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Equiconvergence of spectral decompositions of 1D Dirac operators with regular boundary conditions

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Abstract

One dimensional Dirac operators

$$L_{bc}(v)\, y = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dy}{dx} + v(x)y, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad x \in [0, \pi],$$

considered with L^2-potentials $v(x) = \begin{pmatrix} 0 & P(x) \\ Q(x) & 0 \end{pmatrix}$ and subject to regular boundary conditions (bc), have discrete spectrum. For strictly regular bc, the spectrum of the free operator $L_{bc}(0)$ is simple while the spectrum of $L_{bc}(v)$ is eventually simple, and the corresponding normalized root function systems are Riesz bases. For expansions of functions of bounded variation about these Riesz bases, we prove the uniform equiconvergence property and point-wise convergence on the closed interval $[0, \pi]$. Analogous results are obtained for regular but not strictly regular bc.

Keywords: Dirac operators; Spectral decompositions; Riesz bases; Equiconvergence

1. Introduction

Spectral theory of non-self-adjoint boundary value problems (BVP) for ordinary differential equations on a finite interval I goes back to the classical works of Birkhoff [2,3] and Tamarkin [36–38]. They introduced a concept of regular (R) boundary conditions (bc) and investigated asymptotic behavior of eigenvalues and eigenfunctions of such problems. Moreover, they proved

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that the system of eigenfunctions and associated functions ($SEAF$) of a regular BVP is complete. Detailed presentation of this topic could be found in [31].

More subtle is the question whether $SEAF$ is a basis or an unconditional basis in the Hilbert space $H^0 = L^2(I)$. Mikhailov [27], Keselman [21] and later Dunford [13] proved that the $SEAF$ is an unconditional, or Riesz, basis if bc are strictly regular (SR). This property is lost if bc are $R \setminus SR$, i.e., regular but not strictly regular; unfortunately, this is just the case of periodic (Per^+) and antiperiodic (Per^-) bc. But Shkalikov [32–34] proved that in $R \setminus SR$ cases a proper chosen finite-dimensional projections form a Riesz basis of projections.

Dirac operators

$$Ly = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dY}{dx} + v(x)Y, \qquad Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad v(x) = \begin{pmatrix} 0 & P(x) \\ Q(x) & 0 \end{pmatrix} \tag{1.1}$$

with $P, Q \in L^2(I)$, and more general operators

$$My = iB\frac{dY}{dx} + v(x)Y, \quad Y = (y_j(x))_1^d, \tag{1.2}$$

where B is a $d \times d$-matrix and $v(x)$ is a $d \times d$ matrix-valued $L^2(I)$ function, bring new difficulties. One of them comes from the fact that the values of the resolvent $(\lambda - L_{bc})^{-1}$ are not trace class operators.

For general system (1.2) Malamud and Oridoroga [23–25] proved completeness of SEAF for a wide class of BVP which includes regular (in the sense of [4]) BVP's.

The Riesz basis property for 2×2 Dirac operators (1.1) was proved by Trooshin and Yamamoto [39,40] in the case of separated bc and $v \in L^2$. Hassi and Oridoroga [15] proved the Riesz basis property for (1.2) when $B = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}$, with $a, b > 0$, for separated bc and $v \in C^1(I)$.

Mityagin [29], [30, Theorem 8.8] proved that *periodic (or antiperiodic) bc* give a rise of a Riesz system of 2D projections (or 2D invariant subspaces) under the smoothness restriction $P, Q \in H^\alpha, \ \alpha > 1/2$, on the potentials v in (1.1). The authors removed that restriction in [11], where the same result is obtained for *any* L^2 potential v. This became possible in the framework of the general approach to analysis of invariant (Riesz) subspaces and their closeness to 2D subspaces of the free operator developed and used by the authors in [6,7,9,8,10].

Moreover, in [12] these results are extended to Dirac operators with *any regular bc*. Careful analysis of *regular* and *strictly regular bc* and construction of Riesz bases or Riesz system of projections which is done in [12] give us the background for treating questions on equiconvergence and point-wise convergence of spectral decompositions (or "the development in characteristic functions of the system" as Birkhoff and Langer [4] would say).

These questions for ordinary differential operators were raised by Birkhoff [2,3] and Tamarkin [36–38] as well, or even earlier for second order operators by Steklov et al. A nice survey of further development of equiconvergence theory over the last 100 years (we do not provide the names of authors — any list would be incomplete and unfair) is given by Minkin [28].

In this paper we analyze in detail one-dimensional Dirac operators (1.1); we address the following questions:

(i) for given bc, does the uniform convergence of the Riesz spectral decomposition

$$f(x) = S_N f + \sum_{|k|>N}^{\infty} P_k f, \quad x \in [0, \pi],$$

for an individual $f \in L^2([0,\pi], \mathbb{C}^2)$ depend on the potential v? (Here S_N is the Riesz projection associated to the eigenvalues λ with $|Re\,\lambda| < N+1$, and P_k, $|k| > N$, is the Riesz projection associated with the eigenvalues λ that are "near" to k.)

(ii) for good enough functions f, say f is of bounded variation, do point-wise limits

$$\lim_{m\to\infty}\left(S_N f(x) + \sum_{|k|>N}^{m} P_k f(x)\right) = F(x)$$

exist? If *yes*, how to describe the limit function $F(x)$ in terms of f and bc?

The less rigid questions ask about uniform convergence on compact subsets of $(0,\pi)$. In this case for any complete system $\{u_n(x)\}$ of eigenfunctions of the operator (1.1) with its biorthogonal system $\{\psi_n\}$, let us define

$$\sigma_m(x,f) = \sum_{n\le m} \langle f, \psi_n\rangle u_n$$

and compare these partial sums with

$$S_m(x,f) = \frac{1}{\pi}\int_0^{\pi} \frac{\sin(x-y)}{x-y} f(y)dy,$$

the proxy of partial sums of the standard Fourier series.

Many authors (Il'in [18–20], Horvath [16]) compare σ_m and S_m on compacts in $(0,\pi)$. For example, in [16] it is shown, under the assumption that the system $\{u_j\}$ is a Riesz basis and $v \in L^p$, $p > 2$, that for any compact $K \subset (0,\pi)$ we have

$$\lim_m \left(\sup_{x\in K} |\sigma_m(x,f) - S_m(x,f)|\right) = 0 \quad \forall f \in L^2([0,\pi],\mathbb{C}^2).$$

We focus on questions on equiconvergence and point-wise convergence on the *entire closed interval* $[0,\pi]$. The structure of this paper is the following.

Section 2 reminds elementary facts on Riesz bases and Riesz systems of projections in a Hilbert space, and gives (after [12]) explicitly such bases and systems of projections in the case of free Dirac operators *subject to arbitrary regular bc*, with special attention on their dependence on parameters of boundary conditions.

Any analysis of spectral decompositions requires accurate information on localization of spectra $Sp(L_{bc})$ and good estimates of the resolvent $R_\lambda = (\lambda - L_{bc})^{-1}$ outside the $Sp(L_{bc})$. Such analysis is done in [12], but in Section 3 we carry it in a different way in order to obtain at the same time some basic preliminary inequalities that play an essential role later.

Section 4 is the core of this paper. We study the deviation

$$S_N - S_N^0 = \frac{1}{2\pi i}\int_{\partial R_{NT}} (R_\lambda - R_\lambda^0)d\lambda,$$

where $R_\lambda^0 = (\lambda - L_{bc}^0)^{-1}$ is the resolvent of the free operator L_{bc}^0 and R_{NT} is a rectangle chosen to contain the "first" $2(2N+1)$ eigenvalues (counted with multiplicity). By the perturbation formula $R_\lambda - R_\lambda^0 = \sum_{m=1}^{\infty} R_\lambda^0 (V R_\lambda^0)^m$, where V is the operator of multiplication by the matrix v, we have

$$S_N - S_N^0 = A_N + B_N,$$

where

$$A_N = \frac{1}{2\pi i}\int_{\partial R_{NT}} R_\lambda^0 V R_\lambda^0 d\lambda, \qquad B_N = \frac{1}{2\pi i}\int_{\partial R_{NT}} \sum_{m=2}^{\infty} R_\lambda^0 (V R_\lambda^0)^m d\lambda.$$

It happens that the estimates of the "nonlinear" component B_N (see Proposition 12) are a little bit simpler; they reduce the problem of equiconvergence to questions on behavior of the "linear" component $A_N(F)$ when $n \to \infty$ and its dependence on the smoothness of potentials v or a vector-function F. Theorem 16 and Lemma 19 specify these smoothness conditions and lead to our main result (Theorem 20):

For regular bc, Dirac potentials $v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}$ *with* $P, Q \in L^2([0,\pi])$ *and* $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ *with* $F_1, F_2 \in L^2([0,\pi])$,

$$\left\| \left(S_N - S_N^0\right) F \right\|_\infty \to 0 \quad \text{as } N \to \infty \tag{1.3}$$

whenever one of the following conditions is satisfied:

(a) $\exists \beta > 1$ *such that*

$$\sum_{k\in 2\mathbb{Z}} (|F_{1,k}|^2 + |F_{2,k}|^2)(\log(e+|k|))^\beta < \infty,$$

where $(F_{1,k})_{k\in 2\mathbb{Z}}$ *and* $(F_{2,k})_{k\in 2\mathbb{Z}}$ *are, respectively, the Fourier coefficients of* F_1 *and* F_2 *about the system* $\{e^{ikx},\ k \in 2\mathbb{Z}\}$;

(b) $\exists \beta > 1$ *such that*

$$\sum_{k\in 2\mathbb{Z}} (|p(k)|^2 + |q(k)|^2)(\log(e+|k|))^\beta < \infty,$$

where $(p(k))_{k\in 2\mathbb{Z}}$ *and* $(q(k))_{k\in 2\mathbb{Z}}$ *are, respectively, the Fourier coefficients of* P *and* Q *about the system* $\{e^{ikx},\ k \in 2\mathbb{Z}\}$.

In particular, if F_1, F_2 *are functions of bounded variation or* P, Q *are functions of bounded variation, then* (1.3) *holds.*

This equiconvergence claim reduces (Section 5, Theorem 22) point-wise convergence problem to the case of free operator where we can use explicit information on Riesz bases of root functions and answer question (ii) — see Formulas (5.3) and (5.4).

In Section 6 we consider Dirac operators with more general potential matrices $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ and weighted eigenvalue problems. The results and formulas of Section 5 are properly adjusted to this case.

Finally, in Section 7 we consider Examples (motivated by the paper of Szmytkowski [35]) with self-adjoint separated boundary conditions — see Theorems 29 and 30.

Appendix gives a detailed proof of a technical result (Lemma 19) on C^1-multipliers in the weighted sequence spaces. Discrete Hilbert transform is an essential component of this proof.

2. Preliminaries

1. Riesz bases

Let H be a separable Hilbert space, and let $(e_\gamma,\ \gamma \in \Gamma)$ be an orthonormal basis in H. If $A : H \to H$ is an automorphism, then the system

$$f_\gamma = A e_\gamma, \quad \gamma \in \Gamma, \tag{2.1}$$

is an unconditional basis in H. Indeed, for each $x \in H$ we have

$$x = A(A^{-1}x) = A\left(\sum_{\gamma} \langle A^{-1}x, e_\gamma \rangle e_\gamma\right) = \sum_{\gamma} \langle x, (A^{-1})^* e_\gamma \rangle f_\gamma = \sum_{\gamma} \langle x, \tilde{f}_\gamma \rangle f_\gamma,$$

i.e., (f_γ) is a basis, and its biorthogonal system is

$$\tilde{f}_\gamma = (A^{-1})^* e_\gamma, \quad \gamma \in \Gamma. \tag{2.2}$$

Moreover, it follows that

$$0 < c \le \|f_\gamma\| \le C, \quad m^2\|x\|^2 \le \sum_{\gamma} |\langle x, \tilde{f}_\gamma \rangle|^2 \|f_\gamma\|^2 \le M^2 \|x\|^2, \tag{2.3}$$

with $c = 1/\|A^{-1}\|$, $C = \|A\|$, $M = \|A\| \cdot \|A^{-1}\|$ and $m = 1/M$.

A basis of the form (2.1) is called *Riesz basis.* One can easily see that the property (2.3) characterizes Riesz bases, i.e., a basis (f_γ) is a Riesz bases if and only if (2.3) holds with some constants $C \ge c > 0$ and $M \ge m > 0$. Another characterization of Riesz bases gives the following assertion (see [14, Chapter 6, Section 5.3, Theorem 5.2]): *If (f_γ) is a normalized basis (i.e., $\|f_\gamma\| = 1\ \forall \gamma$), then it is a Riesz basis if and only if it is unconditional.*

2. We consider the Dirac operators $L = L(v)$ and $L^0 = L(0)$ given by (1.1) on the interval $I = [0, \pi]$. In the following, the space $L^2(I, \mathbb{C}^2)$ is regarded with the scalar product

$$\left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle = \frac{1}{\pi} \int_0^\pi \left(f_1(x)\overline{g_1(x)} + f_2(x)\overline{g_2(x)} \right) dx. \tag{2.4}$$

A general boundary condition (bc) for the operator $L(v)$ is given by a system of two linear equations

$$\begin{aligned} &a_1 y_1(0) + b_1 y_1(\pi) + a_2 y_2(0) + b_2 y_2(\pi) = 0 \\ &c_1 y_1(0) + d_1 y_1(\pi) + c_2 y_2(0) + d_2 y_2(\pi) = 0 \end{aligned} \tag{2.5}$$

Consider the corresponding operator $L_{bc}(v)$ in the domain $Dom\, L_{bc}(v)$ which consists of all absolutely continuous y such that $y_1', y_2' \in L^2(I, \mathbb{C})$ and (2.5) holds. It is easy to see that $L_{bc}(v)$ is a closed densely defined operator.

