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Arbitrarily Traceable Graphs and Digraphs*

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The work in this paper extends and generalizes earlier work by Ore on arbitrarily traceable Euler graphs, by Harary on arbitrarily traceable digraphs, by Chartrand and White on randomly n-traversable graphs, and by Chartrand and Lick on randomly Eulerian digraphs. Arbitrarily traceable graphs of mixed type are defined and characterized in terms of a class of forbidden graphs. Arbitrarily traceable digraphs of mixed type are also defined and a simply applied characterization is given for them.

1. INTRODUCTION

Ore [8] defines arbitrarily traceable graphs as Euler graphs which have the following property. If one starts at an appropriate point and traces any blocked trail, observing the single rule that at each point one chooses as the next line a line not previously used, one traces the whole graph. Observe that for Euler graphs this property is equivalent to the following property (which generalizes to non-Euler graphs). If the initial point is properly chosen in each blocked trail (which is otherwise arbitrary, subject only to the above rule), the number of these line disjoint blocked trails required to use all of the lines of the graph is always the *minimum* number of line disjoint trails required to use all the lines of the graph. This property defines the class of graphs which we call arbitrarily traceable graphs of mixed type, and generalizes the notions of arbitrary traceability for Euler graphs [8] and random n-traversability for n-traversable graphs [4].

We define an analogous class of arbitrarily traceable digraphs of mixed type in Section 5. This class includes the arbitrarily traceable Euler digraphs as defined by Harary [6] and the randomly traversable digraphs as defined by Chartrand and Lick [3].

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The main results in this paper are:

(1) A characterization, in terms of a class of forbidden graphs, of arbitrarily traceable graphs of mixed type in which each nontrivial component has an odd point (Theorem 12),

(2) A characterization of those odd points from which an arbitrarily traceable graph of mixed type is arbitrarily traceable (Theorem 16),

(3) A characterization of randomly n-traversable graphs in terms of a property of their evenly attached subgraphs (Theorem 18), and

(4) A characterization of arbitrarily traceable digraphs of mixed type in which each nontrivial weak component has a point for which the indegree and outdegree differ (Theorem 20).

2. Terminology

Basically we follow [5]; however, the following definitions will also be useful.

Let G be a graph and let H be a subgraph of G. A *line of attachment of H in G* is any line x in $E(G) - E(H)$ such that x is incident with a point of H. (Contrast with Ore's definition [7, pp. 78–79].) Say that H is *evenly attached in G* (or H is an *evenly attached subgraph of G*) if each nonisolated point of H is incident with an even number of lines of attachment of H in G.

A point of G is *odd* or *even* according as its degree is, respectively, odd or even. The trail with *initial point* v_0 and *terminal point* v_n is denoted

$$T = v_0, v_1, \ldots, v_n = T(v_0, v_n),$$

where the line $v_{i-1}v_i$ is understood to appear between v_{i-1} and v_i for $i = 1, \ldots, n$. If G is a graph and $T(v_0, v_n)$ is a trail in G, then $G(T)$ denotes the graph whose set of points is the set of points of T and whose set of lines is the set of lines of T. Sometimes T will be used in place of $G(T)$; however, no confusion should result. In particular, if G is a graph and T is a trail in G, then $G - T$ will always be used to denote the graph $G - G(T)$, obtained by deleting the lines of T from G. Following Tutte [9], a trail $T(v_0, v_n)$ is *blocked in G at v_n*, or simply *blocked*, if all lines incident with v_n in G appear in T. Equivalently, T is blocked if and only if $\deg(G, v_n) = \deg(G(T), v_n)$. If $T(a, b)$ is a trail in G and $S(b, c)$ is a trail in $G - T$, then TS denotes the trail

$$TS = a, \ldots, b, \ldots, c.$$

The line disjoint trails $T_1, \ldots, T_n$ *partition* $E(G)$ if

$$E(G) = E(T_1) \cup \cdots \cup E(T_n).$$

A graph obtained by identifying the initial points of n disjoint paths and then identifying their terminal points is an *n-skein*. (Contrast with Ore's definition [8].) Following Chartrand and White [4], a connected graph G with odd points is *n-traversable* if there are n open trails which partition $E(G)$, but any m trails, with $m < n$, fail to partition $E(G)$. Let $n \geqslant 1$. Then a connected graph G is n-traversable if and only if G has precisely $2n$ odd points [5, p. 65].

Many of the definitions for digraphs are analogous to those just given for graphs. Again, basic terminology follows [5]. The symbols $\mathrm{od}(v)$ and $\mathrm{id}(v)$ are used to denote, respectively, the outdegree of v and the indegree of v. Define the function $\mathrm{dif}(v)$ on the points of the digraph D by: $\mathrm{dif}(v) = \mathrm{od}(v) - \mathrm{id}(v)$. A (directed) trail $T(v_0, v_n)$ in digraph D is *blocked at* v_n (or simply *blocked*) if $\mathrm{od}(v_n)$ in T is the same as $\mathrm{od}(v_n)$ in D.

3. Arbitrarily Traceable Graphs

Let G be a graph which is not totally disconnected. Then it is possible to partition $E(G)$ with a collection of line disjoint trails $\{T_1, \ldots, T_n\}$. Let $m(G)$ be the minimum number of line disjoint trails required to partition $E(G)$. Then $m(G) = 1$ if G is an Euler graph, and $m(G) = k + n$ if G has $2k$ odd points and n Euler components, [5, p. 65, Corollary 7.1(a)].

