mathcal{P}_\alpha$ and hence of $\mathcal{B}_\alpha$ into exactly 2^λ clopen sets. As for the ordinals between α and λ, let φ_B be a bijection from ${}^{\lambda}2$ to the family of 2^λ clopen sets into which each member B of $\mathcal{B}_\alpha$ has been partitioned. Then, use the full binary tree of height λ to define the partitions $\mathcal{B}_{\alpha+\xi}$ such that $\xi < \lambda$: given a function $f:\xi \to 2$, and $B \in \mathcal{B}_\alpha$, let

$$A(B,f) = \bigcup\{\varphi_B(h):\ h \in {}^{\lambda}2,\ f \subset h\}$$

and let

$$\mathcal{B}_{\alpha+\xi} = \{A(B,f):\ B \in \mathcal{B}_\alpha,\ f \in {}^{<\lambda}2\}.$$

If $\alpha < \kappa$ is a limit ordinal and $\mathcal{B}_\beta$ has been defined for all $\beta < \alpha$ and $\mathcal{P}_\beta$ has been defined for cofinally many $\beta < \alpha$, we take advantage of the way $\mathcal{P}_\beta$ is defined from $\mathcal{B}_\beta$ to argue that every maximal chain in $\bigcup\{\mathcal{B}_\beta:\ \beta < \alpha\}$ has nonempty intersection, which is clopen since $\alpha < \kappa$. Now let $\mathcal{B}_\alpha$ be the set of all such intersections. We can now finish the argument as in (2) $\Rightarrow$ (3) of Theorem 3.5 to conclude that X is homeomorphic to ${}^{\kappa}2$. Now Theorem 3.8 assures us that κ is weakly compact. □

Since total disconnectedness of X is automatic if κ is uncountable, there follows:

Corollary 3.10. *Let* κ *be an uncountable weakly compact cardinal. A space is homeomorphic to* ${}^{\kappa}2$ *if, and only if, it is* κ*-compact, has no isolated points, and is* κ*-metrizable by a spherically complete inverse* κ*-metric.*

The spherical completeness condition in Theorem 3.9 cannot be eliminated. For instance, the subspace of ${}^{\kappa}2$ which consists of all points with finite support (set of nonzero coordinates) has no isolated points and is closed in ${}^{\kappa}2$ if κ is an uncountable cardinal, and is therefore κ-compact, but it is of cardinality κ and so cannot be homeomorphic to ${}^{\kappa}2$. Of course, one could try to find a simpler property to substitute for spherical completeness, but the following example shows that many natural candidates for this property will not work.

Example 3.11. Let κ be an uncountable weakly compact cardinal. The subspace $\bigcup\{{}^{\kappa}n\colon\ n \in \omega\}$ of ${}^{\kappa}\omega$ is closed and therefore κ-compact. It is also of cardinality 2^{κ}, homogeneous, and homeomorphic to each of its nonempty clopen subsets. However, any compatible inverse κ-metric will give a nested family of closed balls B_n such that $B_n \cap {}^{\kappa}n = \emptyset$. Then

$$\bigcap\{B_n\colon\ n \in \omega\} = \emptyset$$

and so $\bigcup\{{}^{\kappa}n\colon\ n \in \omega\}$ is not homeomorphic to ${}^{\kappa}2$.

We are now ready for a general classification theorem, whose proof is a routine application of Theorems 3.5, 3.6 and the paragraph that precedes it, Theorem 3.9, and the fact that every space with a compatible inverse γ-metric is ultraparacompact.

Theorem 3.12. *The following are equivalent for a nonempty space X.*

(1) *X is homeomorphic to a space of the form $D \times {}^{\delta}\lambda$, where D is discrete and ${}^{\delta}\lambda$ has the initial agreement topology.*

(2) *X is spherically complete with respect to some inverse γ-metric, and either* (a) *γ is weakly compact and X is locally γ-compact or* (b) *there exists $\overline{\lambda}$ such that every point of X has a neighborhood which can be partitioned into $\overline{\lambda}$ open sets but not more than $\overline{\lambda}$ open sets, and such that every nonempty open set can be partitioned into $\overline{\lambda}$ open sets.*

Of course, in Case (2)(a), X is homeomorphic to ${}^{\gamma}2$ if it has no closed discrete subspace of cardinality $|\gamma|$, and to $D \times {}^{\gamma}2$ otherwise, where $|D|$ is the maximum cardinality of a closed discrete subspace of X. Of course, the maximum cardinality is always attained if X is not $|\gamma|$-compact.