Let A_{ij} denote the 2×2 matrix formed by the i-th and j-th columns of the matrix $\begin{bmatrix} a_1 & b_1 & a_2 & b_2 \\ c_1 & d_1 & c_2 & d_2 \end{bmatrix}$,and let $|A_{ij}|$ denote the determinant of the matrix A_{ij}. Each solution of the equation $L^0 y = \lambda y$ has the form $y = \begin{pmatrix} \xi e^{-i\lambda x} \\ \eta e^{i\lambda x} \end{pmatrix}$. It satisfies the boundary condition (2.5) if and only if (ξ, η) is a solution of the system of two linear equations

$$\begin{aligned} &\xi(a_1 + b_1 z^{-1}) + \eta(a_2 + b_2 z) = 0 \\ &\xi(c_1 + d_1 z^{-1}) + \eta(c_2 + d_2 z) = 0 \end{aligned} \tag{2.6}$$

with $z = \exp(i\pi\lambda)$. Therefore, there is a nonzero solution y if and only if the determinant of (2.6) is zero, i.e.,

$$|A_{14}|z^2 + (|A_{13}| + |A_{24}|)z + |A_{23}| = 0. \tag{2.7}$$

Definition 1. The boundary condition (2.5) is called: *regular* if

$$|A_{14}| \ne 0, \qquad |A_{23}| \ne 0, \tag{2.8}$$

and *strictly regular* if additionally

$$(|A_{13}| + |A_{24}|)^2 \neq 4|A_{14}||A_{23}|. \tag{2.9}$$

Further only regular boundary conditions are considered. A multiplication from the left of the system (2.5) by the matrix A_{14}^{-1} gives an equivalent to (2.5) system

$$\begin{aligned} &y_1(0) + by_1(\pi) + ay_2(0) = 0, \\ &dy_1(\pi) + cy_2(0) + y_2(\pi) = 0. \end{aligned} \tag{2.10}$$

So, without loss of generality one may consider only bc of the form (2.10). The boundary conditions (2.10) are uniquely determined by the matrix of coefficients $\begin{pmatrix} 1 & b & a & 0 \\ 0 & d & c & 1 \end{pmatrix}$. Then the boundary conditions are *regular* if

$$bc - ad \neq 0, \tag{2.11}$$

and *strictly regular* if additionally

$$(b - c)^2 + 4ad \neq 0. \tag{2.12}$$

The characteristic equation (2.7) becomes

$$z^2 + (b + c)z + bc - ad = 0. \tag{2.13}$$

In the case of strictly regular boundary bc (2.11) and (2.12) guarantee that (2.13) has two distinct nonzero roots z_1 and z_2 (i.e., the matrix $A_{23} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$ has two *distinct* eigenvalues $-z_1, -z_2$). Let us fix a pair of corresponding eigenvectors $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ and $\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$. Then the matrix $\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$ is invertible, and we set

$$\begin{pmatrix} \alpha_1' & \alpha_2' \\ \beta_1' & \beta_2' \end{pmatrix} := \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}^{-1}. \tag{2.14}$$

Let τ_1 and τ_2 be chosen so that

$$z_1 = e^{i\pi\tau_1}, \qquad z_2 = e^{i\pi\tau_2}, \quad |Re\,\tau_1 - Re\,\tau_2| \leq 1. \tag{2.15}$$

Then the eigenvalues of L^0_{bc} are $\lambda^0_{k,\nu} = k + \tau_\nu,\ \nu \in \{1, 2\}, k \in 2\mathbb{Z}$, and a corresponding system of eigenvectors is $\Phi = \{\varphi^1_k, \varphi^1_k,\ k \in 2\mathbb{Z}\}$, where

$$\varphi^1_k := \begin{pmatrix} \alpha_1 e^{i\tau_1(\pi - x)} e^{-ikx} \\ \alpha_2 e^{i\tau_1 x} e^{ikx} \end{pmatrix}, \qquad \varphi^2_k := \begin{pmatrix} \beta_1 e^{i\tau_2(\pi - x)} e^{-ikx} \\ \beta_2 e^{i\tau_2 x} e^{ikx} \end{pmatrix}. \tag{2.16}$$

If bc is regular but not strictly regular, then (2.11) holds but (2.12) fails, i.e.,

$$(b + c)^2 - 4(bc - ad) = (b - c)^2 + 4ad = 0. \tag{2.17}$$

In this case the Eq. (2.13) has a double root $z_* = -(b+c)/2 \neq 0$ (because $bc - ad \neq 0$). Choose τ_* so that $z_* = \exp(i\pi\tau_*),\ |\tau_*| \leq 1$. Then each eigenvalue of L^0_{bc} is of algebraic multiplicity 2 and has the form $\tau_* + k, k \in 2\mathbb{Z}$.

We call the boundary conditions given by the system (2.10) *periodic-type* if

$$b = c, \qquad a = 0, \qquad d = 0, \tag{2.18}$$

holds. The condition (2.18) takes place if and only if $A_{23} + z_* I$ is the zero matrix, so then any two linearly independent vectors $\binom{\alpha_1}{\alpha_2}$ and $\binom{\beta_1}{\beta_2}$ are eigenvectors of A_{23}. With any choice of such vectors, the system Φ given by (2.16) but with $\tau_2 = \tau_1 = \tau_*$, consists of corresponding eigenfunctions of L^0_{bc}.

Next we consider the case when (2.17) holds but (2.18) fails, i.e.,

$$|b - c| + |a| + |d| > 0. \tag{2.19}$$

In this case each eigenvalue of L^0_{bc} is of algebraic multiplicity 2 but of geometric multiplicity 1, i.e., associated eigenvectors appear. Here we have the following subcases:

(i) $a = 0$, then (2.17) implies $b = c$, and by (2.19) we have $d \neq 0$;
(ii) $d = 0$, then (2.17) implies $b = c$, and by (2.19) we have $a \neq 0$;
(iii) $a, d \neq 0$, then (2.17) implies $b \neq c$.

Now we set

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & \pi b \\ d & 0 \end{pmatrix} & \text{for (i)}, \\ \begin{pmatrix} a & 0 \\ \dfrac{c-b}{2} & \pi b \end{pmatrix} & \text{for (ii), (iii).} \end{cases} \tag{2.20}$$

A corresponding system of eigenvectors is given by

$$\Phi^1 = \{\varphi^1_k,\ k \in 2\mathbb{Z}\}, \qquad \varphi^1_k = \begin{pmatrix} \alpha_1 e^{i\tau_*(\pi - x)} e^{-ikx} \\ \alpha_2 e^{i\tau_* x} e^{ikx} \end{pmatrix}, \tag{2.21}$$

and

$$\Phi^2 = \{\varphi^2_k,\ k \in 2\mathbb{Z}\}, \qquad \varphi^2_k = \begin{pmatrix} (\beta_1 - \alpha_1 x) e^{i\tau_*(\pi - x)} e^{-ikx} \\ (\beta_2 + \alpha_2 x) e^{i\tau_* x} e^{ikx} \end{pmatrix} \tag{2.22}$$

is a system of corresponding associated vectors.

Remark 2. Components of the systems (2.16) and (2.21), (2.22) are uniformly bounded, i.e.,

$$\sup_{k \in 2\mathbb{Z}, \nu \in \{1,2\}} \sup_{[0,\pi]} \left|\varphi^\nu_k(x)\right| = C = C(bc) < \infty. \tag{2.23}$$

Here and thereafter, we denote by C any constant that is absolute up to dependence on bc.

Theorem 3. (a) *For strictly regular or periodic type bc, the system Φ given by* (2.16) *is a Riesz basis in the space $L^2(I, \mathbb{C}^2)$, $I = [0, \pi]$. Its biorthogonal system is $\tilde{\Phi} = \{\tilde{\varphi}^1_k, \tilde{\varphi}^2_k, k \in 2\mathbb{Z}\}$, where*

$$\tilde{\varphi}^1_k := \begin{pmatrix} \overline{\alpha'_1} e^{i\overline{\tau_1}(\pi - x)} e^{-ikx} \\ \overline{\alpha'_2} e^{i\overline{\tau_1} x} e^{ikx} \end{pmatrix}, \qquad \tilde{\varphi}^2_k := \begin{pmatrix} \overline{\beta'_1} e^{i\overline{\tau_2}(\pi - x)} e^{-ikx} \\ \overline{\beta'_2} e^{i\overline{\tau_2} x} e^{ikx} \end{pmatrix}, \tag{2.24}$$

with $\alpha'_1, \alpha'_2, \beta'_1, \beta'_2$ coming, respectively, from (2.14) *for strictly regular bc or periodic type bc.*

(b) *For regular but not strictly regular or periodic type bc, the system* $\Phi = \Phi^1 \cup \Phi^2$ *given in* (2.21) *and* (2.22) *is a Riesz basis in the space* $L^2(I, \mathbb{C}^2)$*. Its biorthogonal system is* $\tilde{\Phi} = \{\tilde{\varphi}_k^1, \tilde{\varphi}_k^2, k \in 2\mathbb{Z}\}$*, where*

$$\tilde{\varphi}_k^1 = \begin{pmatrix} \bar{\Delta}^{-1}\overline{\alpha_2} e^{i\overline{\tau_*}(\pi - x)} e^{-ikx} \\ \bar{\Delta}^{-1}\overline{\alpha_1} e^{i\overline{\tau_*}x} e^{ikx} \end{pmatrix},$$

$$\tilde{\varphi}_k^2 = \begin{pmatrix} \bar{\Delta}^{-1}[\overline{\beta_2} + \overline{\alpha_2}(\pi - x)] e^{i\overline{\tau_*}(\pi - x)} e^{-ikx} \\ \bar{\Delta}^{-1}[\overline{\beta_1} - \overline{\alpha_1}(\pi - x)] e^{i\overline{\tau_*}x} e^{ikx} \end{pmatrix} \tag{2.25}$$

with $\Delta = \alpha_1\beta_2 - \alpha_2\beta_1 + \pi\alpha_1\alpha_2$.

Proof. (a) First we consider the case of strictly regular or periodic type *bc*. The system

$$E = \{e_k^1, e_k^2, \ k \in 2\mathbb{Z}\}, \quad \text{where } e_k^1 := \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix}, \ e_k^2 := \begin{pmatrix} 0 \\ e^{ikx} \end{pmatrix}, \tag{2.26}$$

is an orthonormal basis in $L^2(I, \mathbb{C}^2)$, $I = [0, \pi]$. We have $\Phi = A(E)$, where the operator $A : L^2(I, \mathbb{C}^2) \to L^2(I, \mathbb{C}^2)$ is defined by

$$A \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \alpha_1 e^{i\tau_1(\pi - x)} f(\pi - x) \\ \alpha_2 e^{i\tau_1 x} f(x) \end{pmatrix} + \begin{pmatrix} \beta_1 e^{i\tau_2(\pi - x)} g(\pi - x) \\ \beta_2 e^{i\tau_2 x} g(x) \end{pmatrix}. \tag{2.27}$$

Since the functions $e^{i\tau_\nu x}$ and $e^{i\tau_\nu(\pi - x)}$, $\nu = 1, 2$, are bounded, it follows that A is bounded operator. In view of (2.14) its inverse operator A^{-1} is given by

$$A^{-1} \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} e^{-i\tau_1 x}[\alpha_1' F(\pi - x) + \alpha_2' G(x)] \\ e^{-i\tau_2 x}[\beta_1' F(\pi - x) + \beta_2' G(x)] \end{pmatrix}, \tag{2.28}$$

so one can easily see that A^{-1} is bounded as well. Thus, A is an isomorphism, which proves that the system Φ is a Riesz basis. By (2.2), its biorthogonal system is $\tilde{\Phi} = (A^{-1})^*(E)$, so we obtain (2.24) by a direct computation.

(b) In the case of regular but not strictly regular or periodic type *bc* we consider the operator $A : L^2(I, \mathbb{C}^2) \to L^2(I, \mathbb{C}^2)$ defined by

$$A \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \alpha_1 e^{i\tau_*(\pi - x)} f(\pi - x) \\ \alpha_2 e^{i\tau_* x} f(x) \end{pmatrix} + \begin{pmatrix} (\beta_1 - \alpha_1 x) e^{i\tau_*(\pi - x)} g(\pi - x) \\ (\beta_2 + \alpha_2 x) e^{i\tau_* x} g(x) \end{pmatrix}. \tag{2.29}$$

Then $\Phi = A(E)$, where E is the orthonormal basis (2.26). One can easily see that the operator A is bounded and its inverse operator

$$A^{-1} \begin{pmatrix} F \\ G \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} [(\beta_2 + \alpha_2 x) F(\pi - x) - (\beta_1 - \pi\alpha_1 + \alpha_1 x) G(x)]\, e^{-i\tau_* x} \\ [-\alpha_2 F(\pi - x) + \alpha_1 G(x)]\, e^{-i\tau_* x} \end{pmatrix} \tag{2.30}$$

is also bounded. Thus the system Φ given by (2.21) and (2.22) is a Riesz basis. A direct computation of $\tilde{\Phi} = (A^{-1})^*(E)$ shows that (2.25) holds. □

Lemma 4. *Let bc be given by* (2.10)*, and let A be defined, respectively, by* (2.27) *if bc is strictly regular or periodic type and by* (2.29) *if bc is regular but not strictly regular or periodic type. If* f, g, F, G *are continuous functions on* $[0, \pi]$ *such that* $\binom{F}{G} = A\binom{f}{g}$*, then* $\binom{F}{G}$ *satisfies bc if and only if* $\binom{f}{g}$ *satisfies the periodic boundary conditions* $f(0) = f(\pi)$, $g(0) = g(\pi)$.

Proof. In view of (2.27) and (2.29), the operator A acts "point-wise" on vector-functions $\binom{f}{g}$ multiplying components f and g by smooth functions. Therefore, A generates a linear operator $\tilde{A} : \mathbb{C}^4 \to \mathbb{C}^4$ such that

$$\tilde{A}(f(0), f(\pi), g(0), g(\pi)) = (F(0), F(\pi), G(0), G(\pi)).$$

Since A is invertible, $\tilde{A}$ is also invertible.

The periodic boundary conditions $f(0) = f(\pi)$, $g(0) = g(\pi)$ define a two-dimensional subspace $E_{Per} \subset \mathbb{C}^4$, and the boundary conditions (2.10) define a two-dimensional subspace $E_{bc} \subset \mathbb{C}^4$. In fact, the lemma claims that $\tilde{A} E_{Per} = E_{bc}$. Since $\tilde{A}$ is an isomorphism, is enough to show that $\tilde{A} E_{Per} \subset E_{bc}$. In other words, we need only to prove that if $\binom{f}{g}$ satisfies the periodic boundary conditions then $\binom{F}{G} = A\binom{f}{g}$ satisfies (2.10).