Put $m = m(G)$.

Definition 1. A *tracing* of G is an ordered $(3m + 1)$-tuple

$$(G_1, v_1, T_1, \ldots, G_m, v_m, T_m, G_{m+1}),$$

where T_i is a blocked trail in G_i with initial point v_i, for $i = 1, \ldots, m$, and $G_1, \ldots, G_{m+1}$ are defined by

$$G_1 = G,$$
$$G_i = G_{i-1} - T_{i-1}, \qquad \text{for} \quad 2 \leqslant i \leqslant m + 1.$$

The *ith stage of the tracing* is the ordered triple (v_i, T_i, G_{i+1}), and G_{i+1} is *the graph remaining at the ith stage of the tracing*. Say the tracing is *successful* if $\{T_1, T_2, \ldots, T_m\}$ partitions $E(G)$ (equivalently, if G_{m+1} is totally disconnected), otherwise the tracing is *unsuccessful*. Note that it is always possible to obtain at least one successful tracing of G.

DEFINITION 2. Let G be a graph. Say G is *arbitrarily traceable of mixed type from the point* v_1 if it is totally disconnected and $v_1 \in V(G)$, or else if it is possible at each stage, $i = 1, 2, \ldots, m$, of a tracing of G to choose the point v_i in $V(G_i)$ so that no matter what blocked trail T_i with initial point v_i is chosen, the tracing

$$(G_1, v_1, T_1, \ldots, G_m, v_m, T_m, G_{m+1})$$

is successful. Say G is *arbitrarily traceable of mixed type* if there is a point v in $V(G)$ so that G is arbitrarily traceable of mixed type from v.

Note that a graph G is arbitrarily traceable of mixed type if and only if each component of G is arbitrarily traceable of mixed type. Note also that if G is arbitrarily traceable of mixed type and if H is obtained from G by adding or deleting isolated points, then H is also arbitrarily traceable of mixed type (from any point that works for G).

The Euler arbitrarily traceable graphs studied by Ore [8], Harary [6], and Bäbler [1] and the randomly n-traversable graphs of Chartrand and White [4] are all arbitrarily traceable of mixed type. It will be convenient to define two subclasses of arbitrarily traceable graphs of mixed type:

(1) the *Euler arbitrarily traceable graphs*, which are both Euler graphs and arbitrarily traceable graphs of mixed type; and

(2) the *arbitrarily traceable graphs* (without any modifier), which are arbitrarily traceable graphs of mixed type in which each nontrivial component has odd points.

The connection between randomly n-traversable graphs and arbitrarily traceable graphs is examined next.

DEFINITION 3 (Chartrand and White [4]). Let G be an n-traversable graph. Say G is *randomly n-traversable from the odd point v* if for each sequence $v_1, v_2, \ldots, v_n$ of n odd points of G for which $v_1 = v$, and for every n trails $T_1, T_2, \ldots, T_n$ such that T_1 is a blocked trail from v_1 in G and T_i is a blocked trail from v_i in $G - (G(T_1) \cup \cdots \cup G(T_{i-1}))$, it follows that $E(G) = E(T_1) \cup \cdots \cup E(T_n)$. If G is randomly n-traversable from *each* odd point, then G is randomly n-traversable.

The following theorem is immediate from Definitions 2 and 3.

THEOREM 1. *Let G be an n-traversable graph. Then G is randomly n-traversable from v if and only if:*

(1) *G is arbitrarily traceable from v, and*

(2) *if $(G_1, v_1, T_1, \ldots, G_n, v_n, T_n, G_{n+1})$ is any tracing of G, where $v_1 = v$ and each v_i is odd in G_i for $i = 1, 2, \ldots, n$, then each G_j is arbitrarily traceable from each of its odd points for $j = 1, 2, \ldots, n$.*

Ore's characterization theorem [8, Theorem 4] states that an Euler graph is Euler arbitrarily traceable from the point v if and only if each cycle in G contains v. In view of this characterization and the observation that a graph is arbitrarily traceable of mixed type if and only if each component is arbitrarily traceable of mixed type, it is sufficient to study only arbitrarily traceable graphs in attempting to characterize arbitrarily traceable graphs of mixed type.

It will be useful to reformulate Definition 2 for the case of arbitrarily traceable graphs. Note first that if a graph is arbitrarily traceable, then each v_i in a successful tracing must be an odd point.

REFORMULATION 1. Let G be a graph with $2k$ odd points. Then G is arbitrarily traceable if and only if:

(1) $k = 0$ and G is totally disconnected, or

(2) $k \geqslant 1$ and there exists an odd point v in $V(G)$ such that whenever T is a blocked trail from v, the graph $G' = G - T$ is arbitrarily traceable.

Note that addition or deletion of isolated points does not affect the arbitrary traceability of a graph.

Consider the case $k = 1$. Let G be a graph with precisely two odd points, u and v. Then G is arbitrarily traceable from v if and only if it has only one nontrivial component and this component (which must be 1-traversable) is randomly 1-traversable from v.

The following proposition is Theorem 2 of Chartrand and White [4].

PROPOSITION. *Let u and v be the two odd points of a 1-traversable graph G. Then G is randomly 1-traversable from u if and only if every cycle of G contains v.*

THEOREM 2. *Let G be a graph with precisely two odd points, u and v. Then G is arbitrarily traceable from u if and only if every cycle of G contains v.*

Proof. Suppose G has precisely two odd points u and v, and that G is arbitrarily traceable from u. Then G has only one nontrivial component (for each nontrivial component contains an odd point) and thus this component is randomly 1-traversable from u. Thus by the preceding proposition, each cycle in G must contain v.