As we have seen, Case (2)(b) is disjoint from Case (2)(a) and X is determined up to homeomorphism in Case (2)(b) by its character $\chi(X)$, its weight $w(X)$, and by $\overline{\lambda}$, as long as we use the convention that a discrete space is of character 1. In general, $\chi(X)$ is regular and equals $\operatorname{cf}\gamma$, and it either equals 1 (whence $\overline{\lambda} = 1$ and we have a discrete space) or it is infinite and then $w(X) \geqslant \overline{\lambda} \geqslant \chi(X)$. Also, a routine ramification argument using spherical completeness gives $\overline{\lambda}^{<\chi(X)} = \overline{\lambda}$. With these restrictions, all choices of these cardinals can be realized by some X. In particular, $w(X) > \overline{\lambda}$ gives us $D \times {}^{\chi(X)}\overline{\lambda}$ for some discrete space D, while $w(X) = \overline{\lambda}$ gives ${}^{\chi(X)}\overline{\lambda}$.

Another point of similarity of ${}^{\kappa}2$ for weakly compact κ with the Cantor set is that ${}^{\kappa}2$ is closed in any κ-metrizable space containing it. In fact, there is a general theory of κ-compactness behind this last result:

Definition 3.13. A topological space X is *κ-additive* if every intersection of fewer than κ open sets is open.

Of course, every κ-metrizable space is κ-additive. The following theorem has a simple proof just like the one that shows every compact Hausdorff space is closed in any Hausdorff space containing it:

Theorem 3.14. *A κ-compact κ-additive space is closed in any Hausdorff κ-additive space containing it.*

On the other hand, if X is κ-metrizable and not κ-compact, then we can split it into exactly κ clopen sets, and add a point p whose neighborhoods are those subsets of $X \cup \{p\} = Y$ which include p together with all but $< \kappa$ members of the clopen partition of X. Then Y is κ-metrizable and X is not closed in Y. Other elementary results along these lines can be found in [19]. Thanks in part to Theorem 3.14, we have not only an analogue of the Cantor set but also of the two other absolute F_σ-sets characterized by Alexandroff and Urysohn. It is quite easy to see that, if κ is weakly compact, there is only one locally κ-compact non-κ-compact space of weight κ without isolated points that can be given a spherically complete inverse κ-metric. It is, up to homeomorphism, the unique non-κ-compact open subset of ${}^{\kappa}2$.

There is also an easily defined analogue of Alexandroff and Urysohn's Ψ that hews closely to their original description: let $\Psi(\kappa)$ be the space of all points of ${}^{\kappa}\kappa$ which eventually take on only the values of 0 and 1. This is the union of κ copies of ${}^{\kappa}2$, the ξth copy A_ξ being the set of all points which take on no values besides 0 and 1 from the ξth term on. Indeed, A_ξ is a copy of ${}^{\kappa}2$, because it is the union of $< \kappa$ relatively clopen subsets $A_\xi(f) = \{g \in A_\xi\colon\ g \upharpoonright \xi = f\}$ as f ranges over ${}^{\xi}\kappa$. Of course, $A_\xi(f)$ is a copy of ${}^{\kappa}2$, and so is A_ξ, since it satisfies the conditions of the characterization in Theorem 3.9.

Thus $\Psi(\kappa)$ is an absolute F_κ-set in the category of Hausdorff κ-additive spaces. In other words, if $\Psi(\kappa)$ is embedded in a Hausdorff κ-additive space Y, it is the union of $\leqslant \kappa$ closed subsets of Y. Other properties generalizing the ones Alexandroff and Urysohn gave for $\Psi(\aleph_0)$ are also easy to show. Because $\Psi(\kappa)$ is dense in ${}^{\kappa}\kappa$, it is nowhere locally κ-compact, and clearly every nonempty open subset has cardinality $> \kappa$ (in fact, it has cardinality ${}^{\kappa}2$). Finally, zero-dimensionality is automatic when κ is uncountable. Another space easily seen to have these properties is the subspace $\bigcup\{{}^{\kappa}\lambda\colon\ \lambda < \kappa\}$ of ${}^{\kappa}\kappa$.