Since

$$\begin{pmatrix} F \\ G \end{pmatrix} = A \begin{pmatrix} f \\ g \end{pmatrix} = A \begin{pmatrix} f \\ 0 \end{pmatrix} + A \begin{pmatrix} 0 \\ g \end{pmatrix},$$

it is enough to show that $A\binom{f}{0}$ and $A\binom{0}{g}$ satisfy bc. Set $\binom{y_1}{y_2} := A\binom{f}{0}$.

If bc is strictly regular or periodic type boundary condition, then by (2.15) and (2.27) it follows that

$$y_1(0) = \alpha_1 z_1 f(\pi), \qquad y_1(\pi) = \alpha_1 f(0), \qquad y_2(0) = \alpha_2 f(0),$$
$$y_2(\pi) = \alpha_2 z_1 f(\pi).$$

Therefore, taking into account that $f(\pi) = f(0)$, we obtain that $\binom{y_1}{y_2}$ satisfies the boundary conditions (2.10):

$$\begin{pmatrix} 1 & b & a & 0 \\ 0 & d & c & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 z_1 f(\pi) \\ \alpha_1 f(0) \\ \alpha_2 f(0) \\ \alpha_2 z_1 f(\pi) \end{pmatrix} = f(0) \begin{pmatrix} b + z_1 & a \\ d & c + z_1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0$$

$\Big($due to the definition of $\binom{\alpha_1}{\alpha_2}$, see the lines prior to (2.14) and after (2.18)$\Big)$.

If bc is regular but not strictly regular or periodic type, then the characteristic equation (2.13) has a double root $z_* = -(b + c)/2 = \exp(i\pi\tau_*)$, and by (2.29) we have

$$y_1(0) = \alpha_1 z_* f(\pi), \qquad y_1(\pi) = \alpha_1 f(0), \qquad y_2(0) = \alpha_2 f(0),$$
$$y_2(\pi) = \alpha_2 z_* f(\pi).$$

Using (2.20), one can easily verify that $\binom{y_1}{y_2}$ satisfies the boundary conditions (2.10).

The proof that $A\binom{0}{g}$ satisfies the boundary conditions (2.10) is similar; we omit the details. □

3. Estimates for the resolvent of L_{bc} and localization of spectra

The operator $L_{bc}(v)$ maybe considered as a perturbation of the free operator L^0_{bc}. We study $L_{bc}(v)$ by considering its Fourier matrix representation with respect to the Riesz basis $\Phi = \Phi(bc)$ consisting of root functions of L^0_{bc}, which was constructed in Theorem 3.

For strictly regular bc, the spectrum of L^0_{bc} consists of two disjoint sequences of simple eigenvalues

$$Sp(L^0_{bc}) = \{\tau_1 + k,\ k \in 2\mathbb{Z}\} \cup \{\tau_2 + k,\ k \in 2\mathbb{Z}\},$$

where τ_1, τ_2 depend on bc. For regular but not strictly regular bc the spectrum of L^0_{bc} consists of eigenvalues of algebraic multiplicity 2 and has the form

$$Sp\,(L^0_{bc}) = \{\tau_* + k,\ k \in 2\mathbb{Z}\},$$

where τ_* depends on bc. In both cases the resolvent operator $R^0_{bc}(\lambda) = (\lambda - L^0_{bc})^{-1}$ is well defined for $\lambda \notin Sp(L^0_{bc})$, and we have

$$R^0_{bc}(\lambda)\varphi^\nu_k = \frac{1}{\lambda - \tau_\nu - k}\varphi^\nu_k, \quad k \in 2\mathbb{Z},\ \nu \in \{1, 2\}, \tag{3.1}$$

where we put $\tau_1 = \tau_2 = \tau_*$ in the case of regular but not strictly regular bc. Moreover, $R^0_{bc}(\lambda)$ acts continuously from $L^2([0, \pi], \mathbb{C}^2)$ into $L^\infty([0, \pi], \mathbb{C}^2)$. Indeed, if $F = \sum_{k,\nu} F^\nu_k \varphi^\nu_k$, then

$$R^0_{bc}(\lambda)F = \sum_{k,\nu} \frac{F^\nu_k}{\lambda - k - \tau_\nu}\varphi^\nu_k.$$

By (2.23), $\|R^0_{bc}(\lambda)F\|_\infty \le \sum_{k,\nu} \frac{C|F^\nu_k|}{|\lambda-k-\tau_\nu|}$, so by the Cauchy inequality,

$$\|R^0_{bc}(\lambda)F\|_\infty \le C\left(\sum_{k,\nu} \frac{1}{|\lambda - k - \tau_\nu|^2}\right)^{1/2}\left(\sum_{k,\nu}|F^\nu_k|^2\right)^{1/2}.$$

Thus, in view of (2.3),

$$\|R^0_{bc}(\lambda)\|_{L^2\to L^\infty} \le C_1[a(\lambda - \tau_1) + a(\lambda - \tau_2)]^{1/2}, \tag{3.2}$$

where $C_1 = C_1(bc)$ and $a(\lambda)$ is defined and estimated in the following lemma.

Lemma 5. *Consider the function*

$$a(\lambda) = \sum_{k\in 2\mathbb{Z}} \frac{1}{|\lambda - k|^2}, \quad \lambda \notin 2\mathbb{Z}. \tag{3.3}$$

If $\lambda = m + \xi + it$ with $m \in 2\mathbb{Z}$ and $-1 < \xi < 1$, then

$$a(\lambda) \le \frac{1}{\xi^2 + t^2} + \frac{8}{1 + 2|t|}. \tag{3.4}$$

Proof. Since $a(\lambda + 2) = a(\lambda)$, it is enough to consider the case where $m = 0$. Then we have

$$a(\lambda) = \sum_{k\in 2\mathbb{Z}} \frac{1}{(\xi - k)^2 + t^2} \le \frac{1}{\xi^2 + t^2} + 2\sum_{k\in 2\mathbb{N}} \frac{1}{(k - 1)^2 + t^2}.$$

Since $(k - 1)^2 + t^2 \ge \frac{1}{2}(k - 1 + |t|)^2$, we obtain

$$\sum_{k\in 2\mathbb{N}} \frac{1}{(k - 1)^2 + t^2} \le \int_{3/2}^\infty \frac{2}{(u - 1 + |t|)^2}du = \frac{4}{1 + 2|t|},$$

which completes the proof. $\square$

Let $V : L^2(I,\mathbb{C}^2) \to L^2(I,\mathbb{C}^2)$ be the operator of multiplication by the matrix $v(x) = \begin{bmatrix} 0 & P(x) \\ Q(x) & 0 \end{bmatrix}$, i.e., $V\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} Py_2 \\ Qy_1 \end{pmatrix}$. The operator V could be unbounded in $L^2(I,\mathbb{C}^2)$ but it acts continuously from $L^\infty(I,\mathbb{C}^2)$ into $L^2(I,\mathbb{C}^2)$. Indeed, if $y_1, y_2 \in L^\infty$ then

$$\left\| \begin{pmatrix} Py_2 \\ Qy_1 \end{pmatrix} \right\|^2 = \frac{1}{\pi}\int_0^\pi (|Qy_1|^2 + |Py_2|^2)dx \le \|v\|^2 \cdot \left\| \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\|_\infty^2,$$

where we set for convenience

$$\|v\|^2 = \|P\|^2 + \|Q\|^2, \qquad \left\| \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\|_\infty = \max(\|y_1\|_\infty, \|y_2\|_\infty). \tag{3.5}$$

Therefore,

$$\|V\|_{L^\infty \to L^2} \le \|v\|. \tag{3.6}$$

The following lemma follows immediately from the explicit form of the bases Φ and their biorthogonal systems given in Theorem 3.

Lemma 6. *The matrix representation of V with respect to the basis Φ has the form*

$$V \sim \begin{bmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{bmatrix}, \qquad V^{\eta\nu} = \left(V^{\eta\nu}_{jk}\right)_{j,k\in 2\mathbb{Z}}, \quad \eta,\nu \in \{1,2\}, \tag{3.7}$$

$$V^{\eta\nu}_{jk} = \langle V\varphi^\nu_k, \tilde{\varphi}^\eta_j \rangle = w^{\eta\nu}(j+k), \tag{3.8}$$

where

$$w^{\eta\nu} = \left(w^{\eta\nu}(m)\right) \in \ell^2(2\mathbb{Z}), \quad \|w^{\eta\nu}\|_{\ell^2} \le C\|v\|, \tag{3.9}$$

with $C = C(bc)$.

Proof. Indeed, in view of the explicit formulas for the basis $\Phi = \{\varphi^\nu_k\}$ and its biorthogonal system $\tilde{\Phi} = \{\tilde{\varphi}^\nu\}$ it follows that (3.8) holds with

$$w^{\eta\nu}(m) = p^{\eta\nu}(-m) + q^{\eta\nu}(m), \tag{3.10}$$

where $p^{\eta\nu}(k),\ q^{\eta\nu}(k),\ k \in 2\mathbb{Z}$ are, respectively, the Fourier coefficients of functions of the form

$$g^{\eta\nu}(x)P(x),\ h^{\eta\nu}(x)Q(x), \quad g^{\eta\nu}, h^{\eta\nu} \in C^\infty([0,\pi]),\ \eta,\nu = 1,2. \quad \square \tag{3.11}$$

For convenience, we set

$$r(m) = \max\{|w^{\mu\nu}(m)|,\ \mu,\nu = 1,2\}, \quad m \in 2\mathbb{Z}; \tag{3.12}$$

then

$$r = (r(m)) \in \ell^2(2\mathbb{Z}), \quad \|r\| \le C\|v\|. \tag{3.13}$$

The standard perturbation formula for the resolvent

$$R_{bc}(\lambda) = R^0_{bc}(\lambda) + R^0_{bc}(\lambda)VR^0_{bc}(\lambda) + \sum_{m=2}^\infty R^0_{bc}(\lambda)(VR^0_{bc}(\lambda))^m \tag{3.14}$$

is valid if the series on the right converges. From (3.2) and (3.6) it follows that $VR^0_{bc}(\lambda)$ is a continuous operator in $L^2([0,\pi],\mathbb{C}^2)$ which norm does not exceed

$$\begin{aligned} \|VR^0_{bc}(\lambda)\| &\leq \|V\|_{L^\infty\to L^2}\|R^0_{bc}(\lambda)\|_{L^2\to L^\infty} \\ &\leq C_1\|v\|\cdot[a(\lambda-\tau_1)+a(\lambda-\tau_2)]^{1/2}. \end{aligned} \tag{3.15}$$

But this estimate does not guarantee the convergence in (3.14) for λ's which are close to the real line. Therefore, our next goal is to estimate the norms $\|R^0_{bc}(\lambda)VR^0_{bc}(\lambda)\|_{L^2\to L^\infty}$.

But first we introduce some notations. For each ℓ^2–sequence $x=(x(j))_{j\in 2\mathbb{Z}}$ and $m>0$ we set

$$\mathcal{E}_m(x)=\left(\sum_{|j|\geq m}|x(j)|^2\right)^{1/2}. \tag{3.16}$$

In a case of strictly regular bc, we subdivide the complex plane $\mathbb{C}$ into strips

$$H_m=\left\{z\in\mathbb{C}:\ -1\leq Re\left(z-m-\frac{\tau_1+\tau_2}{2}\right)\leq 1\right\},\quad m\in 2\mathbb{Z}, \tag{3.17}$$

and set

$$H^N=\bigcup_{|m|\leq N}H_m, \tag{3.18}$$

$$\rho:=\frac{1}{2}\min(1-|Re(\tau_1-\tau_2)|/2,\ |\tau_1-\tau_2|/2), \tag{3.19}$$

$$D^\mu_m=\{z\in\mathbb{C}:|z-\tau_\mu-m|<\rho\},\quad m\in 2\mathbb{Z} \tag{3.20}$$

and

$$R_{NT}=\left\{z=x+it:\ \left|x-Re\,\frac{\tau_1+\tau_2}{2}\right|<N+1,\ |t|<T\right\}, \tag{3.21}$$

where $N\in 2\mathbb{N}$ and $T>0$.

In case of regular but not strictly regular boundary conditions we subdivide the complex plane $\mathbb{C}$ into strips

$$H_m=\{z\in\mathbb{C}:-1\leq Re\,(z-m-\tau_*)\leq 1\},\quad m\in 2\mathbb{Z}, \tag{3.22}$$

and set

$$H^N=\bigcup_{|m|\leq N}H_m, \tag{3.23}$$

$$D_m=\{z\in\mathbb{C}:|z-\tau_*-m|<1/4\},\quad m\in 2\mathbb{Z} \tag{3.24}$$

and

$$R_{NT}=\{z=x+it:|x-Re\,\tau_*|<N+1,\ |t|<T\}, \tag{3.25}$$

where $N\in 2\mathbb{N}$ and $T>0$.

Lemma 7. (a) *For $\lambda \in H_m \setminus (D_m^1 \cup D_m^2)$ in a case of strictly regular bc, or for $\lambda \in H_m \setminus D_m$ in a case of regular but not strictly regular bc, we have*

$$\|R_{bc}^0(\lambda) V R_{bc}^0(\lambda)\|_{L^2 \to L^\infty} \le C \left(\frac{\|v\|}{\sqrt{|m|}} + (\mathcal{E}_{|m|}(r)) \right) \quad \text{for } m \neq 0, \tag{3.26}$$

where $C = C(bc)$.

(b) *If $T \ge 1 + 2|\tau_1| + 2|\tau_2|$, then*

$$\|R_{bc}^0(\lambda) V R_{bc}^0(\lambda)\|_{L^2 \to L^\infty} \le C \frac{\|v\|}{T} \quad \text{for } |Im\,\lambda| \ge T, \tag{3.27}$$

where $C = C(bc)$.

Proof. In view of the matrix representations of the operators V and $R_{bc}^0(\lambda)$ given in (3.1) and Lemma 6, if $F = \sum_{k,\nu} F_k^\nu \varphi_k^\nu$ then

$$R_{bc}^0(\lambda) V R_{bc}^0(\lambda) F = \sum_{k,\nu} \sum_{j,\eta} \frac{w^{\eta\nu}(j+k) F_k^\nu}{(\lambda - j - \tau_\eta)(\lambda - k - \tau_\nu)} \varphi_k^\nu,$$

where $\tau_1 = \tau_2 = \tau_*$ in case of regular but not strictly regular bc.