Conversely, suppose G has precisely two odd points u and v, and suppose that each cycle of G contains v. Then G has only one nontrivial component and this component is randomly 1-traversable from u by the preceding proposition. Thus G is arbitrarily traceable from u. ∎

Those 1-traversable graphs which are randomly 1-traversable from both odd points have also been characterized by Chartrand and White [4, Corollary 2a]. This result translates as follows.

THEOREM 3. *Let u and v be the two odd points of a 1-traversable graph G. Then G is arbitrarily traceable from both u and v if and only if each cycle of G contains both u and v.*

Note that these graphs are $(2n + 1)$-skeins.

A construction producing all arbitrarily traceable 1-traversable graphs can now be given. It is analogous to Ore's construction for Euler arbitrarily traceable graphs [8].

THEOREM 4. *All 1-traversable graphs G with distinct odd points u and v, which are arbitrarily traceable from u are obtainable from cycle free graphs F without isolates by specifying the point u in $V(F)$ and joining the points in F to a new point v in such a way that all of the points in the resulting graph are even, except for u and v which are odd:*

(1) *If* $\deg(F, w)$ *is even* $(w \neq u)$, *do not join w to v;*

(2) *if* $\deg(F, w)$ *is odd* $(w \neq u)$, *join w to v by a line; and*

(3) *join u and v if necessary to cause* $\deg(u)$ *and* $\deg(v)$ *to be odd in G.*

Proof. If G is arbitrarily traceable from u, then $G - v$ must be cycle-free by Theorem 2.

Conversely, suppose a cycle-free graph F without isolates is specified and the above construction is performed. In the resulting graph G, all cycles pass through v, the only points of odd degree are u and v, and G is connected. Thus G is arbitrarily traceable from u. ∎

4. FORBIDDEN GRAPHS

Now consider arbitrarily traceable graphs with $2k$ odd points ($k \geqslant 0$). Recall that each nontrivial component will have odd points.

The next two propositions are, respectively, Theorem 4 and Corollary 4a of Chartrand and White [4].

PROPOSITION. *Let G be an n-traversable graph. Let v be an odd point of G and suppose that each cycle of G contains at least n odd points other than v. Then G is randomly n-traversable from v.*

PROPOSITION. *Let G be an n-traversable graph. If each cycle of G*

contains at least $n + 1$ odd points, then G is randomly n-traversable (i.e., randomly n-traversable from each odd point).

By Theorem 1 the following two theorems are immediate corollaries of the preceding two propositions.

THEOREM 5. *Let G be an n-traversable graph. Let v be an odd point of G and suppose each cycle of G contains at least n odd points other than v. Then G is arbitrarily traceable from v.*

THEOREM 6. *Let G be an n-traversable graph. If each cycle of G contains at least $n + 1$ odd points, then G is arbitrarily traceable from each odd point.*

The next result is an immediate corollary of Theorem 6.

COROLLARY 6. *Any forest is arbitrarily traceable from each odd point.*

Three simple lemmas follow.

LEMMA 1. *Let G and H be homeomorphic graphs. Then G is arbitrarily traceable if and only if H is arbitrarily traceable.*

LEMMA 2. *Let G be arbitrarily traceable from the odd point a. Let T be any (nonblocked) trail from a to an even point c of G. Then $G - T$ is arbitrarily traceable from c.*

Proof. Note first that $\deg(G - T, c)$ is odd. Let Q be any blocked trail from c in $G - T$. Set $T' = TQ$. Then T' is a blocked trail from a in G. Thus $G - T'$ is arbitrarily traceable. But $G - T' = (G - T) - Q$. So by Reformulation 1, $G - T$ is arbitrarily traceable from c. ∎

LEMMA 3. *Let G be arbitrarily traceable from the odd point a. Let $P(b, v)$ be a path disjoint from G. Let G^* be obtained from G and P by identifying v and a. Then G^* is arbitrarily traceable from b.*

Proof. Let T be a blocked trail from b in G^*. Then $T \cap G$ is the graph of a blocked trail from a in G. Since $G^* - T$ and $G - (G \cap T)$ differ only by isolated points, $G^* - T$ is arbitrarily traceable. Thus by Reformulation 1, G^* is arbitrarily traceable from b. ∎

THEOREM 7. *Let G be arbitrarily traceable from the odd point u. Let H be an Euler subgraph of G. Then $G - H$ is arbitrarily traceable from u.*

Proof. (The proof is by induction on k, where G has $2k$ odd points.)

Basis. Let $k = 1$. Let u and v be the two odd points of G. Then u and v belong to the same component C of G, and all other components of G are trivial. Now H is an Euler subgraph of G, thus H is a subgraph of C. Thus assume without loss of generality that G is connected. Put $G^* = G - H$. Since H is an Euler subgraph of G, the graph G^* must have precisely two odd points, namely u and v. Since, by Theorem 2, all cycles of G contain v, it follows that all cycles of G^* contain v. Thus, by Theorem 2, G^* is arbitrarily traceable from u.

Inductive step. Let $k \geqslant 1$. Suppose that when G is arbitrarily traceable from u, G has fewer than $2k$ odd points, H is an Euler subgraph of G, and $G^* = G - H$, it follows that G^* is arbitrarily traceable from u.