In view of Example 3.11, it seems unrealistic to conjecture that the foregoing properties characterize these spaces up to homeomorphism. We can, however, draw on other properties of these spaces to arrive at several characterizations.

Definition 3.15. A κ-metric space is *initially spherically complete* if every nested collection of closed spheres whose inverse radii are bounded below κ has nonempty intersection.

Theorem 3.16. *Let κ be an uncountable weakly compact cardinal. The following are equivalent for a space X.*

(1) *X is homeomorphic to $\bigcup\{{}^{\kappa}\lambda\colon\ \lambda < \kappa\}$.*

(2) *X is the union of an ascending κ-chain of copies of ${}^{\kappa}2$, each of which is nowhere dense in all succeeding copies, and X is initially spherically complete with respect to an inverse κ-metric.*
(3) *X is a nowhere locally κ-compact space that is the union of κ copies of ${}^{\kappa}2$ and is initially spherically complete with respect to an inverse κ-metric.*
(4) *X is an absolute F_{κ}-set in the category of regular κ-additive spaces, and X is nowhere locally κ-compact, initially spherically complete with respect to an inverse κ-metric, and has the property that each nonempty open set contains a dense-in-itself spherically complete subset.*

Before proving this theorem, we give a lemma whose proof incorporates some ideas that will be used in proving the theorem.

Lemma 3.17. *Let X be as in* (4) *of Theorem* 3.16. *Then every κ-compact subset of X is contained as a nowhere dense subset in some copy of ${}^{\kappa}2$ in X.*

Proof. First we note that every nonempty open subset of X contains a copy of ${}^{\kappa}2$. Indeed, any dense-in-itself spherically complete κ-metric space M contains a copy of ${}^{\kappa}2$. This can be seen by a simple induction picking a binary subtree of the tree $\{B(x,\alpha)\colon x \in M,\ \alpha < \kappa\}$, and using spherical completeness to show that the subspace of points whose base comes from this subtree is homeomorphic to the branch space, which in turn is a copy of ${}^{\kappa}2$.

Let K be a κ-compact subset of X. By nowhere local κ-compactness and the preceding paragraph, we may assume $\{B(z,\alpha+1)\colon z \in B(x,\alpha)\}$ is of cardinality at least κ, for each $x \in X$ and each $\alpha < \kappa$. Also, inasmuch as K is κ-compact, fewer than κ of the $\{B(z,\alpha+1)\colon z \in B(x,\alpha)\}$ meet K, and so we can take a copy $K(x,\alpha)$ of ${}^{\kappa}2$ from one of the balls $B(z,\alpha+1)$ that does not meet K. Let

$$\mathcal{B} = \{B(x,\alpha)\colon x \in K,\ \alpha < \kappa\}.$$

Each maximal chain in $\mathcal{B}$ of cardinality $\leqslant \kappa$ has nonempty intersection, a ball $B(z,\gamma)$ where γ is the supremum of the inverse diameters of the balls making up the chain. For each such $B(z,\gamma)$ that does not meet K, let $L(z,\gamma)$ be a copy of ${}^{\kappa}2$ in $B(z,\gamma)$ and let $\mathcal{L}$ be the set of all such $L(z,\gamma)$. Let

$$K^* = \bigcup\{K(x,\alpha)\colon B(x,\alpha)\cap K \neq \emptyset\} \cup \Big(\bigcup \mathcal{L}\Big).$$

Now any cover of $K^* \cup K$ by clopen balls contains fewer than κ members meeting K and so there is some $\alpha < \kappa$ such that the $B(x,\alpha)$ that meet K cover it. The part of K^* outside the union of these $B(x,\alpha)$ is the union of fewer than κ subsets $K(x,\beta)$ $(\beta < \alpha)$ and $L(z,\gamma)$ $(\gamma < \alpha)$, each of which is κ-compact, so $K^* \cup K$ is κ-compact. It also has no isolated points since none of the $K(x,\alpha)$ or $L(z,\gamma)$ has any, and each point of K is a limit point of the points of K^*. Every κ-compact subset H of a κ-metrizable space is complete; indeed, every Cauchy filter on H contains a family of relatively clopen sets of arbitrarily large inverse diameters, and if the intersection were empty, the complements would be a clopen cover without a subcover of cardinality $< \kappa$. Hence in particular $K \cup K^*$ is complete, and by the first paragraph, each $K(x,\alpha)$ and each $L(z,\gamma)$ can be taken to be spherically complete. Finally, any nest in $\mathcal{B}$ has nonempty intersection with $K^* \cup K$ and thus this space is spherically complete. $\square$

Proof of Theorem 3.16. (1) $\Rightarrow$ (2) is easy: let the αth member of the sequence be ${}^{\kappa}\alpha$ and use Theorem 3.6.