For convenience, we set

$$g_k = \max(|F_k^1|, |F_k^2|), \quad k \in 2\mathbb{Z}. \tag{3.28}$$

Then $g = (g_k) \in \ell^2$ and $\|g\| \le C\|F\|$, where $C = C(bc)$.

By (2.23) and (3.12),

$$\|R_{bc}^0(\lambda) V R_{bc}^0(\lambda) F\|_\infty \le C \sum_{k,\nu} \sum_{j,\eta} \frac{r(j+k)|g_k|}{|\lambda - j - \tau_\eta \| \lambda - k - \tau_\nu|}. \tag{3.29}$$

Let $\lambda \in H_m$. Then $\lambda = m + \xi + Re\frac{\tau_1+\tau_2}{2} + it$ with $-1 < \xi \le 1$ (in the case of regular but not strictly regular $bc\tau_1 = \tau_2 = \tau_*$). Therefore, in view of (2.15), we have for $k \neq m$

$$|\lambda - k - \tau_\nu| \ge |m-k| - 1 - \frac{1}{2}|Re(\tau_1 - \tau_2)| \ge |m-k| - \frac{3}{2} \ge \frac{1}{4}|m-k|. \tag{3.30}$$

For strictly regular bc and $\lambda \in H_m \setminus (D_m^1 \cup D_m^2)$, (3.30) implies

$$\|R_{bc}^0(\lambda) V R_{bc}^0(\lambda) F\|_\infty \le C \left(\frac{r(2m) g_m}{\rho^2} + \frac{4}{\rho}\sigma_1 + \frac{4}{\rho}\sigma_2 + 16\sigma_3 \right) \tag{3.31}$$

where $\sigma_1 = \sum_{k \neq m} \frac{r(m+k) g_k}{|m-k|}$,

$$\sigma_2 = \sum_{j \neq m} \frac{r(j+m) g_m}{|j-m|}, \qquad \sigma_3 = \sum_{k \neq m} \sum_{j \neq m} \frac{r(j+k) g_k}{|m-j \| m-k|}.$$

We have

$$\sum_{k \neq m} \frac{r(m+k)}{|m-k|} = \sum_{|k-m|>|m|} \frac{r(m+k)}{|m-k|} + \sum_{|k-m| \le |m|} \frac{r(m+k)}{|m-k|}.$$

By the Cauchy inequality,

$$\sum_{|k-m|>|m|} \frac{r(m+k)}{|m-k|} \le \|r\| \cdot \left(\sum_{|k-m|>|m|} \frac{1}{(m-k)^2} \right)^{1/2} \le \frac{\|r\|}{\sqrt{|m|}}.$$

On the other hand, if $|k-m| \le |m|$ then $|k+m| \ge 2|m| - |m-k| \ge |m|$. Therefore

$$\sum_{|k-m|\le|m|} \frac{r(m+k)}{|m-k|} \le \left(\sum_{|k-m|\le|m|} r(m+k)\,|^2 \right)^{1/2} \left(\sum_{k\neq m} \frac{1}{(m-k)^2} \right)^{1/2}$$
$$\le \left(\sum_{j\ge|m|} r(j)\,|^2 \right)^{1/2} \cdot \frac{\pi}{\sqrt{3}} \le 2\mathcal{E}_{|m|}(r).$$

Since $|g_k| \le \|g\|$, the above inequalities imply that

$$\sigma_\alpha \le \|g\| \left(\frac{\|r\|}{\sqrt{|m|}} + 2\mathcal{E}_{|m|}(r) \right), \quad \alpha = 1, 2. \tag{3.32}$$

Next we estimate $\sigma_3 \le \sigma_3^1 + \sigma_3^2 + \sigma_3^3$, where

$$\sigma_3^1 = \sum_{|m-k|>\frac{|m|}{2}} \sum_{j\neq m} \frac{r(j+k)g_k}{|m-j\,\|\,m-k|}, \qquad \sigma_3^2 = \sum_{k\neq m} \sum_{|m-j|>\frac{|m|}{2}} \frac{r(j+k)g_k}{|m-j\,\|\,m-k|},$$
$$\sigma_3^3 = \sum_{|m-k|\le\frac{|m|}{2}} \sum_{|m-j|\le\frac{|m|}{2}} \frac{r(j+k)g_k}{|m-j\,\|\,m-k|}.$$

By the Cauchy inequality,

$$(\sigma_3^1)^2 \le \sum_k \left(|g_k|^2 \sum_j |r(j+k)|^2 \right) \cdot \sum_{j\neq m} \frac{1}{(j-m)^2} \sum_{|m-k|>\frac{|m|}{2}} \frac{1}{(k-m)^2}$$
$$\le \|r\|^2 \cdot \|g\|^2 \frac{\pi^2}{3} \cdot \frac{4}{|m|} \le \frac{16}{|m|} \|r\|^2 \|g\|^2.$$

The same argument gives the same estimate for σ_3^2.

On the other hand, if $|j-m| \le |m|/2$ and $|k-m| \le |m|/2$, then $|j+k| \ge 2|m| - |m-j| - |m-k| \ge |m|$. Therefore, by the Cauchy inequality we obtain

$$(\sigma_3^3)^2 \le \sum_{|k-m|\le\frac{|m|}{2}} \left(|g_k|^2 \sum_{|j-m|\le\frac{|m|}{2}} |r(j+k)|^2 \right) \cdot \sum_{j\neq m} \frac{1}{(j-m)^2} \sum_{k\neq m} \frac{1}{(k-m)^2}$$
$$\le \|g\|^2 (\mathcal{E}_{|m|}(r))^2 \frac{\pi^2}{3} \frac{\pi^2}{3} \le 16\|g\|^2 (\mathcal{E}_{|m|}(r))^2.$$

From the above estimates it follows

$$\sigma_3 \le 4\|g\| \left(\frac{2\|r\|}{\sqrt{|m|}} + \mathcal{E}_{|m|}(r) \right). \tag{3.33}$$

Now, in view of (3.31), the estimates (3.32) and (3.33) imply (3.26) in the case of strictly regular bc. In the case of regular but not strictly regular bc the proof is the same because for $\lambda \in H_m \setminus D_m$ (3.29) implies (3.31) with $\rho = 1/4$.

Finally, we prove (b). By (3.2) and (3.6), it follows that

$$\|R_{bc}^0(\lambda) V R_{bc}^0(\lambda)\|_{L^2\to L^\infty} \le C\|v\| \cdot [a(\lambda-\tau_1) + a(\lambda-\tau_2)].$$

If $T > 1 + 2|\tau_1| + 2|\tau_2|$ and $|Im\,\lambda| \geq T$, then

$$|Im\,(\lambda - \tau_\nu)| \geq T - |\tau_\nu| \geq T/2, \quad \nu = 1, 2.$$

Therefore, by Lemma 5 we obtain, for $|Im\,\lambda| \geq T$,

$$a(\lambda - \tau_1) + a(\lambda - \tau_2) \leq \frac{2}{(T/2)^2} + \frac{16}{T} \leq \frac{24}{T},$$

which implies (3.27). □

By (3.6) we have

$$\|V R^0_{bc}(\lambda) V R^0_{bc}(\lambda)\|_{L^2 \to L^2} \leq \|v\| \cdot \|R^0_{bc}(\lambda) V R^0_{bc}(\lambda)\|_{L^2 \to L^\infty}.$$

Therefore, in view of (3.26) and (3.27), the following holds.

Corollary 8. *There are $N_0 \in 2\mathbb{N}$ and $T_0 > 0$ such that if $|m| \geq N$ and $\lambda \in H_m \setminus (D^1_m \cup D^2_m)$ in case of strictly regular bc or $\lambda \in H_m \setminus D_m$ in case of regular but not strictly regular bc, or if $|Im\,\lambda| \geq T_0$,*

$$\|V R^0_{bc}(\lambda) V R^0_{bc}(\lambda)\|_{L^2 \to L^2} \leq 1/2. \tag{3.34}$$

By (3.14), the validity of (3.34) for some λ means that the resolvent $R_{bc}(\lambda)$ exists for that λ, which leads to the following localization of spectra assertion.

Lemma 9. (a) *For strictly regular bc there is an $N_0 = N_0(v, bc) \in 2\mathbb{N}$ and $T_0 = T_0(v, bc) > 0$ such that if $N \geq N_0,\ N \in 2\mathbb{N}$, and $T \geq T_0$ then*

$$Sp\,(L_{bc}(v_\zeta)) \subset R_{NT} \cup \bigcup_{|m|>N} \left(D^1_m \cup D^2_m\right) \quad \text{for } v_\zeta = \zeta v,\ |\zeta| \leq 1. \tag{3.35}$$

(b) *For regular but not strictly regular bc there is an $N_0 = N_0(v, bc) \in 2\mathbb{N}$ and $T_0 = T_0(v, bc) > 0$ such that if $N \geq N_0,\ N \in 2\mathbb{N}$, and $T \geq T_0$ then*

$$Sp\,(L_{bc}(v_\zeta)) \subset R_{NT} \cup \bigcup_{|m|>N} D_m \quad \text{for } v_\zeta = \zeta v, \quad |\zeta| \leq 1. \tag{3.36}$$

For strictly regular bc, consider the Riesz projections associated with $L = L_{bc}$

$$S_N = \frac{1}{2\pi i} \int_{\partial R_{NT}} (\lambda - L)^{-1} d\lambda, \qquad P_{n,\alpha} = \frac{1}{2\pi i} \int_{\partial D^\alpha_n} (\lambda - L)^{-1} d\lambda, \quad \alpha = 1, 2, \tag{3.37}$$

and let S^0_N and $P^0_{n,\alpha}$ be the Riesz projections associated with the free operator L^0_{bc}. A standard argument (continuity about the parameter ζ in (3.35)) shows that

$$\dim P_{n,\alpha} = \dim P^0_{n,\alpha} = 1, \qquad \dim S_N = \dim S^0_N = 2N + 2. \tag{3.38}$$

If bc are regular but not strictly regular, consider the Riesz projections associated with $L = L_{bc}$

$$S_N = \frac{1}{2\pi i} \int_{\partial R_{NT}} (\lambda - L)^{-1} d\lambda, \qquad P_n = \frac{1}{2\pi i} \int_{\partial D_n} (\lambda - L)^{-1} d\lambda, \tag{3.39}$$

and let S_N^0 and P_n^0 be the Riesz projections associated with the free operator L_{bc}^0. The same argument, as in the case of strictly regular bc, shows that

$$\dim P_n = \dim P_n^0 = 2, \qquad \dim S_N = \dim S_N^0 = 2N + 2. \tag{3.40}$$

Further analysis of the Riesz projections leads to the following result — see [12, Theorems 15 and 20].

Theorem 10. *Suppose v is an L^2-Dirac potential.*

(a) *If bc is strictly regular then there are $N_0 = N_0(v, bc) \in 2\mathbb{N}$ and $T_0 = T_0(v, bc) > 0$ such that if $N \geq N_0$, $N \in 2\mathbb{N}$, and $T \geq T_0$ then the Riesz projections S_N, S_N^0 and $P_{n,\alpha}$, $P_{n,\alpha}^0$, $n \in 2\mathbb{Z}$, $|n| > N$, are well defined by (3.37), and we have*

$$\sum_{|n|>N} \|P_{n,\alpha} - P_{n,\alpha}^0\|^2 < \infty, \quad \alpha = 1, 2. \tag{3.41}$$

Moreover, the system $\{S_N;\ P_{n,\alpha},\ n \in 2\mathbb{Z},\ |n| > N,\ \alpha = 1, 2\}$ is a Riesz basis of projections in $L^2([0, \pi], \mathbb{C}^2)$, i.e.,

$$\mathbf{f} = S_N(\mathbf{f}) + \sum_{\alpha=1}^{2} \sum_{|n|>N} P_{n,\alpha}(\mathbf{f}) \quad \forall \mathbf{f} \in L^2([0, \pi], \mathbb{C}^2), \tag{3.42}$$

where the series converge unconditionally.

(b) *If bc is regular but not strictly regular, there are $N_0 = N_0(v, bc) \in 2\mathbb{N}$ and $T_0 = T_0(v, bc) > 0$ such that if $N \geq N_0$, $N \in 2\mathbb{N}$, and $T \geq T_0$ then the Riesz projections S_N, S_N^0 and P_n, P_n^0, $n \in 2\mathbb{Z}$, $|n| > N$, are well defined by (3.39), and we have*

$$\sum_{|n|>N} \|P_n - P_n^0\|^2 < \infty. \tag{3.43}$$

Moreover, the system $\{S_N;\ P_n,\ n \in 2\mathbb{Z},\ |n| > N\}$ is a Riesz basis of projections in $L^2([0, \pi], \mathbb{C}^2)$, i.e.,

$$\mathbf{f} = S_N(\mathbf{f}) + \sum_{|n|>N} P_n(\mathbf{f}) \quad \forall \mathbf{f} \in L^2([0, \pi], \mathbb{C}^2), \tag{3.44}$$

where the series converge unconditionally.

Since the projections $P_{n,\alpha}$ in (3.42) are one-dimensional, we obtain the following.

Corollary 11. *If bc is strictly regular, then there exists a Riesz basis in $L^2([0, \pi], \mathbb{C}^2)$ consisting of eigenfunctions and at most finitely many associated functions of the Dirac operator $L_{bc}(v)$.*

4. Equiconvergence

By Theorem 3, for every regular bc there is a Riesz basis $\Phi = \{\varphi_k^\nu\}$ in $L^2([0, \pi], \mathbb{C}^2)$ consisting of root functions of the free Dirac operator L_{bc}^0. In the next section, we study the point-wise convergence of the L^2-expansions with respect to the basis $\Phi = \Phi(bc)$

$$\sum_{k\in 2\mathbb{Z}} \sum_{\nu=1}^{2} \langle F, \tilde{\varphi}_k^\nu \rangle \varphi_k^\nu. \tag{4.1}$$

By Corollary 11, for strictly regular bc there is a Riesz basis $\Psi = \{\psi_k^\nu\}$ in $L^2([0, \pi], \mathbb{C}^2)$ consisting of root functions of the Dirac operator $L_{bc}(v)$. The point-wise convergence of the corresponding L^2-expansions

$$F = \sum_{k \in \mathbb{Z}} \sum_{\nu=1}^{2} \langle F, \tilde{\psi}_k^\nu \rangle \psi_k^\nu \tag{4.2}$$

is closely related to the point-wise convergence in (4.1) because (under some assumptions on F or v)

$$\left\| \sum_{|k| \le N} \sum_{\nu=1}^{2} \langle F, \tilde{\psi}_k^\nu \rangle \psi_k^\nu - \sum_{|k| \le N} \sum_{\nu=1}^{2} \langle F, \tilde{\varphi}_k^\nu \rangle \varphi_k^\nu \right\|_\infty \to 0 \quad \text{as } N \to \infty. \tag{4.3}$$

Further we refer to (4.3) as *equiconvergence* of spectral decompositions (4.1) and (4.2).