Suppose that G is arbitrarily traceable from the odd point u, G has $2k$ odd points, H is an Euler subgraph of G, and set $G^* = G - H$. Let T be a blocked trail from u in G^*. Let v be the terminal point of T and consider two cases, according as v is or is not an element of $V(H)$.

Case I. If $v \in V(H)$, then $G(T) \cup H$ is the graph of a blocked trail Q from u in G. Thus by Reformulation 1, $G - Q$ is arbitrarily traceable. But $G^* - T = G - Q$. So, $G^* - T$ is arbitrarily traceable.

Case II. If $v \notin V(H)$, then T is a blocked trail in G. Put $G - T = G'$. Then G' is arbitrarily traceable and has $2k - 2$ odd points. Furthermore, H is an Euler subgraph of G'. Thus by the induction hypothesis, $G' - H$ is arbitrarily traceable. But $G' - H = G^* - T$, so $G^* - T$ is arbitrarily traceable.

In either case, $G^* - T$ is arbitrarily traceable; thus, by Reformulation 1, G^* is arbitrarily traceable from u. ∎

THEOREM 8. *Let G be a graph which is arbitrarily traceable from a. Let v be an even point of G. Let G^* be formed by splitting v into v' and v'', where each line of G that was incident with v is incident with exactly one of v' and v''. Then G^* is arbitrarily traceable. Furthermore, G^* is arbitrarily traceable from a.*

Proof. The proof is by induction on k where G^* has $2k$ odd points.

Basis. Let $k = 0$. Then G^* is totally disconnected and the result follows.

Inductive step. Let $k \geqslant 1$. Suppose the proposition is true for all graphs G^* satisfying the hypotheses and having fewer than $2k$ odd points. Let G^* satisfy the hypotheses and have $2k$ odd points. Since G^* has an odd point, G has a nontrivial component, and thus G has an odd point. Thus G is arbitrarily traceable from some odd point. So we may assume without loss of generality that a is not an isolate in G^*. We may also assume without loss of generality that both v' and v'' have positive degree in G^*. Let T be a blocked trail from a in G^*. Consider two cases according as T is blocked in $\{v', v''\}$ or not.

Case I. If T is blocked at v' (respectively v''), then both v' and v'' are odd in G^*. Note that T corresponds to a nonblocked trail $T'(a, v)$ in G, where in T' the point v'' (respectively v') has been reidentified with v. Since v is even in G, it follows from Lemma 2 that $G - T'$ is arbitrarily traceable from v'' (respectively v'). But $G^* - T$ and $G - T'$ differ only by the isolated point v' (respectively v'') of $G^* - T$, and thus $G^* - T$ is arbitrarily traceable from v'' (respectively v').

Case II. If T is blocked at some point w where $w \in \{v', v''\}$, let T' be the trail in G obtained by reidentifying v' and v''. Note that $G^* - T$ has fewer than $2k$ odd points. Now $G^* - T$ is obtained from $G - T'$ by splitting the even point v into v' and v''. Thus by the induction hypothesis, $G^* - T$ is arbitrarily traceable from a.

In either case then, G^* is arbitrarily traceable from a. ∎

In general, splitting odd points does not preserve arbitrary traceability. The graph of Fig. 1 is arbitrarily traceable, while the graph of Fig. 2 is not arbitrarily traceable (although it is arbitrarily traceable of mixed type).

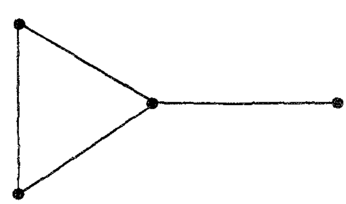

FIG. 1. An arbitrarily traceable graph.

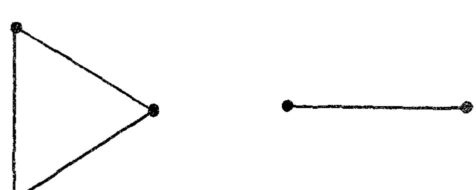

FIG. 2. A graph which is not arbitrarily traceable.

The following result shows that the odd point a can be split when $\deg(G, a)$ is positive and G is arbitrarily traceable from a.

THEOREM 9. *Let G be arbitrarily traceable from odd point a. (So* $\deg(a)$ *is positive.) Let G^* be obtained from G by splitting a into two points a' and a'' with* $\deg(G^*, a')$ *odd. Then G^* is arbitrarily traceable from a'.*

Proof. Let T^* be a blocked trail from a' in G^*. Let T be the corresponding trail in G obtained by identifying a' and a''. Then $G - T$ is arbitrarily traceable. Also, $\deg(G - T, a)$ is even. Since $G^* - T^*$ is obtained from $G - T$ by splitting a, it follows from Theorem 8 that $G^* - T^*$ is arbitrarily traceable. ▌

THEOREM 10. *Let G^* be a graph. Let F be a forest disjoint from G^*. Let b^* be an even point of G^* and let b be an even point of F. Let G be formed from G^* and F by identifying b^* and b. Then G is arbitrarily traceable if and only if G^* is arbitrarily traceable.*

Proof. Suppose G is arbitrarily traceable. The even point $b^* = b$ of G can be split into two points in such a way that G^* is a union of connected components of the new graph. Thus by Theorem 8, G^* is arbitrarily traceable.

The other half of the theorem is proved by induction on k, where G has $2k$ odd points.

Basis. Let $k = 0$. Then G is totally disconnected. Thus, by Reformulation 1, G is arbitrarily traceable.