(2) $\Rightarrow$ (3) Any κ-compact clopen subset of X is complete (see the proof of Lemma 3.17) and so by initial spherical completeness of X it would be spherically complete and thus homeomorphic to ${}^{\kappa}2$. Now use the obvious analogue of the Baire Category Theorem to conclude that a clopen subset of ${}^{\kappa}2$ is not the union of κ nowhere dense sets.

(3) $\Rightarrow$ (4) is clear from Theorem 3.14 and the fact that every nonempty clopen subset of ${}^{\kappa}2$ is homeomorphic to ${}^{\kappa}2$.

(4) $\Rightarrow$ (3) First we show that X has weight $\leqslant \kappa$. If its weight were greater, we could split it into exactly κ^+ clopen sets, and add a point p whose neighborhoods are those subsets of $X \cup \{p\} = Y$ which include p together with all but $\leqslant \kappa$ members of the clopen partition of X. Then Y is κ-additive and X is not the union of fewer than κ^+ closed subsets of Y.

We can thus embed X in ${}^{\kappa}2$. Any subset of X that is closed in ${}^{\kappa}2$ is κ-compact, and the union of fewer than κ of these is κ-compact, so X is the union of a κ-chain of κ-compact subsets.

(3) $\Rightarrow$ (2) is a routine application of Lemma 3.17 and transfinite induction, again using the fact that a union of fewer than κ-many κ-compact sets is κ-compact.

Finally, to show (2) $\Rightarrow$ (1), we will show that any two spaces with the properties stated in (2) are homeomorphic. Let X and Y be two such spaces and let $\langle K_\alpha\colon \alpha < \kappa\rangle$ and $\langle L_\alpha\colon \alpha < \kappa\rangle$ be the respective κ-chains. We will replace these with κ-chains of κ-compact subspaces $\langle X_\alpha\colon \alpha < \kappa\rangle$ and $\langle Y_\alpha\colon \alpha < \kappa\rangle$ by induction so that $K_\alpha \subset X_\alpha$ and $L_\alpha \subset Y_\alpha$ for all successor ordinals, while $X_\gamma = \bigcup\{X_\alpha\colon \alpha < \gamma\}$ for all limit ordinals γ, and similarly for Y_γ. We will define a homeomorphism $\varphi : X \to Y$ by defining homeomorphisms $\varphi_\alpha : X_\alpha \to Y_\alpha$ so that $\varphi_\alpha \restriction X_\xi = \varphi_\xi$ whenever $\xi < \alpha$. Each basic clopen set B of X will be associated with a disjoint family $\bigcup \mathcal{D}$ of fewer than κ basic clopen subsets of Y in such a way that

$$\text{if } X_\xi \cap B \neq \emptyset, \text{ then } \varphi_\xi^{\rightarrow}(X_\xi \cap B) = Y_\xi \cap \Big(\bigcup \mathcal{D}(B)\Big). \tag{$*$}$$

By making sure $(*)$ holds at successor steps of the induction, and letting $\varphi_\gamma = \bigcup\{\varphi_\alpha\colon \alpha < \gamma\}$ for limit ordinals α, we insure that $\varphi = \bigcup\{\varphi_\alpha\colon \alpha < \kappa\}$ is a homeomorphism; bijectivity is automatic.