Let R_{NT} be the rectangle defined in (3.21) and let S_N and S_N^0 be the corresponding projections defined by (3.37). Then (4.3) can be written as

$$\left\| \left(S_N - S_N^0 \right) F \right\|_\infty \to 0 \quad \text{as } N \to \infty. \tag{4.4}$$

This form of the equiconvergence statement is suitable for regular but not strictly regular bc as well. Of course, $S_N = S_N(v, bc)$ but for the sake of simplicity the dependence on v and bc is suppressed in notations. Further, we also write R_λ instead of $R_{bc}(\lambda)$.

Since

$$R_\lambda - R_\lambda^0 = R_\lambda^0 V R_\lambda^0 + \sum_{m=2}^{\infty} R_\lambda^0 (V R_\lambda^0)^m,$$

we have

$$S_N - S_N^0 = \frac{1}{2\pi i} \int_{\partial R_{NT}} (R_\lambda - R_\lambda^0) d\lambda = A_N + B_N, \tag{4.5}$$

where

$$A_N = \frac{1}{2\pi i} \int_{\partial R_{NT}} R_\lambda^0 V R_\lambda^0 d\lambda \tag{4.6}$$

$$B_N = \frac{1}{2\pi i} \int_{\partial R_{NT}} \sum_{m=2}^{\infty} R_\lambda^0 (V R_\lambda^0)^m d\lambda. \tag{4.7}$$

Next we show that all restrictions on the class of functions for which (4.4) holds come from analysis of the operators A_N.

Proposition 12. *For every regular bc, L^2-potential v and $F \in L^2([0, \pi], \mathbb{C}^2)$,*

$$\|B_N F\|_\infty \to 0 \quad \text{as } N \to \infty \ (N \in 2\mathbb{N}). \tag{4.8}$$

Proof. To prove (4.8) it is enough to show that

(i) there is a constant $K > 0$ such that

$$\|B_N\|_{L^2 \to L^\infty} \le K \quad \forall N \ge N_0; \tag{4.9}$$

(ii) $\|B_N G\|_\infty \to 0$ as $N \to \infty$ for functions G in a dense subset of $L^2([0, \pi], \mathbb{C})$.

Then a standard argument shows that (i) and (ii) imply (4.8).

First we prove (i). The integral in (4.7) does not depend on the choice of $T > T_0$ because the integrand depends analytically on λ if $|Im\,\lambda| > T_0$. Therefore, for every $T > T_0$,

$$\|B_N\|_{L^2\to L^\infty} \le \frac{1}{2\pi}\int_{\partial R_{NT}} \left\| \sum_{m=2}^{\infty} R_\lambda^0 (V R_\lambda^0)^m \right\|_{L^2\to L^\infty} d|\lambda|. \tag{4.10}$$

In view of (3.2), to estimate the latter integral we need to find estimates from above of $a(\lambda-\tau_1)+a(\lambda-\tau_2)$ for $\lambda \in \partial R_{NT}$, where $a(\lambda)$ is the function defined in Lemma 5.

The boundary ∂R_{NT} consist of four segments $\Delta_1^-, \Delta_1^+, \Delta_2^-, \Delta_2^+$, where $\Delta_1^\pm = \{\lambda = x+it :\ -N-1+Re\frac{\tau_1+\tau_2}{2} \le x \le N+1+Re\frac{\tau_1+\tau_2}{2},\ t=\pm T\}$, $\Delta_2^\pm = \{\lambda = x+it :\ x = \pm(N+1)+Re\frac{\tau_1+\tau_2}{2},\ -T\le t\le T\}$. If $\lambda\in\Delta_1^\pm$, then $|Im\,\lambda| = T$, so for $T > T_0 \ge 1+2|\tau_1|+2|\tau_2|$

$$|Im\,(\lambda-\tau_\nu)| \ge T - |\tau_\nu| \ge T/2, \quad \nu = 1,2.$$

Then, by Lemma 5,

$$a(\lambda-\tau_1)+a(\lambda-\tau_2) \le \frac{2}{(T/2)^2}+\frac{16}{T} \le \frac{24}{T}, \quad \lambda\in\Delta_1^\pm. \tag{4.11}$$

If $\lambda\in\Delta_2^\pm$, then $\lambda-\tau_\alpha = \pm(N+1)+(-1)^\alpha Re\,\frac{\tau_1-\tau_2}{2} + i(t-Im\,\tau_\alpha), \alpha = 1,2$. By (2.15), it follows that

$$\lambda-\tau_\alpha = m_\alpha+\xi_\alpha+i(t-Im\,\tau_\alpha), \quad m_\alpha\in 2\mathbb{Z},\ \xi_\alpha\in[-1,-1/2]\cup[1/2,1],\ \alpha=1,2.$$

Therefore, by Lemma 5 we obtain

$$a(\lambda-\tau_1)+a(\lambda-\tau_2) \le h(t), \quad t = Im\,\lambda,\ \lambda\in\Delta_2^\pm, \tag{4.12}$$

where

$$h(t) := \sum_{\alpha=1}^{2}\left(\frac{1}{1/4+|t-Im\,\tau_\alpha|^2}+\frac{8}{1+2|t-Im\,\tau_\alpha|}\right). \tag{4.13}$$

The integrand of the integral in (4.10) does not exceed

$$\left\|\sum_{m=2}^{\infty} R_\lambda^0(VR_\lambda^0)^m\right\|_{L^2\to L^\infty} \le \|R_\lambda^0 V R_\lambda^0 V R_\lambda^0\|_{L^2\to L^\infty}\sum_{m=0}^{\infty}\|(VR_\lambda^0)^m\|_{L^2\to L^2}.$$

By (3.15), $\|VR_\lambda^0\| \le C\|v\|\cdot[a(\lambda-\tau_1)+a(\lambda-\tau_2)]^{1/2}$, so (4.11) and (4.12) yield

$$\|VR_\lambda^0\| \le \begin{cases} 5C\|v\|/\sqrt{T} & \text{if } \lambda\in\Delta_1^\pm,\\ C\|v\|\sqrt{h(t)} & \text{if } \lambda\in\Delta_2^\pm,\ t=Im\,\lambda.\end{cases}$$

By (4.13), $h(t)\to 0$ as $|t|\to\infty$. Therefore, there is a constant $C_1 = C_1(bc) > 0$ such that

$$\sup\{\|VR_\lambda^0\|_{L^2\to L^2} : \lambda\in\partial R_{NT}\} \le C_1\|v\|. \tag{4.14}$$

In view of Corollary 8 — see (3.34) — if $T > T_0, N > N_0$ and $\lambda\in\partial R_{NT}$ then $\|VR_\lambda^0 VR_\lambda^0\|_{L^2\to L^2}\le 1/2$, so

$$\sum_{m=0}^{\infty}\|(VR_\lambda^0)^m\|_{L^2\to L^2} \le \sum_{s=1}^{\infty}(1+C_1\|v\|)\|(VR_\lambda^0 VR_\lambda^0)^s\|_{L^2\to L^2} \le 1+C_1\|v\|. \tag{4.15}$$

On the other hand, (3.2) and (3.6) imply that

$$\|R_\lambda^0 V R_\lambda^0 V R_\lambda^0\|_{L^2\to L^\infty} \le C\|v\|^2[a(\lambda-\tau_1)+a(\lambda-\tau_2)]^{3/2}. \tag{4.16}$$

Therefore, in view of (4.11), (4.12) and (4.15), it follows that

$$\left\|\sum_{m=2}^{\infty} R_\lambda^0 (V R_\lambda^0)^m\right\|_{L^2\to L^\infty} \le \begin{cases} C_2(24/T)^{3/2} & \text{if } \lambda\in\Delta_1^\pm, \\ C_2[h(t)]^{3/2} & \text{if } \lambda\in\Delta_2^\pm,\ t=Im\lambda \end{cases}$$

with $C_2 = C(1+C_1\|v\|)\|v\|^2$. Now, choosing $T = N$ and taking into account that $\int_{-\infty}^{\infty}[h(t)]^{3/2}dt < \infty$ (see (4.13)), we obtain that the integrals in (4.10) are uniformly bounded. This completes the proof of (i).

Next we prove (ii). Fix a regular bc, and let $\Phi = \{\varphi_k^\nu\}$ be the corresponding Riesz basis of eigenfunctions of the free operator L_{bc}^0 given in Theorem 3. Since the linear combinations of φ_k^ν are dense in $L^2([0,\pi],\mathbb{C}^2)$, it is enough to prove (ii) for $G = \varphi_k^\nu$.

Fix $k \in 2\mathbb{Z}$ and $\nu\in\{1,2\}$. By (4.7), we have for every $T > T_0$

$$\|B_N\varphi_k^\nu\|_\infty \le \frac{1}{2\pi}\int_{\partial R_{NT}}\left\|\sum_{m=2}^{\infty} R_\lambda^0 (V R_\lambda^0)^m \varphi_k^\nu\right\|_\infty d|\lambda|. \tag{4.17}$$

The integrand does not exceed

$$\left\|\sum_{m=2}^{\infty} R_\lambda^0 (V R_\lambda^0)^m \varphi_k^\nu\right\|_\infty \le \|R_\lambda^0 V R_\lambda^0\|_{L^2\to L^\infty}\sum_{m=0}^{\infty}\|(V R_\lambda^0)^m\|_{L^2\to L^2}\|V R_\lambda^0\varphi_k^\nu\|.$$

By (3.1), (3.6) and (2.23),

$$\|V R_\lambda^0 \varphi_k^\nu\| \le \|v\|\cdot\|R_\lambda^0\varphi_k^\nu\|_\infty \le \frac{C\|v\|}{|\lambda-k-\tau_\nu|}.$$

If $\lambda\in\Delta_1^\pm$, then $|Im\,\lambda| = T$ so for large enough T it follows $|\lambda-k-\tau_\nu|\ge T/2$. If $\lambda\in\Delta_2^\pm$, then $\lambda = \pm(N+1)+Re\frac{\tau_1+\tau_2}{2}+it$, so for large enough N we obtain

$$|\lambda-k-\tau_\nu| \ge \frac{1}{\sqrt{2}}\left(N+1-|k|-\left|Re\frac{\tau_1-\tau_2}{2}\right|\right)+\frac{1}{\sqrt{2}}|t-Im\,\tau_\nu| \ge \frac{N+|t|}{2}.$$

Therefore, for large enough N,

$$\|V R_\lambda^0\varphi_k^\nu\| \le \begin{cases} \dfrac{2C\|v\|}{T} & \text{for } \lambda\in\Delta_1^\pm, \\ \dfrac{2C\|v\|}{N+|t|} & \text{for } \lambda\in\Delta_2^\pm. \end{cases}$$

On the other hand, (3.2) and (3.6) imply that

$$\|R_\lambda^0 V R_\lambda^0\|_{L^2\to L^\infty} \le C\|v\|[a(\lambda-\tau_1)+a(\lambda-\tau_2)].$$

Therefore, by (4.11), (4.12) and (4.15) it follows that

$$\left\|\sum_{m=2}^{\infty} R_\lambda^0 (V R_\lambda^0)^m\varphi_k^\nu\right\|_\infty \le \begin{cases} C_3 T^{-2} & \text{if } \lambda\in\Delta_1^\pm, \\ C_3\dfrac{h(t)}{N+|t|} & \text{if } \lambda\in\Delta_2^\pm,\ Im\,\lambda = t, \end{cases}$$

where $C_3 = C_3(\|v\|, bc)$. The integral in (4.17) is sum of integrals on $\Delta_1^{\pm}$ and $\Delta_2^{\pm}$. In view of the latter estimates, if we choose $T = N$, then the integrals over $\Delta_1^{\pm}$ go to zero as $N \to \infty$. On the other hand, by (4.13) $h(t) \asymp \frac{1}{1+|t|}$, so the integrals over $\Delta_2^{\pm}$ do not exceed a multiple of

$$\int_{-\infty}^{\infty} \frac{1}{(1+|t|)(N+|t|)} dt = 2\frac{\log N}{N-1} \to 0 \quad \text{as } N \to \infty.$$

This completes the proof. □

Corollary 13. *In the above notations, for every regular bc, L^2-potential v and $F \in L^2([0,\pi], \mathbb{C}^2)$*

$$\lim_N \left\| \left(S_N - S_N^0\right) F \right\|_\infty = 0 \Longleftrightarrow \lim_N \|A_N F\|_\infty = 0. \tag{4.18}$$

Next we give conditions on v or F that guarantee the existence of the right-hand limit in (4.18). Let us fix a regular bc, and let $\Phi = \{\varphi_k^{\nu}\}$ be the corresponding Riesz basis of eigenfunctions of the free operator L_{bc}^0 given in Theorem 3.