Inductive step. Let $k \geqslant 1$. Suppose the result is true for any graph G satisfying the hypotheses and having fewer than $2k$ odd points. Let G be formed as in the statement of the theorem where G has $2k$ odd points and G^* is arbitrarily traceable from u. If G^* is trivial, then G is arbitrarily traceable by Corollary 6. Thus assume that G^* is not trivial. It follows that G^* is arbitrarily traceable from some *odd* point. Thus assume without loss of generality that $\deg(G, u)$ is odd. Let $T(u, v)$ be a blocked trail from u in G. Consider two cases according as v is or is not in $V(G^*)$.

Case I. Suppose $v \in V(G^*)$. Then T is a blocked trail in G^*. Thus $G^* - T$ is arbitrarily traceable. Furthermore, $G - T$ is formed from the disjoint graphs $G^* - T$ and F be identifying the even points b^* of $G^* - T$ and b of F. Since $G - T$ has fewer than $2k$ odd points, it follows by the induction hypothesis that $G - T$ is arbitrarily traceable.

Case II. Suppose $v \notin V(G^*)$. Then T is blocked in F and there is a trail $T'(u, b^*)$ in G^* satisfying $G^*(T') = G(T) \cap G^*$. Now T' is a nonblocked trail from u to b^* in G^* and $\deg(G^*, b^*)$ is even, so by Lemma 2, $G^* - T'$ is arbitrarily traceable from b^*. Put $F' = F - (F \cap T)$. Since $\deg(G, b^*)$ is even and $\deg(G^* - T', b^*)$ is odd, it follows that $\deg(F', b)$

is odd. Thus there is a path $P(a, b)$ in F' where a is an endpoint of F'. Let H be the graph formed from $G^* - T'$ and $P(a, b)$ by identifying b^* and b. By Lemma 3, H is arbitrarily traceable from a. Furthermore, $b^* = b$ is even in H. Let $F'' = F' - P(a, b)$. Then $\deg(F'', b)$ is even. Now $G - T$ is the graph obtained from H and F'' by identifying b^* with b. But $G - T$ has fewer than $2k$ odd points, so by the induction hypothesis, $G - T$ is arbitrarily traceable.

In either case, $G - T$ is arbitrarily traceable; thus, by Reformulation 1, G is arbitrarily traceable. ∎

THEOREM 11. *Let G be an evenly attached subgraph of an arbitrarily traceable graph G^*. Suppose $F = G^* - G$ is a forest. Then G is arbitrarily traceable.*

Proof. The proof is by induction on k where G^* has $2k$ odd points.

Basis. Let $k = 0$. Then G^* is totally disconnected, and thus G is totally disconnected. By Reformulation 1, G is arbitrarily traceable.

Inductive step. Let $k \geqslant 1$. Suppose the result holds whenever G^* and G satisfy the hypotheses and G^* has fewer than $2k$ odd points. Suppose G^* and G satisfy the hypotheses and that G^* has $2k$ odd points. Let G^* be arbitrarily traceable from a. Since G^* has odd points, assume without loss of generality that a is odd in G^*. Distinguish two cases according as a is incident with a line of G or not.

Case I. Suppose a is incident with a line of G. Let $T(a, w)$ be a blocked trail from a in G. Then all lines at w in G are in T. Thus $G - T$ is evenly attached in $G^* - T$. It follows that there is a blocked trail T' from a in G^* satisfying $G^*(T') \cap G = T$. Note that lines of attachment of $G - T$ in $G^* - T$ occur in pairs in T'. Thus $G - T$ is evenly attached in $G^* - T'$. Since T' is a blocked trail from a in G^*, it follows that $G^* - T'$ is arbitrarily traceable. Also, $G^* - T'$ has precisely $2k - 2$ odd points and

$$(G^* - T') - (G - T)$$

is a forest. Thus by the induction hypothesis, $G - T$, and hence G, is arbitrarily traceable.

Case II. Suppose a is not incident with a line of G. Since $\deg(G^*, a)$ is odd, it follows that $\deg(F, a)$ is odd. Thus there is a path $P = P(a, b)$ in F, where b is an endpoint of F and $b \neq a$. Since P is a blocked trail in G^*, it follows that $G^* - P$ is arbitrarily traceable. Since each nonisolate of G has even degree in P, it follows that G is evenly attached in $G^* - P$. But $G^* - P$ has exactly $2k - 2$ odd points. So by the induction hypothesis, G is arbitrarily traceable. ∎

Now it is possible to define a class of forbidden graphs such that if F is an evenly attached forbidden subgraph of G, then G is not arbitrarily traceable.

DEFINITION 4. A connected graph F is a *forbidden graph if:*

(1) F is not arbitrarily traceable; and

(2) whenever H is a proper, evenly attached subgraph of F, then H is arbitrarily traceable.

Note that a forbidden graph is nontrivial.

The next result shows the connection between forbidden graphs and arbitrarily traceable graphs.