Let $X_0 = K_0$, $Y_0 = L_0$, and let $\varphi_0 : X_0 \to Y_0$ be any homeomorphism. For $B = X$ we let $\mathcal{D}(B) = \{Y\}$. Of course, $B(x,0) = X$ for all $x \in X$, and we always have $B(x,\gamma)$ $(= \{x' \in X\colon \mu(x,x') \geqslant \gamma\}) = \bigcap\{B(x,\xi)\colon \xi < \gamma\}$ whenever γ is a limit ordinal. Suppose $\alpha < \kappa$ and $\mathcal{D}(B)$ has been defined for all $B = B(x,\xi)$ such that $\xi < \alpha$ and $B \cap X_0 \neq \emptyset$. If α is a limit ordinal, and $x \in X_0$, and $B(x,\alpha) \cap X_0 \neq \emptyset$, then $\bigcup\mathcal{D}(B')$ is a clopen subset of Y for all $B' = B(x,\xi)$ such that $\xi < \alpha$, and the intersection $D(B)$ of the sets $\bigcup\mathcal{D}(B')$ is clopen in Y. Let $\mathcal{D}(B(x,\alpha))$ be a disjoint family of basic clopen subsets of $D(B(x,\alpha))$ covering Y_0, each of which meets Y_0. Then, because Y_0 is κ-compact, $|\mathcal{D}(B(x,\alpha))| < \kappa$. If $\alpha = \xi + 1$ and $x \in X_0$ and $D \in \mathcal{D}(B(x,\xi))$, let $\mathcal{D}_D(B(x,\alpha))$ be a disjoint family of proper basic clopen subsets of D covering $Y_0 \cap D$. Let

$$\mathcal{D}\big(B(x,\alpha)\big) = \bigcup\big\{\mathcal{D}_D\big(B(x,\alpha)\big)\colon\ D \in \mathcal{D}\big(B(x,\xi)\big)\big\}.$$

Then $|\mathcal{D}(B(x,\alpha))| < \kappa$ because Y_0 is κ-compact.

If φ_ξ has been defined for all $\xi \leqslant \beta$, and $\gamma < \kappa$, let $\mathcal{Q}_\beta(\gamma)$ (respectively $\mathcal{B}_\beta(\gamma)$) be the collection of all $B = B(x,\gamma)$ such that $X_\beta \cap B \neq \emptyset$ (respectively such that $X_\beta \cap B = \emptyset$, $X_\beta \cap B(x,\delta) \neq \emptyset$ whenever $\delta < \gamma$, and $K_{\beta+1} \cap B \neq \emptyset$). Similarly, define $\mathcal{R}_\beta(\gamma)$ (respectively $\mathcal{C}_\beta(\gamma)$) to be the collection of all $B = B(y,\gamma)$ such that $Y_\beta \cap B \neq \emptyset$ (respectively such that $Y_\beta \cap B = \emptyset$, $Y_\beta \cap B(x,\delta) \neq \emptyset$ whenever $\delta < \gamma$, and $L_{\beta+1} \cap B \neq \emptyset$). Note that if $x \in K_{\beta+1} \setminus X_\beta$ then there is a unique ordinal γ such that $B(x,\gamma) \in \mathcal{B}_\beta(\gamma)$. A similar result obtains for $y \in L_{\beta+1} \setminus Y_\beta$. Part of our induction hypothesis is that, $\mathcal{D}(B)$ has been defined for all $B \in \bigcup\{\mathcal{Q}_\beta(\alpha)\colon\ \alpha < \kappa\}$, and each member of $\mathcal{D}(B)$ is in $\mathcal{R}_\beta(\gamma)$, as is any basic clopen ball containing it.

If $B(x,\delta) \in \mathcal{Q}_\beta(\delta)$, let

$$\mathfrak{B} = \big\{B \in \mathcal{B}_\beta(\delta+1)\colon\ B \subset B(x,\delta)\big\}.$$

If $B(y,\gamma) \in \mathcal{D}(B(x,\delta))$, let $\mathcal{C}_\beta(y,\gamma) = \{B \in \mathcal{C}_\beta(\gamma+1)\colon\ B \subset B(y,\gamma)\}$ and let

$$\mathfrak{C} = \mathfrak{C}_\beta(x,\delta) = \bigcup\big\{\mathcal{C}_\beta(y,\gamma)\colon\ B(y,\delta) \in \mathcal{D}\big(B(x,\delta)\big)\big\}.$$