Lemma 14. *Under the above assumptions, for every $\varphi_k^{\nu} \in \Phi$*

$$\|A_N \varphi_k^{\nu}\|_\infty \to 0 \quad \text{as } N \to \infty. \tag{4.19}$$

Proof. Fix φ_k^{ν} and consider $N > |k|$. By (3.1) and Lemma 6,

$$(A_N \varphi_k^{\nu})(x) = \frac{1}{2\pi i} \int_{\partial R_{NT}} \sum_{\eta=1}^{2} \sum_{j \in 2\mathbb{Z}} \frac{w^{\eta\nu}(j+k)}{(\lambda - j - \tau_\eta)(\lambda - k - \tau_\nu)} \varphi_j^{\eta}(x) d\lambda,$$

so by the Residue Theorem

$$(A_N \varphi_k^{\nu})(x) = \sum_{\eta=1}^{2} \sum_{|j|>N} \frac{w^{\eta\nu}(j+k)}{(k+\tau_\nu) - (j+\tau_\eta)} \varphi_j^{\eta}(x).$$

If $j, k \in 2\mathbb{Z}$, $j \neq k$, then

$$|k - j + \tau_\nu - \tau_\eta| \geq |k-j| - |Re(\tau_\nu - \tau_\eta)| \geq |k-j| - 1 \geq \frac{|k-j|}{2}. \tag{4.20}$$

Therefore, in view of (2.23) and (3.12), it follows that

$$\|A_N \varphi_k^{\nu}\|_\infty \leq 4C \sum_{|j|>N} \frac{r(j+k)}{|k-j|}.$$

Thus, the Cauchy inequality implies

$$\|A_N \varphi_k^{\nu}\|_\infty \leq 4C\|r\| \left(\sum_{|j|>N} \frac{1}{|k-j|^2} \right)^{1/2} \leq \frac{4C\|r\|}{(N-|k|)^{1/2}} \to 0$$

as $N \to \infty$, which completes the proof. □

Fix an L^2-function $F : [0, \pi] \to \mathbb{C}^2$ and consider

$$(A_N F)(x) = \frac{1}{2\pi i} \int_{\partial R_{NT}} R^0_\lambda V R^0_\lambda F d\lambda, \tag{4.21}$$

Let $F = \sum_{\nu=1}^2 \sum_{k\in 2\mathbb{Z}} F_k^\nu \varphi_k^\nu$ be the expansion of F about the basis $\{\varphi_k^\nu\}$. By the matrix representation of the operators V and R^0_{bc} it follows that

$$(A_N F)(x) = \frac{1}{2\pi i} \int_{\partial R_{NT}} \sum_{\nu,\eta=1}^{2} \sum_{k\in 2\mathbb{Z}} \sum_{j\in 2\mathbb{Z}} \frac{w^{\eta\nu}(j+k) F_k^\nu}{(\lambda - j - \tau_\eta)(\lambda - k - \tau_\nu)} \varphi_j^\eta(x) d\lambda. \tag{4.22}$$

The Residue Theorem implies

$$\begin{aligned}(A_N F)(x) = &\sum_{\nu,\eta=1}^{2} \sum_{|k|\le N} \sum_{|j|>N} \frac{w^{\eta\nu}(j+k) F_k^\nu}{(k+\tau_\nu) - (j+\tau_\eta)} \varphi_j^\eta(x) \\ &+ \sum_{\nu,\eta=1}^{2} \sum_{|k|>N} \sum_{|j|\le N} \frac{w^{\eta\nu}(j+k) F_k^\nu}{(j+\tau_\eta) - (k+\tau_\nu)} \varphi_j^\eta(x).\end{aligned}$$

We set

$$g_m = \max\left\{ \left|F_m^1\right|, \left|F_m^2\right| \right\}; \tag{4.23}$$

then $g = (g_m) \in \ell^2(2\mathbb{Z})$ and $\|g\| \le const \cdot \|F\|$. Therefore, in view of (2.23), (3.12), (4.20) and (4.23), it follows that

$$\|A_N F\|_\infty \le 8C(\sigma_1(N) + \sigma_2(N)), \tag{4.24}$$

where

$$\sigma_1(N) = \sum_{|k|\le N} \sum_{|j|>N} \left| \frac{r(j+k) g_k}{j-k} \right|, \qquad \sigma_2(N) = \sum_{|k|>N} \sum_{|j|\le N} \left| \frac{r(j+k) g_k}{k-j} \right|, \tag{4.25}$$

and j, k, N are even numbers.

Lemma 15. *In the above notations, the following holds:*

(a) *If* $g = (g_k) \in \ell^p(2\mathbb{Z}),\ p \in (1, 2)$ *and* $r = (r(k)) \in \ell^2(2\mathbb{Z})$, *then*

$$\sigma_\mu(N) \le C(p)\|r\|\|g\|_p, \quad \mu = 1, 2. \tag{4.26}$$

(b) *If* $g = (g_k) \in \ell^2(2\mathbb{Z})$ *and* $r = (r(k)) \in \ell^p(2\mathbb{Z}),\ p \in (1, 2)$, *then*

$$\sigma_\mu(N) \le C(p)\|r\|_p\|g\|, \quad \mu = 1, 2. \tag{4.27}$$

(c) *If* $\exists \delta > 1 :\ |g|_\delta^2 := \sum_k |g_k|^2[\log(|k| + e)]^\delta < \infty$ *and* $r \in \ell^2(2\mathbb{Z})$, *then*

$$\sigma_\mu(N) \le C(\delta)\|r\||g|_\delta, \quad \mu = 1, 2. \tag{4.28}$$

(d) *If* $\exists \delta > 1 :\ |r|_\delta^2 := \sum_k |r(k)|^2[\log(|k| + e)]^\delta < \infty$ *and* $g \in \ell^2(2\mathbb{Z})$, *then*

$$\sigma_\mu(N) \le C(\delta)|r|_\delta\|g\|, \quad \mu = 1, 2. \tag{4.29}$$

Proof. Throughout the proof $j, k \in 2\mathbb{Z}$ and $N \in 2\mathbb{N}$. We will use the following inequalities: if $|k| \le N$ then for $s \ge 2$

$$\sum_{|j|>N} \frac{1}{|j-k|^s} \le \int_{N+1-|k|}^{\infty} \frac{1}{x^s} dx \le \frac{1}{(N+1-|k|)^{s-1}}, \tag{4.30}$$

and if $|k| > N$ then for $s \ge 2$

$$\sum_{|j|\le N} \frac{1}{|k-j|^s} \le \int_{|k|-N-1}^{\infty} \frac{1}{x^s} dx \le \frac{1}{(|k|-N-1)^{s-1}}. \tag{4.31}$$

Suppose $|k| \le N$. In case (a), the Cauchy inequality and (4.30) with $s = 2$ imply

$$\sum_{|j|>N} \left| \frac{r(j+k)}{j-k} \right| \le \|r\| \left(\sum_{|j|>N} \frac{1}{(j-k)^2} \right)^{1/2} \le \|r\| \frac{1}{(N+1-|k|)^{1/2}}.$$

Therefore, by the Hölder inequality (with $q = p/(p-1) > 2$), it follows

$$\begin{aligned} \sigma_1(N) &\le \|r\| \sum_{|k|\le N} \frac{|g_k|}{(N+1-|k|)^{1/2}} \\ &\le \|r\| \left(\sum_{|k|\le N} |g_k|^p \right)^{1/p} \left(\sum_{|k|\le N} (N+1-|k|)^{-q/2} \right)^{1/q} \le C(p)\|r\|\|g\|_p, \end{aligned}$$

where $C(p) = \left(\sum_{m=1}^{\infty} 2m^{-q/2} \right)^{1/q}, \ q = (p-1)/p.$

In case (b), the Hölder inequality (with $q > 2$) and (4.30) with $s = q$ imply

$$\sigma_1(N) \le \sum_{|k|\le N} |g_k| \, \|r\|_p \left(\sum_{|j|>N} \frac{1}{|j-k|^q} \right)^{1/q} \le \sum_{|k|\le N} \frac{|g_k| \, \|r\|_p}{(N+1-|k|)^{1-1/q}}.$$

By the Cauchy inequality, we obtain

$$\sigma_1(N) \le \|r\|_p \|g\| \left(\sum_{m=1}^{\infty} \frac{2}{m^{2-2/q}} \right)^{1/2} \le C(p)\|r\|_p\|g\|$$

with $C(p) = \left(\sum_{m=1}^{\infty} \frac{2}{m^{2/p}} \right)^{1/2}$.

In case (c), the Cauchy inequality and (4.30) with $s = 2$ imply

$$\sigma_1(N) \le \|r\| \sum_{|k|\le N} \frac{|g_k|}{(N+1-|k|)^{1/2}}.$$

If $|k| \le N/2$, then $N + 1 - |k| \ge N + 1 - N/2 = (N+2)/2$, so applying again the Cauchy inequality we obtain

$$\sum_{|k|\le N/2} \frac{|g_k|}{(N+1-|k|)^{1/2}} \le \|g\| \left(\sum_{|k|\le N/2} \frac{1}{N+1-|k|} \right)^{1/2} \le 2\|g\|.$$

On the other hand, if $|k| > N/2$ then $|k| \geq N+2-|k|$, so

$$\sum_{N/2<|k|\leq N} \frac{|g_k|}{(N+1-|k|)^{1/2}} \leq \sum_{N/2<|k|\leq N} \frac{|g_k|[\log(e+|k|)]^{\delta/2}}{(N+1-|k|)^{1/2}[\log(N+2-|k|)]^{\delta/2}}$$

$$\leq |g|_\delta \left(\sum_{N/2<|k|\leq N} \frac{1}{(N+1-|k|)[\log(N+2-|k|)]^\delta} \right)^{1/2}$$

$$\leq C_1(\delta)|g|_\delta$$

with $C_1(\delta) = \left(\sum_{m=1}^\infty \frac{2}{m(\log(m+1))^\delta}\right)^{1/2}$. Since $\|g\| \leq |g|_\delta$, it follows that

$$\sigma_1(N) \leq C(\delta)\|r\||g|_\delta \quad \text{with } C(\delta) = 2 + C_1(\delta).$$

In case (d), the Cauchy inequality implies

$$\sum_{|j|>N} \left| \frac{r(j+k)}{j-k} \right| \leq \mathcal{E}_{N+2-|k|}(r) \left(\sum_{|j|>N} \frac{1}{(j-k)^2} \right)^{1/2}, \tag{4.32}$$

where $\mathcal{E}_m(r) = \left(\sum_{|i|\geq m} |r(i)|^2\right)^{1/2}$. Since

$$(\mathcal{E}_m(r))^2 \leq \frac{1}{(\log m)^\delta} \sum_{|k|\geq m} |r(k)|^2[\log(e+|k|)]^\delta \leq \frac{|r|_\delta^2}{(\log m)^\delta},$$

in view of (4.32) and (4.30) with $s = 2$ it follows

$$\sigma_1(N) \leq |r|_\delta \sum_{|k|\leq N} \frac{|g_k|}{(N+1-|k|)^{1/2}(\log(N+2-|k|))^{\delta/2}}$$

$$\leq |r|_\delta \|g\| \cdot \left(\sum_{m=1}^\infty \frac{2}{m(\log(m+1))^\delta} \right)^{1/2} \leq C(\delta)|r|_\delta \|g\|$$

with $C(\delta) = \left(\sum_{m=1}^\infty \frac{2}{m(\log(m+1))^\delta}\right)^{1/2}$.

This completes the proof of (4.26)–(4.29) for $\sigma_1(N)$. The proof in the case of sums $\sigma_2(N)$ is essentially the same — but then $|k| \geq N+2$ and $|j| \leq N$, so one has to use (4.31) instead of (4.30) and replace in all formulas $N+1-|k|$ by $|k|-N-1$. For example, in case (c), the Cauchy inequality and (4.31) with $s = 2$ imply

$$\sigma_2(N) \leq \|r\| \sum_{|k|>N} \frac{|g_k|}{(|k|-N-1)^{1/2}}.$$

Therefore, again by the Cauchy inequality, it follows

$$\sigma_2(N) \leq \|r\| \sum_{|k|>N} \frac{|g_k|[\log(e+|k|)]^{\delta/2}}{(|k|-N-1)^{1/2}[\log(|k|-N)]^{\delta/2}}$$

$$\leq \|r\|\,|g|_\delta \left(\sum_{|k|>N} \frac{1}{(|k|-N-1)[\log(|k|-N)]^\delta} \right)^{1/2} \leq C_1(\delta)\|r\|\,|g|_\delta$$

with $C_1(\delta) = \left(\sum_{m=1}^\infty \frac{2}{m(\log(m+1))^\delta}\right)^{1/2}$. We omit the details about cases (a), (b) and (d). $\square$

Theorem 16. *Given a regular bc, L^2-potential v and $F \in L^2([0,\pi], \mathbb{C}^2)$, let $g = (g_k)_{k\in 2\mathbb{Z}}$ be defined by (4.23) and let $r = (r(k))_{k\in 2\mathbb{Z}}$ be defined by Lemma 6 and (3.12). Then*

$$\|A_N F\|_\infty \to 0 \quad \text{as } N \to \infty \tag{4.33}$$

whenever one of the following conditions holds:

(a) $\exists p \in (1,2)$ *such that* $(g_k) \in \ell^p(2\mathbb{Z})$;
(b) $\exists p \in (1,2)$ *such that* $(r(k)) \in \ell^p(2\mathbb{Z})$;
(c) $\exists \delta > 1$ *such that* $\sum_k |g_k|^2 [\log(|k|+e)]^\delta < \infty$;
(d) $\exists \delta > 1$ *such that* $\sum_k |r(k)|^2 [\log(|k|+e)]^\delta < \infty$.

Moreover, each of the conditions (a)–(d) guarantees that

$$\|(S_N - S_N^0)F\|_\infty \to 0 \quad \text{as } N \to \infty. \tag{4.34}$$

Proof. Suppose $F \in L^2([0,\pi], \mathbb{C}^2)$ and $F = \sum_{k\in 2\mathbb{Z}} \sum_{\nu=1}^2 F_k^\nu \varphi_k^\nu$ is the expansion of F about the basis Φ. Let (a) holds (with $g = (g_k)$ defined by $g_k = \max(|F_k^1|, |F_k^2|)$, i.e., by (4.23)).

Fix $\varepsilon > 0$ and choose an $N_1 \in 2\mathbb{Z}$ such that $\sum_{|k|>N_1} |g_k|^p < \varepsilon^p$. Set

$$\tilde{F} = \sum_{|k|>N_1} \sum_{\nu=1}^{2} F_k^\nu \varphi_k^\nu, \quad \tilde{g} = (\tilde{g}_k),\ \tilde{g}_k = 0 \text{ if } |k| \le N_1,\ \tilde{g}_k = g_k \text{ if } |k| > N_1.$$

By (4.24) and Lemma 15(a), it follows that

$$\|A_N \tilde{F}\|_\infty \le 16CC(p)\|r\| \|\tilde{g}\|_p \le 16CC(p)\|r\| \cdot \varepsilon.$$

On the other hand, since $F - \tilde{F}$ is a finite linear combination of basis functions φ_k^ν, Lemma 14 implies that

$$\|A_N(F - \tilde{F})\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Therefore, $\limsup \|A_N F\|_\infty \le 16CC(p)\|r\| \cdot \varepsilon$ for every $\varepsilon > 0$, so (4.33) holds.