THEOREM 12. *Let G be a graph. Then G is arbitrarily traceable if and only if G contains no forbidden graph H as an evenly attached subgraph.*

Proof. Assume first that G contains a forbidden evenly attached subgraph H. If $H = G$, then G is not arbitrarily traceable by (1) of Definition 4. Thus suppose $H \neq G$. It is sufficient to show that the component G' of G which contains H is not arbitrarily traceable. We do this by contradiction. Thus assume G' is arbitrarily traceable. Then G' has odd points. Consider $G' - H$. By Theorem 11, if $G' - H$ is a forest and G' is arbitrarily traceable, then H must also be arbitrarily traceable, a contradiction. Thus assume $G' - H$ is not a forest. Let $E_1, E_2, \ldots, E_t$ be Euler subgraphs of $G' - H$ such that

$$(G' - H) - (E_1 \cup \cdots \cup E_t)$$

is a forest. By t applications of Theorem 7, the graph

$$G'' = G' - (E_1 \cup \cdots \cup E_t)$$

is arbitrarily traceable. Each point of G' has even degree in $E_1 \cup \cdots \cup E_t$. Thus H is an evenly attached subgraph of G''. Furthermore, $G'' - H$ is a forest. Thus by Theorem 11, H must be arbitrarily traceable, a contradiction. Therefore G' can not be arbitrarily traceable.

Conversely, assume that G is not arbitrarily traceable. Then some component G' of G is not arbitrarily traceable. If (2) of Definition 4 holds for $F = G'$, then G' is forbidden. Thus G' is the desired evenly attached forbidden subgraph. If (2) of Definition 4 fails for $F = G'$, let $\mathscr{F}$ be the set of all connected evenly attached proper subgraphs of G' which are not arbitrarily traceable. Then each graph in $\mathscr{F}$ is nontrivial. Let F' be any element of $\mathscr{F}$ with fewest lines. Let H be any proper, evenly attached subgraph of F'. Then H is evenly attached in G'. By minimality of F', it follows that H must be arbitrarily traceable. Thus F' is forbidden and F' is the desired subgraph. ∎

THEOREM 13. *Let G be a graph. Then G is arbitrarily traceable if and only if each evenly attached subgraph H of G is arbitrarily traceable.*

Proof. The nontrivial half is shown as follows. Suppose that G contains an evenly attached subgraph H which is not arbitrarily traceable. Then, by Theorem 12, H contains an evenly attached forbidden subgraph F. But F is evenly attached in G also. So by Theorem 12, G is not arbitrarily traceable. ∎

Note that any cycle is a forbidden graph. This observation leads to the following theorem.

THEOREM 14. *Let G be arbitrarily traceable with $2k$ odd points. Then every cycle of G contains an odd point.*

Proof. If some cycle C of G contains only even points, then C is an evenly attached forbidden subgraph of G. Then by Theorem 12, G can not be arbitrarily traceable, a contradiction. ∎

The condition that every cycle contain an odd point is not a sufficient condition for G to be arbitrarily traceable. For example, the graph of Fig. 3 is not arbitrarily traceable. It is, in fact forbidden. Notice that in the graph G of Fig. 3, each odd point is the only odd point on some cycle. This property is the reason that G is not arbitrarily traceable from either of its odd points. This property generalizes as follows.

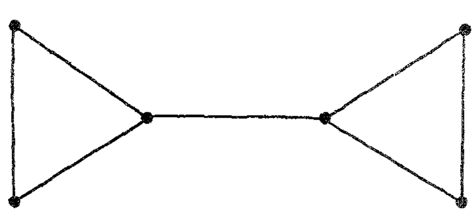

FIG. 3. A forbidden graph.

THEOREM 15. *Let a be an odd point of the graph G. Let F be a forbidden subgraph of G containing a as an even point. Suppose that whenever $v \in V(F) - \{a\}$, then v is incident with an even number of lines of attachment of F in G. Then G is not arbitrarily traceable from a.*

Proof. Construct a blocked trail T from a as follows. Choose as the first line of T a line not in F. Then any time a point of F is encountered in the further construction of T, choose a line not in F as the next line. If T is thus constructed, then the graph $G - T$ contains F as an evenly attached subgraph. Thus by Theorem 12, $G - T$ is not arbitrarily traceable. So by Reformulation 1, G is not arbitrarily traceable from a. ∎

Now we characterize those odd points a from which an arbitrarily traceable graph is arbitrarily traceable.

THEOREM 16. *Let G be an arbitrarily traceable graph. Then G is arbitrarily traceable from the odd point a if and only if, whenever F is a forbidden subgraph of G containing a as an even point, there is another point v of F for which* $\deg(G, v) - \deg(F, v)$ *is odd.*

Proof. Suppose G is arbitrarily traceable from a. Then Theorem 15 implies that each forbidden subgraph F of G which contains a as an even point must have another point v at which it has an odd number of lines of attachment in G. Thus $\deg(G, v) - \deg(F, v)$ is odd. Now suppose that G is arbitrarily traceable, but not from a. Then there is a blocked trail T from a in G so that $G - T$ contains an evenly attached forbidden subgraph F which is not evenly attached in G. Thus F must contain a as an even point, and F must have an even number of lines of attachment in G at each of its other points. But then $\deg(G, v) - \deg(F, v)$ is even for all $v \neq a$ in $V(F)$. ∎

COROLLARY 16. *If G is arbitrarily traceable and a is any endpoint of G, then G is arbitrarily traceable from a.*

Proof. The condition of Theorem 16 is satisfied vacuously since the only connected subgraph of G containing a as an even point is the trivial graph, which is not forbidden. ∎

Now we give a refinement of Theorem 13.