Because X is nowhere locally compact of weight κ, we may assume $\{B(z,\alpha+1)\colon\ z \in B(x,\alpha)\}$ is of cardinality κ, for each $x \in X$ and each $\alpha < \kappa$, with a similar result holding in Y. Now let $\lambda = \max\{|\mathfrak{B}|, |\mathfrak{C}|\} < \kappa$ and let $\mathfrak{B}^\#$ and $\mathfrak{C}^\#$ be collections of λ closed balls, such that (i) $\mathfrak{B} \subset \mathfrak{B}^\#$; (ii) $\mathfrak{C} \subset \mathfrak{C}^\#$; (iii) each member of $\mathfrak{B}^\#$ is a ball $B(x',\delta+1)$ such that $x' \in B(x,\delta)$ and $B(x',\delta+1) \cap X_\beta = \emptyset$ and (iv) each member of $\mathfrak{C}^\#$ is a ball $B(y,\gamma+1)$ such that $B(y,\gamma) \in \mathcal{D}(B(x,\delta))$ and $B(y,\gamma+1) \cap Y_\beta = \emptyset$. If $B \in \mathfrak{B}^\# \setminus \mathfrak{B}$, then $B \cap K_{\beta+1} = \emptyset$, and we let $X_{\beta+1}(B)$ be a copy of ${}^\kappa 2$ in B, taken from some later K_η which does meet B; of course, every nonempty relatively clopen subset of K_η is homeomorphic to ${}^\kappa 2$. The rest of our construction will insure that $X_{\beta+1}(B) = X_{\beta+1} \cap B$. Similarly if $C \in \mathfrak{C}^\# \setminus \mathfrak{C}$, then $C \cap L_{\beta+1} = \emptyset$, and we let $Y_{\beta+1}(C)$ be a copy of ${}^\kappa 2$ in C, and the rest of our construction will insure that $Y_{\beta+1}(C) = Y_{\beta+1} \cap C$. On the other hand, if $B \in \mathfrak{B}$, we will insure that $X_{\beta+1} \cap B = K_{\beta+1} \cap B$, and similarly with $C \in \mathfrak{C}$. Assuming this, let $f : \mathfrak{B}^\# \to \mathfrak{C}^\#$ be a bijection, let $\mathcal{D}(B) = \{f(B)\}$ for all $B \in \mathfrak{B}^\#$ and let φ_B be a homeomorphism whose domain is $X_{\beta+1} \cap B$ and whose range is $Y_{\beta+1} \cap f(B)$. The remainder of this $\beta \to \beta+1$ induction step will insure that $\varphi_{\beta+1} \restriction B = \varphi_B$.

Since the various $\mathfrak{C}^\#_\beta(x,\delta)$ are disjoint from each other as δ ranges over κ, these definitions of $X_{\beta+1}$ do not conflict with each other for distinct $B(x,\delta)$ and $B(x',\delta')$. We are also still free to define $X_\beta + 1 \cap B(x,\sigma)$ whenever σ is a limit ordinal and $B(x,\xi) \cap X_\beta \neq \emptyset$ for all $\xi \leqslant \sigma$. Let $\mathfrak{B}_1$ be the set of all such $B(x,\sigma)$. If $B(x,\sigma) \cap K_{\beta+1}$

is nonempty, we let it equal $B(x,\sigma)\cap X_{\beta+1}$, while if it is empty, we pick a copy of ${}^{\kappa}2$ in $B(x,\sigma)$ and let that be $B(x,\sigma)\cap X_{\beta+1}$. Over in Y, we have already defined $\mathcal{D}(B(x,\xi))$ for all $\xi<\sigma$, and so we can define $\mathcal{D}(B(x,\sigma))$ to be the set of all intersections of maximal chains in $\bigcup\{\mathcal{D}(B(x,\xi))\colon\ \xi<\sigma\}$. By spherical completeness of Y and the fact that $|\mathcal{D}(B(x,\xi))|\leqslant\kappa$ for all $\xi\leqslant\sigma$, these intersections are all nonempty, and there are fewer than κ of them. Of course, $D\cap Y_\beta=\emptyset$ for all $D\in\mathcal{D}(B(x,\sigma))$ since any point in the intersection would equal $\varphi_\sigma(x)$ for some $x\in B(x,\sigma)$, which is impossible. Now if $D\in\mathcal{D}(B(x,\sigma))$ and $D\cap L_{\beta+1}\neq\emptyset$, we let that be $D\cap Y_{\beta+1}$, while if $D\cap L_{\beta+1}=\emptyset$, we pick a copy of ${}^{\kappa}2$ in D and let that be $D\cap Y_{\beta+1}$. When this is done for all $D\in\mathcal{D}(B(x,\sigma))$, the space $Y_{\beta+1}\cap\bigcup\{\mathcal{D}(B(x,\sigma))\}$ is a copy of ${}^{\kappa}2$ and we let $\varphi_{B(x,\sigma)}$ be a homeomorphism to it from $B(x,\sigma)\cap X_{\beta+1}$.