The proof is similar in the cases (b)–(d) — we use Lemma 14 and, respectively, parts (b)–(d) of Lemma 15.

Of course, in view of Proposition 12, each of the conditions (a)–(d) implies that $\lim_N \|(S_N - S_N^0)F\|_\infty = 0$. $\square$

For a given regular *bc*, Theorem 16 gives sufficient conditions for equiconvergence in terms of matrix representation of the potential $v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}$ and coefficients of the expansion of $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ about the basis Φ (which consists of root functions of L^0_{bc}).

In particular, for periodic (Per^+) or antiperiodic (Per^-) boundary conditions,

$$Per^\pm : y_1(\pi) = \pm y_1(0), \qquad y_2(\pi) = \pm y_2(0),$$

we may consider the following bases of eigenfunctions:

$$\Phi_{Per^+} = \left\{ \varphi_k^1 = \begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix},\ \varphi_k^2 = \begin{pmatrix} 0 \\ e^{ikx} \end{pmatrix},\ k \in 2\mathbb{Z} \right\},$$

$$\Phi_{Per^-} = \left\{ \varphi_k^1 = \begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix},\ \varphi_k^2 = \begin{pmatrix} 0 \\ e^{ikx} \end{pmatrix},\ k \in 1 + 2\mathbb{Z} \right\}.$$

Now the matrix representation of the operator of multiplication by v is

$$V \sim \begin{bmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{bmatrix}, \qquad V^{11}_{jk} = V^{22}_{jk} = 0, \qquad V^{12}_{jk} = p(-j-k), \qquad V^{21}_{jk} = q(j+k),$$

where $p(m)$ and $q(m)$, $m \in 2\mathbb{Z}$ are, respectively, the Fourier coefficients of the functions P and Q about the system $\{e^{imx}, m \in 2\mathbb{Z}\}$, and the corresponding sequence $(r(m))$ (compare with Lemma 6 and (3.12)) is

$$r(m) = \max(|p(-m)|, |q(m)|), \quad m \in 2\mathbb{Z}.$$

Therefore, the following holds (compare with [30, Proposition 7.3]).

Corollary 17. *Let* $v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}$. *Suppose there is* $p \in (1,2)$ *such that*

$$\sum_{m \in 2\mathbb{Z}} (|p(m)|^p + |q(m)|^p) < \infty,$$

or there is $\delta > 1$ *such that*

$$\sum_{m \in 2\mathbb{Z}} (|p(m)|^2 + |q(m)|^2)[\log(e + |m|)]^\delta < \infty,$$

where $p(m)$ *and* $q(m)$ *are, respectively, the Fourier coefficients of the functions* P *and* Q *about the system* $\{e^{imx}, m \in 2\mathbb{Z}\}$. *Then we have, for periodic* Per^+ *or antiperiodic* Per^- *boundary conditions,*

$$\|(S_N - S_N^0)F\|_\infty \to 0 \quad \text{as } N \to \infty \; \forall F \in L^2([0,\pi], \mathbb{C}^2).$$

However, one needs to impose on F conditions depending on bc to guarantee equiconvergence for every L^2-potential v. For example, if $bc = Per^\pm$, Theorem 16 implies the following.

Corollary 18. *Suppose* v *is an* L^2*-potential matrix. Then*

$$\|(S_N - S_N^0)F\|_\infty \to 0 \quad \text{as } N \to \infty$$

providing one of the following holds:

(i) $bc = Per^+$, *and there is* $p \in (1,2)$ *such that*

$$\sum_{m \in 2\mathbb{Z}} (|F_{1,m}|^p + |F_{1,m}|^p) < \infty,$$

or there is $\delta > 1$ *such that*

$$\sum_{m \in 2\mathbb{Z}} (|F_{1,m}|^2 + |F_{1,m}|^2)[\log(e + |m|)]^\delta < \infty,$$

where $F_{1,m}$ *and* $F_{2,m}$ *are, respectively, the Fourier coefficients of the functions* F_1 *and* F_2 *about the system* $\{e^{imx}, m \in 2\mathbb{Z}\}$;

(ii) $bc = Per^-$, *and there is* $p \in (1,2)$ *such that*

$$\sum_{m \in 1+2\mathbb{Z}} (|F_{1,m}|^p + |F_{1,m}|^p) < \infty,$$

or there is $\delta > 1$ *such that*

$$\sum_{m \in 1+2\mathbb{Z}} (|F_{1,m}|^2 + |F_{1,m}|^2)[\log(e + |m|)]^\delta < \infty,$$

where $F_{1,m}$ and $F_{2,m}$ are, respectively, the Fourier coefficients of the functions F_1 and F_2 about the system $\{e^{imx}, \ m \in 1 + 2\mathbb{Z}\}$.

Next we discuss what conditions guarantee equiconvergence property simultaneously for all regular bc.

Recall that if $\Omega = (\Omega(k))_{k \in 2\mathbb{Z}}$ is a sequence of positive numbers (weight sequence), one may consider the weighted sequence space

$$\ell^2(\Omega, 2\mathbb{Z}) = \left\{ x = (x_k) : \sum_{k \in 2\mathbb{Z}} (|x_k| \Omega(k))^2 < \infty \right\}$$

and the corresponding Sobolev space

$$H(\Omega) = \left\{ f = \sum_{k \in 2\mathbb{Z}} f_k e^{ikx} : (f_k) \in \ell^2(\Omega) \right\}. \tag{4.35}$$

In particular, consider the Sobolev weights

$$\Omega_\alpha(k) = (1 + k^2)^{\alpha/2}, \quad k \in 2\mathbb{Z}, \tag{4.36}$$

and the logarithmic weights

$$\omega_\beta(k) = (\log(e + |k|))^\beta, \quad k \in 2\mathbb{Z}, \ \beta \in \mathbb{R}. \tag{4.37}$$

Let H^α and h^β denote the corresponding Sobolev spaces (4.35). Of course, $H^\alpha \subset h^\beta$ if $\alpha > 0$ and $h^\beta \subset H^\alpha$ if $\alpha < 0$ for any β.

Lemma 19. *Let $g \in C^1([0, \pi])$.*

(a) *If $f \in H^\alpha, \ -1/2 < \alpha < 1/2$, then $f \cdot g \in H^\alpha$.*
(b) *If $f \in h^\beta, \ -\infty < \beta < \infty$, then $f \cdot g \in h^\beta$.*

Proof is given in Appendix. These statements are straightforward corollaries or partial cases of the discrete Muckenhoupt theorem [17, Theorem 10]. Prior to 1972/73 the case (b), i.e., boundedness of discrete Hilbert transform in the spaces h^β had been proven by Carleson [5, (10.1)]. The recent book [26] gives many further results on multipliers in Sobolev spaces of functions of one or several variables.

Theorem 20. *For regular bc, Dirac potentials $v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}$ with $P, Q \in L^2([0, \pi])$ and $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ with $F_1, F_2 \in L^2([0, \pi])$,*

$$\left\| \left(S_N - S_N^0 \right) F \right\|_\infty \to 0 \quad \text{as } N \to \infty \tag{4.38}$$

whenever one of the following conditions is satisfied:

(a) $\exists \beta > 1$ *such that*

$$\sum_{k \in 2\mathbb{Z}} (|F_{1,k}|^2 + |F_{2,k}|^2)(\log(e + |k|))^\beta < \infty,$$

where $(F_{1,k})_{k \in 2\mathbb{Z}}$ and $(F_{2,k})_{k \in 2\mathbb{Z}}$ are, respectively, the Fourier coefficients of F_1 and F_2 about the system $\{e^{ikx}, \ k \in 2\mathbb{Z}\}$;

(b) $\exists \beta > 1$ *such that*

$$\sum_{k \in 2\mathbb{Z}} (|p(k)|^2 + |q(k)|^2)(\log(e + |k|))^\beta < \infty,$$

where $(p(k))_{k \in 2\mathbb{Z}}$ *and* $(q(k))_{k \in 2\mathbb{Z}}$ *are, respectively, the Fourier coefficients of* P *and* Q *about the system* $\{e^{ikx}, \ k \in 2\mathbb{Z}\}$.

In particular, if F_1, F_2 *are functions of bounded variation or* P, Q *are functions of bounded variation, then* (4.38) *holds.*

Proof. Suppose a regular bc is fixed. Let $\Phi = (\varphi_k^\nu)$ and $\tilde{\Phi} = (\tilde{\varphi}_k^\nu)$ be the corresponding Riesz basis of eigenvectors of L_{bc}^0 and its biorthogonal system constructed in Theorem 10.

Suppose (a) holds for a function F. In view of the explicit formulas (2.24) and (2.25) for the biorthogonal system $\tilde{\Phi}$ it follows that the expansion coefficients $F_k^\nu = \langle F, \tilde{\varphi}_k^\nu \rangle$ can be represented as

$$F_k^\nu = f_{1,-k}^\nu + f_{2,k}^\nu, \quad k \in 2\mathbb{Z}, \ \nu = 1, 2,$$

where $f_{1,k}^\nu$ and $f_{2,k}^\nu$ are the Fourier coefficients of functions of the form

$$f_1^\nu(x) = h_1^\nu(x) F_1(x) + h_2^\nu(x) F_2(x), \quad \nu = 1, 2,$$

with $h_1^\nu(x), h_2^\nu(x) \in C^\infty([0, \pi])$. Now Lemma 19(b) implies that

$$\sum_{k \in 2\mathbb{Z}} (|F_k^1|^2 + |F_k^2|^2)(\log(e + |k|))^\beta < \infty,$$

but then Condition (c) of Theorem 16 holds, hence (4.38) follows.

Suppose (b) holds with some $\beta > 1$. In view of Lemma 6 and its proof, the sequences $(w^{\eta\nu}(m))_{m \in 2\mathbb{Z}}$, which generate the matrix representation of the operator V (see (3.8) and (3.9)) are given by (3.10) in terms of the Fourier coefficients of some products of P and Q by C^∞-functions (see (3.11)). Therefore, by Lemma 19 we have $(w^{\eta\nu}(m)) \in \ell^2(\omega_\beta)$. In view of (3.12), this implies that Condition (d) in Theorem 16 holds, hence (4.38) follows.

It is well-known (see [41, Chapter 2, Section 4, Theorem 4.12]), that if $f : [0, \pi]$ is a function of bounded variation then its Fourier coefficients $f_k = \frac{1}{\pi} \int_0^\pi f(x) e^{-ikx} dx$ satisfy $|f_k| \leq C/|k|, \ k \neq 0$, where $C = C(f)$. Therefore, if F_1, F_2 are functions of bounded variation then (a) holds, and if P, Q are functions of bounded variation then (b) holds, so in both cases (4.38) follows, which completes the proof.

Of course, one can handle the case of functions of bounded variation directly, without using Lemma 19. Indeed, the matrix representation coefficients of V and the expansion coefficients of F about the basis $\Phi = \{\varphi_k^\nu\}$ are coming from the Fourier coefficients of products of P, Q and F_1, F_2 with some smooth functions. Since a product of a function of bounded variation with a smooth function is also a function of bounded variation, the corresponding sequences $(r(m))$ and (g_k) are dominated by const$/|k|$, so they are in the space $\ell^2(\omega_\beta)$ for $\beta > 1$. Thus, respectively, (c) or (d) in Theorem 16 holds, so the claim follows. □

5. Point-wise convergence of spectral decompositions

It is well-known that if f is a function of bounded variation on $[0, \pi]$ then its Fourier series with respect to the system $\{e^{imx}, \ m \in 2\mathbb{Z}\}$ converges point-wise to $\frac{1}{2}[f(x-0) + f(x+0)]$ for $x \in (0, \pi)$, and to $\frac{1}{2}[f(\pi - 0) + f(0 + 0)]$ for $x = 0, \pi$. More precisely, the following holds.

Lemma 21 (*See [41, Vol 1, Theorem 8.14]*). *If $f : [0,\pi] \to \mathbb{C}$ is a function of bounded variation, then*

$$\lim_{M\to\infty} \sum_{m=-M}^{M} \langle f(x), e^{imx}\rangle e^{imx} = \begin{cases} \frac{1}{2}[f(x-0)+f(x+0)] & \text{if } x \in (0,\pi), \\ \frac{1}{2}[f(\pi-0)+f(0+0)] & \text{if } x = 0,\pi. \end{cases} \tag{5.1}$$

Moreover, the convergence is uniform on every closed subinterval of $(0,\pi)$ on which f is continuous, and the convergence is uniform on the closed interval $[0,\pi]$ if and only if f is continuous and $f(0) = f(\pi)$.

For systems of o.d.e., an interesting point-wise convergence statement has been proven in [4, pp. 127–128], but under rather restrictive assumptions. For example, the matrices W_a and W_b (see [4], lines 5–6 on p. 64 and Formula (7) there, or Formula (46) on p. 87) are assumed to be invertible but this never happens in the case of separated bc like (7.3) in Section 7 below. Another strong assumption is that the spectrum $Sp(L_{bc})$ is eventually simple (see [4, lines 9–11 on p. 98]). In general, this assumption is not satisfied for regular but not strictly regular bc.

We do not impose any assumption on the boundary conditions but regularity. The main result of this section is the following.