THEOREM 17. *Let G be arbitrarily traceable from odd point a. Let H be an evenly attached subgraph of G and suppose $a \in V(H)$. Then H is arbitrarily traceable from a.*

Proof. By Theorem 13, H is arbitrarily traceable. Since H is evenly attached in G and $\deg(G, a)$ is odd, it follows that $\deg(H, a)$ is odd. Let F be a forbidden subgraph of H which contains a as an even point. Then F is a forbidden subgraph of G which contains a as an even point. Since G is arbitrarily traceable from a, it follows by Theorem 16 that F has another point $v \neq a$ for which $\deg(G, v) - \deg(F, v)$ is odd. Since an even number (possibly zero) of lines in G incident with v are also lines of attachment of H in G, it follows that $\deg(H, v) - \deg(F, v)$ is odd. Thus by Theorem 16, H is arbitrarily traceable. ∎

THEOREM 18. *Let G be an n-traversable graph. Then G is randomly n-traversable if and only if for each nontrivial, connected, evenly attached subgraph H of G, there is an integer $k \geqslant 1$ such that H is randomly k-traversable.*

Proof. Half is trivial. We prove the nontrivial half as follows.

Assume that G is randomly n-traversable. Then by Theorem 1, G is arbitrarily traceable. Let H be any nontrivial, connected, evenly attached subgraph of G. Then by Theorem 13 and Definition 2, H has an odd point. Thus there is an integer $k \geqslant 1$ so that H is k-traversable. Since G is randomly n-traversable, it follows from Definition 3 and Theorem 1 that G is arbitrarily traceable from each odd point. Since H is evenly attached in G, it follows from Theorem 17 that H is arbitrarily traceable from each of its odd points.

Let

$$(H_1, v_1, T_1, \ldots, H_k, v_k, T_k, H_{k+1})$$

be any tracing of $H_1 = H$ in which each v_i is odd in H_i. We show that H_{k+1} is totally disconnected, arguing by contradiction. Assume that H_{k+1} is not totally disconnected. Then H_{k+1} contains an Euler component K. Define the first k stages of a tracing of $G_1 = G$ inductively as follows. Since H_1 is evenly attached in G_1, there is a blocked trail T_1' from v_1 in G_1 such that $H_1 \cap G(T_1') = T_1$. Set $G_2 = G_1 - T_1'$. Then $H_2 = H_1 - T_1$ is evenly attached in G_2, and since $H_{k+1} \subseteq H_2$, it follows that H_2 is not totally disconnected. Let i be an integer $2 \leqslant i \leqslant k$. Assume that H_i is evenly attached in G_i. Then there is a blocked trail T_i' from v_i in G_i such that $H_i \cap G(T_i') = T_i$. Set $G_{i+1} = G_i - T_i'$. Then $H_{i+1} = H_i - T_i$ is evenly attached in G_{i+1}. Thus H_{k+1} is evenly attached in G_{k+1}. But G_{k+1} is arbitrarily traceable, and H_{k+1} contains an Euler component K. Since K is evenly attached in H_{k+1}, it follows that K is evenly attached in G_{k+1}, a contradiction. Thus H_{k+1} must be totally disconnected. Therefore, each tracing of $H = H_1$ in which each v_i is odd is successful. It follows by Theorem 1 that H is randomly k-traversable. ∎

5. Arbitrarily Traceable Digraphs

The analogous problem for digraphs will now be considered. Let D be a digraph with a nonempty set of arcs. Let $m(D)$ be the minimum number of arc disjoint trails which partition the set of arcs of D. Then $m(D) = 1$ if D is an Euler digraph and, more generally, $m(D) = k + n$, where n is the number of Euler components of D and (see [2, p. 37])

$$k = (1/2) \sum |\operatorname{dif}(v)| \qquad (v \in V(D)).$$

Put $m = m(D)$.

DEFINITION 5. A tracing of D is an ordered $(3m + 1)$-tuple

$$(D_1, v_1, T_1, \ldots, D_m, v_m, T_m, D_{m+1}),$$

where, for each $i = 1, 2, \ldots, m$, T_i is a blocked trail in D_i with initial point v_i, and $D_1, D_2, \ldots, D_{m+1}$ are defined by

$$D_1 = D,$$
$$D_i = D_{i-1} - T_{i-1} \qquad \text{for} \quad 2 \leqslant i \leqslant m + 1.$$

The *ith stage of the tracing* is the ordered triple (v_i, T_i, D_{i+1}). The tracing is *successful* if $\{T_1, T_2, \ldots, T_m\}$ partitions the set of arcs of D. Otherwise the tracing is *unsuccessful.*

DEFINITION 6. Let D be a digraph. Say D is *arbitrarily traceable of mixed type from the point* v_1 if D is totally disconnected and $v_1 \in V(D)$, or else if D has a nonempty set of arcs and it is possible at each stage, $i = 1, 2, \ldots, m$, of a tracing of D to choose the point v_i in $V(D_i)$ so that no matter what blocked trail T_i with initial point v_i is chosen, the tracing

$$(D_1, v_1, T_1, \ldots, D_m, v_m, T_m, D_{m+1})$$

is successful. Say that D is *arbitrarily traceable of mixed type* if there exists a point v so that D is arbitrarily traceable of mixed type from the point v.

Note that we must have $\operatorname{dif}(v_i) \geqslant 0$, for $i = 1, \ldots, m$, in a successful tracing. Note also that if D is any digraph, then D is arbitrarily traceable if and only if each weak component of D is arbitrarily traceable.

We define two subclasses of arbitrarily traceable digraphs of mixed type: the *Euler arbitrarily traceable digraphs,* which are Euler digraphs in addition to being arbitrarily traceable digraphs of mixed type; and the *arbitrarily traceable digraphs,* which are arbitrarily traceable digraphs of mixed type in which each nontrivial weak component has a point v with $\operatorname{dif}(v) \neq 0$.