When this is done for all $B(x,\sigma)\in\mathfrak{B}_1$, we let $X_{\beta+1}$ be the union of X_β with $\bigcup\{X_{\beta+1}\cap B\colon\ B\in\mathfrak{B}^{\#}\cup\mathfrak{B}_1\}$ and let $\varphi_{\beta+1}$ be defined *mutatis mutandis*, letting $Y_{\beta+1}$ be the range of $\varphi_{\beta+1}$. It is routine to verify that $\varphi_{\beta+1}$ is a homeomorphism extending φ_β, that the various equalities that we promised to insure are insured, and that $(*)$ continues to hold for all B for which $\mathcal{D}(B)$ has been defined up to now.

This $\beta\to\beta+1$ induction step is completed by defining $\mathcal{D}(B)$ for all $B\in\mathcal{Q}_{\beta+1}(\gamma)\setminus\mathcal{Q}_\beta(\gamma)$ for all $\gamma<\kappa$. To do this, we treat each $B^{\#}\in\mathfrak{B}^{\#}\cup\mathfrak{B}_1$ the way we treated X_0 at the beginning of the induction. Besides this substitution of the various $B^{\#}$ for X_0, the only other change required is the substitution of $\varphi_{B^{\#}}(X_\beta+1\cap B^{\#})$ for Y_0. □

Problem 3.18. Can initial spherical completeness be eliminated from any one of (2), (3), or (4) in Theorem 3.16?

It is especially tempting to conjecture that it can be eliminated from (2). It is easy to produce a κ-sequence of homeomorphisms between corresponding members of the ascending κ-chains of κ-compact spaces in two spaces satisfying (2), because of the following lemma, which can be proven like (2) $\Rightarrow$ (1), and one could even use a much simpler back-and-forth construction of a homeomorphism somewhat like in the usual proof that any two countable metrizable spaces are homeomorphic.

Lemma 3.19. *Let κ be weakly compact and let K and L be nowhere dense closed sets in copies Z and W of ${}^{\kappa}2$. Any homeomorphism from K to L can be extended to a homeomorphism from Z to W.*

Taking the union of the homeomorphisms at limit ordinals $\gamma<\kappa$ gives a homeomorphism, because each well-ordered net converging to a point in X_γ or Y_γ is the union of fewer than κ nets in the summands, one of which is cofinal in the union. However, I could not see how to guarantee that the ultimate union of the φ_α is a homeomorphism from X to Y in such a process.

We close this paper by calling the reader's attention to a very recent paper [1] giving new topological and uniform-space characterizations for κ-compact κ-metric spaces. In analogy with the well-known fact that a metric space is compact if, and only if, it is complete and totally bounded, one might conjecture that a κ-metric space is κ-compact iff it is κ-totally bounded and complete. One interesting result of [1] is that this is indeed true if κ is weakly compact. However, if κ is the first strongly inaccessible cardinal number then, as we have indicated, ${}^{\kappa}2$ is a counterexample that is even spherically complete. What is needed, as shown in [1], is a different property:

Definition 3.20. A κ-metric space is *supercomplete* if the hyperspace $H(X)$ of all nonempty closed subsets of X is complete in the Hausdorff uniformity.

If μ is an inverse κ-metric for X, then the inverse κ metric on $H(X)$ given by

$$\mu(E,F) = \sup\{\alpha\colon\ E \subset B(F,\alpha) \text{ and } F \subset B(E,\alpha)\}$$

generates the Hausdorff uniformity, where $B(S,\alpha) = \bigcup\{B(x,\alpha)\colon\ x \in S\}$ for all nonempty closed $S \subset X$. In [1] it is shown that a κ-additive uniform space is κ-compact if, and only if, it is supercomplete and κ-totally bounded. (If κ is any regular cardinal, every κ-metrizable space is κ-additive.) Another striking result of [1] is the one mentioned about weakly compact κ, which can be rephrased thus: for κ-metric κ-totally bounded spaces, completeness is equivalent to supercompleteness.

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