Harary [6] and Chartrand and Lick [3] have studied Euler arbitrarily traceable digraphs. Chartrand and Lick [3] also treat the case where D is weakly connected and satisfies

$$(1/2) \sum |\operatorname{dif}(v)| = 1 \qquad (v \in V(D)).$$

Harary [6] observes that Ore's characterization of all Euler arbitrarily traceable graphs generalizes to Euler arbitrarily traceable digraphs. That is, an Euler digraph D is arbitrarily traceable from v if and only if each cycle of D contains v. (The word "cycle" will always mean directed cycle when we talk about digraphs.) Thus, in order to characterize the class of all arbitrarily traceable digraphs of mixed type, it is sufficient to characterize all arbitrarily traceable digraphs.

First a reformulation of the definition is given. Note that if D is arbitrarily traceable from u, and u is not isolated, then $\operatorname{dif}(u) > 0$.

REFORMULATION 2. Let D be a digraph and put

$$k = (1/2) \sum |\operatorname{dif}(v)| \qquad (v \in V(D)).$$

Then D is arbitrarily traceable if and only if:

(1) $k = 0$ and D is totally disconnected, or

(2) $k \geqslant 1$ and there is a point u, with $\operatorname{dif}(u) > 0$, so that whenever T is a blocked trail from u, the digraph $D' = D - T$ is arbitrarily traceable.

The main result of Chartrand and Lick [3] on randomly traversable digraphs is restated here for arbitrarily traceable digraphs.

THEOREM 19. *Let D be a weakly connected digraph for which $k = 1$. Let u and v be the points in $V(D)$ such that* $\operatorname{dif}(u) = -1$ *and* $\operatorname{dif}(v) = 1$. *Then D is arbitrarily traceable from v if and only if u is contained in every cycle of D.*

We now characterize arbitrarily traceable digraphs.

THEOREM 20. *Let D be a digraph. Then D is arbitrarily traceable if and only if each cyle of D contains a point u with* $\operatorname{dif}(u) < 0$. *Furthermore, when D is arbitrarily traceable, then it is arbitrarily traceable from each point a with* $\operatorname{dif}(a) > 0$.

Proof. Let C be a cycle of D for which $\operatorname{dif}(v) \geqslant 0$ for each point v of C. In any successful tracing of D, each blocked trail must begin with a point v_i for which $\operatorname{dif}(v_i) > 0$ holds in the digraph D_i. Since $\operatorname{dif}(v) \geqslant 0$ for each point v of C, it is possible to leave C untraced by choosing an arc off C each time a point of C is encountered in constructing the blocked trails of a tracing. Thus the condition is necessary.

The sufficiency of the condition is established by induction on k, where

$$k = (1/2) \sum |\operatorname{dif}(v)| \qquad (v \in V(D)).$$

Basis. Let $k = 0$. Then the result is vacuously true.

Inductive step. Let $k \geqslant 1$. Suppose the condition is sufficient whenever D satisfies

$$(1/2) \sum |\operatorname{dif}(v)| < k \qquad (v \in V(D)).$$

Let D satisfy the condition and suppose

$$(1/2) \sum |\operatorname{dif}(v)| = k \qquad (v \in V(D)).$$

Then there are points v of D for which $\operatorname{dif}(v) > 0$. Let a be any point of D satisfying $\operatorname{dif}(a) > 0$. Let T be any blocked trail from a in D. Let C be any cycle in $D - T$. Then C is a cycle in D, so there is a point u on C with $\operatorname{dif}(u) < 0$. Clearly $\operatorname{dif}(u) < 0$ in $D - T$ also. Thus the condition is satisfied in $D - T$. Since $|\operatorname{dif}(a)|$ decreases by one from D to $D - T$ and $|\operatorname{dif}(v)|$ is increased for no point, it follows that $D - T$ also satisfies

$$(1/2) \sum |\operatorname{dif}(v)| < k \qquad (v \in V(D)).$$

Thus by the induction hypothesis, $D - T$ is arbitrarily traceable, and by Reformulation 2, D is arbitrarily traceable. ∎

The following corollary is immediate.

COROLLARY 20. *A digraph D is arbitrarily traceable if and only if deleting from D all points v for which* $\operatorname{dif}(v) < 0$ *leaves an acyclic digraph.*

There exists an arbitrarily traceable digraph for which the associated graph is not arbitrarily traceable: The digraph of Fig. 4 is arbitrarily traceable, while the graph of Fig. 3 is not arbitrarily traceable. Also there exists an arbitrarily traceable graph for which it is possible to direct the lines in such a way that the resulting digraph is not arbitrarily traceable. For example, the graph of Fig. 1 is arbitrarily traceable, while the digraph of Fig. 5 is not arbitrarily traceable.

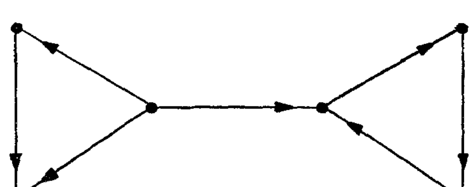

FIG. 4. An arbitrarily traceable digraph.

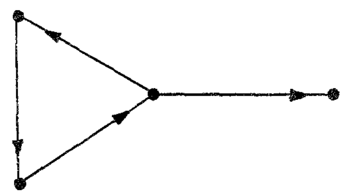

FIG. 5. A digraph corresponding to the graph of Fig. 1.

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