Theorem 22. *Let bc be a regular boundary condition given by (2.10), and let $\Phi = \{\varphi_k^\nu,\ k \in 2\mathbb{Z},\ \nu = 1,2\}$ and $\tilde{\Phi} = \{\tilde{\varphi}_k^\nu,\ k \in 2\mathbb{Z},\ \nu = 1,2\}$ be the corresponding Riesz basis of root functions of L^0_{bc} and its biorthogonal system which are constructed in Theorem 3. If $f, g : [0,\pi] \to \mathbb{C}$ are functions of bounded variation which are continuous at 0 and π, then*

$$\lim_{M\to\infty} \sum_{m=-M}^{M} \sum_{\nu=1}^{2} \left\langle \begin{pmatrix} f \\ g \end{pmatrix},\ \tilde{\varphi}_m^\nu \right\rangle \varphi_m^\nu(x) = \begin{pmatrix} \tilde{f}(x) \\ \tilde{g}(x) \end{pmatrix}, \tag{5.2}$$

where

$$\begin{pmatrix} \tilde{f}(x) \\ \tilde{g}(x) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} f(x-0)+f(x+0) \\ g(x-0)+g(x+0) \end{pmatrix} \quad \text{for } x \in (0,\pi) \tag{5.3}$$

and

$$\begin{pmatrix} \tilde{f}(x) \\ \tilde{g}(x) \end{pmatrix} = \begin{cases} \frac{1}{2} \begin{pmatrix} f(0) - bf(\pi) - ag(0) \\ \frac{d}{bc-ad} f(0) + g(0) - \frac{b}{bc-ad} g(\pi) \end{pmatrix} & \text{if } x = 0, \\ \frac{1}{2} \begin{pmatrix} -\frac{c}{bc-ad} f(0) + f(\pi) + \frac{a}{bc-ad} g(\pi) \\ -df(\pi) - cg(0) + g(\pi) \end{pmatrix} & \text{if } x = \pi. \end{cases} \tag{5.4}$$

Moreover, if both $f(\pi-t)$ and $g(t)$ are continuous on some closed subinterval of $(0,\pi)$, then the convergence (5.2) is uniform on that interval. The convergence is uniform on the closed interval $[0,\pi]$ if and only if f and g are continuous on $[0,\pi]$ and $\binom{f}{g}$ satisfies the boundary condition bc given by (2.10).

Proof. Let A and A^{-1} be the operators defined in the proof of Theorem 3 by (2.27) and (2.28) in case bc is strictly regular or periodic type, and by (2.29) and (2.30) in case bc is regular but not strictly regular or periodic type. The operators A and A^{-1} act on a vector-function $\binom{f(t)}{g(t)}$ by multiplying $f(t),\ g(t),\ f(\pi-t),\ g(\pi-t)$ by some C^∞-functions, so A and A^{-1} are defined point-wise. Recall from the proof of Theorem 3 that

$$\varphi_k^\nu = Ae_k^\nu, \qquad \tilde{\varphi}_k^\nu = (A^{-1})^* e_k^\nu, \quad \text{where } e_k^1 = \begin{pmatrix} e^{ikt} \\ 0 \end{pmatrix},\ e_k^2 = \begin{pmatrix} 0 \\ e^{ikt} \end{pmatrix}. \tag{5.5}$$

Suppose f and g are functions of bounded variation on $[0,\pi]$, and let

$$\begin{pmatrix} F \\ G \end{pmatrix}(t) := A^{-1}\begin{pmatrix} f \\ g \end{pmatrix}(t); \tag{5.6}$$

then F and G are functions of bounded variation on $[0,\pi]$ also (as products of functions of bounded variations by C^∞-functions).

In view of (2.28) and (2.30), the functions $F(t)$ and $G(t)$ are continuous at t if and only if $f(\pi-t)$ and $g(t)$ are continuous at t. Therefore,

(i) if $f(\pi-t)$ and $g(t)$ are continuous on some closed interval $I \subset [0,\pi]$ then F and G are continuous on I as well.
Moreover, by Lemma 4,

(ii) if f and g are continuous on $[0,\pi]$ and $\binom{f}{g}$ satisfies the boundary conditions (2.10), then F and G are continuous on $[0,\pi]$ and $\binom{F}{G}$ satisfies the periodic boundary conditions $F(0)=F(\pi),\ G(0)=G(\pi)$.

By (5.5),

$$\left\langle \begin{pmatrix} f \\ g \end{pmatrix},\ \tilde{\varphi}_m^1 \right\rangle = \left\langle A^{-1}\begin{pmatrix} f \\ g \end{pmatrix},\ e_m^1 \right\rangle = \left\langle \begin{pmatrix} F \\ G \end{pmatrix},\ \begin{pmatrix} e^{imt} \\ 0 \end{pmatrix} \right\rangle = \langle F, e^{imt}\rangle,$$

and similarly, $\left\langle \binom{f}{g}, \tilde{\varphi}_m^2 \right\rangle = \langle G, e^{imt}\rangle$.

By Lemma 21, the Fourier series of F and G with respect to the system $\{e^{imt},\ m \in 2\mathbb{Z}\}$ converge point-wise, and the convergence is uniform in the cases (i) and (ii) mentioned above. Let $\tilde{F}(x)$ and $\tilde{G}(x)$ denote, respectively, the point-wise sums of those series at $x \in [0,\pi]$. Fix a point $x \in [0,\pi]$; then

$$\begin{aligned}\sum_{m=-M}^{M}\sum_{\nu=1}^{2}\left\langle \begin{pmatrix} f \\ g \end{pmatrix}, \tilde{\varphi}_m^\nu \right\rangle \varphi_m^\nu(x) &= A\left(\sum_{-M}^{M}\left[\langle F, e^{imt}\rangle e_m^1 + \langle G, e^{imt}\rangle e_m^2\right]\right)(x) \\ &= A\begin{pmatrix} \sum_{m=-M}^{M}\langle F, e^{imt}\rangle e^{imt} \\ \sum_{m=-M}^{M}\langle G, e^{imt}\rangle e^{imt} \end{pmatrix}(x) \to A\begin{pmatrix} \tilde{F} \\ \tilde{G} \end{pmatrix}(x) \\ &\quad \text{as } M \to \infty.\end{aligned}$$

Therefore, the expansion of $\binom{f}{g}$ about the basis Φ converges point-wise to the vector-function

$$\begin{pmatrix} \tilde{f}(x) \\ \tilde{g}(x) \end{pmatrix} := A\begin{pmatrix} \tilde{F} \\ \tilde{G} \end{pmatrix}(x). \tag{5.7}$$

Moreover, the convergence is uniform on every closed subinterval of $[0,\pi]$ on which both $f(\pi-t)$ and $g(t)$ are continuous, and on the closed interval $[0,\pi]$ if f and g are continuous on $[0,\pi]$ and $\binom{f}{g}$ satisfies the boundary conditions (2.10).

For $x\in(0,\pi)$, Lemma 21 implies that

$$\begin{pmatrix}\tilde F(x)\\ \tilde G(x)\end{pmatrix}=\begin{pmatrix}\frac{1}{2}[F(x-0)+F(x+0)]\\ \frac{1}{2}[G(x-0)+G(x+0)]\end{pmatrix}=\frac{1}{2}\begin{pmatrix}F(x-0)\\ G(x-0)\end{pmatrix}+\frac{1}{2}\begin{pmatrix}F(x+0)\\ G(x+0)\end{pmatrix},$$

so

$$\begin{pmatrix}\tilde f(x)\\ \tilde g(x)\end{pmatrix}=\frac{1}{2}A\begin{pmatrix}F(x-0)\\ G(x-0)\end{pmatrix}+\frac{1}{2}A\begin{pmatrix}F(x+0)\\ G(x+0)\end{pmatrix}. \tag{5.8}$$

In case bc is strictly regular or periodic type the operator A^{-1} is given by (2.28), so

$$\begin{pmatrix}F(x-0)\\ G(x-0)\end{pmatrix}=A^{-1}\begin{pmatrix}f\\ g\end{pmatrix}(x-0)=\begin{pmatrix}e^{-i\tau_1x}[\alpha_1'f(\pi-x+0)+\alpha_2'g(x-0)]\\ e^{-i\tau_2x}[\beta_1'f(\pi-x+0)+\beta_2'g(x-0)]\end{pmatrix}$$

and we obtain by (2.27) and (2.14)

$$\begin{aligned}A\begin{pmatrix}F(x-0)\\ G(x-0)\end{pmatrix}&=\begin{pmatrix}\alpha_1[\alpha_1'f(x+0)+\alpha_2'g(\pi-x-0)]\\ \alpha_2[\alpha_1'f(\pi-x+0)+\alpha_2'g(x-0)]\end{pmatrix}\\ &\quad+\begin{pmatrix}\beta_1[\beta_1'f(x+0)+\beta_2'g(\pi-x-0)]\\ \beta_2[\beta_1'f(\pi-x+0)+\beta_2'g(x-0)]\end{pmatrix}=\begin{pmatrix}f(x+0)\\ g(x-0)\end{pmatrix}.\end{aligned}$$

In an analogous way it follows that $A\begin{pmatrix}F(x+0)\\ G(x+0)\end{pmatrix}(x)=\begin{pmatrix}f(x-0)\\ g(x+0)\end{pmatrix}$. Therefore, in view of (5.8), we obtain that (5.3) holds for strictly regular or periodic type bc.

In case of regular bc which is not strictly regular or periodic type, (2.30) implies that

$$\begin{pmatrix}F(x-0)\\ G(x-0)\end{pmatrix}=\frac{1}{\Delta}\begin{pmatrix}e^{-i\tau_*x}[(\beta_2+\alpha_2x)f(\pi-x+0)-(\beta_1-\pi\alpha_1+\alpha_1x)g(x-0)\\ e^{-i\tau_*x}[-\alpha_2f(\pi-x+0)+\alpha_1g(x-0)],\end{pmatrix},$$

where $\Delta=\alpha_1\beta_2-\alpha_2\beta_1+\pi\alpha_1\alpha_2$. Therefore, by (2.29) it follows that

$$\begin{aligned}A\begin{pmatrix}F(x-0)\\ G(x-0)\end{pmatrix}&=\frac{1}{\Delta}\begin{pmatrix}\alpha_1(\beta_2+\alpha_2\pi-\alpha_2x)f(x+0)-\alpha_1(\beta_1-\alpha_1x)g(\pi-x-0)\\ \alpha_2(\beta_2+\alpha_2x)f(\pi-x+0)-\alpha_2(\beta_1-\pi\alpha_1+\alpha_1x)g(x-0)\end{pmatrix}\\ &\quad+\frac{1}{\Delta}\begin{pmatrix}(\beta_1-\alpha_1x)(-\alpha_2f(x+0)+\alpha_1g(\pi-x-0))\\ (\beta_2+\alpha_2x)(-\alpha_2f(\pi-x+0)+\alpha_1g(x-0))\end{pmatrix}\\ &=\begin{pmatrix}f(x+0)\\ g(x-0)\end{pmatrix}.\end{aligned}$$

Similar calculation shows that $A\begin{pmatrix}F(x+0)\\ G(x+0)\end{pmatrix}(x)=\begin{pmatrix}f(x-0)\\ g(x+0)\end{pmatrix}$. Therefore, (5.3) holds in case bc is regular but not strictly regular or periodic type.

Next we evaluate $\tilde f(0),\tilde f(\pi),\tilde g(0),\tilde g(\pi)$. For convenience, the calculations are presented in a matrix form.

If bc is regular or periodic type, then by (5.7) and (2.27)

$$\begin{pmatrix} \tilde f(0) \\ \tilde f(\pi) \\ \tilde g(0) \\ \tilde g(\pi) \end{pmatrix} = \begin{bmatrix} 0 & \alpha_1 z_1 & 0 & \beta_1 z_2 \\ \alpha_1 & 0 & \beta_1 & 0 \\ \alpha_2 & 0 & \beta_2 & 0 \\ 0 & \alpha_2 z_1 & 0 & \beta_2 z_2 \end{bmatrix} \begin{pmatrix} \tilde F(0) \\ \tilde F(\pi) \\ \tilde G(0) \\ \tilde G(\pi) \end{pmatrix}, \tag{5.9}$$

where $z_1 = e^{i\tau_1\pi}$ and $z_2 = e^{i\tau_2\pi}$ are the roots of (2.13) (in case bc is strictly regular $z_1 \neq z_2$; if bc is periodic type then $z_1 = z_2 = z_*$, and $\tau_1 = \tau_2 = \tau_*$).

In view of Lemma 21, we have

$$\begin{pmatrix} \tilde F(0) \\ \tilde F(\pi) \\ \tilde G(0) \\ \tilde G(\pi) \end{pmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix} \begin{pmatrix} F(0) \\ F(\pi) \\ G(0) \\ G(\pi) \end{pmatrix}. \tag{5.10}$$

On the other hand, by (5.6) and (2.28) it follows that

$$\begin{pmatrix} F(0) \\ F(\pi) \\ G(0) \\ G(\pi) \end{pmatrix} = \begin{bmatrix} 0 & \alpha_1' & \alpha_2' & 0 \\ \alpha_1'/z_1 & 0 & 0 & \alpha_2'/z_1 \\ 0 & \beta_1' & \beta_2' & 0 \\ \beta_1'/z_2 & 0 & 0 & \beta_2'/z_2 \end{bmatrix} \begin{pmatrix} f(0) \\ f(\pi) \\ g(0) \\ g(\pi) \end{pmatrix}. \tag{5.11}$$

Now (5.9)–(5.11) imply

$$\begin{pmatrix} \tilde f(0) \\ \tilde f(\pi) \\ \tilde g(0) \\ \tilde g(\pi) \end{pmatrix} = \frac{1}{2}\mathcal{M} \begin{pmatrix} f(0) \\ f(\pi) \\ g(0) \\ g(\pi) \end{pmatrix}, \tag{5.12}$$

where

$$\mathcal{M} = \begin{bmatrix} \alpha_1\alpha_1' + \beta_1\beta_1' & \alpha_1\alpha_1' z_1 + \beta_1\beta_1' z_2 & \alpha_1\alpha_2' z_1 + \beta_1\beta_2' z_2 & \alpha_1\alpha_2' + \beta_1\beta_2' \\ \dfrac{\alpha_1\alpha_1'}{z_1} + \dfrac{\beta_1\beta_1'}{z_2} & \alpha_1\alpha_1' + \beta_1\beta_1' & \alpha_1\alpha_2' + \beta_1\beta_2' & \dfrac{\alpha_1\alpha_2'}{z_1} + \dfrac{\beta_1\beta_2'}{z_2} \\ \dfrac{\alpha_2\alpha_1'}{z_1} + \dfrac{\beta_2\beta_1'}{z_2} & \alpha_2\alpha_1' + \beta_2\beta_1' & \alpha_2\alpha_2' + \beta_2\beta_2' & \dfrac{\alpha_2\alpha_2'}{z_1} + \dfrac{\beta_2\beta_2'}{z_2} \\ \alpha_2\alpha_1' + \beta_2\beta_1' & \alpha_2\alpha_1' z_1 + \beta_2\beta_1' z_2 & \alpha_2\alpha_2' z_1 + \beta_2\beta_2' z_2 & \alpha_2\alpha_2' + \beta_2\beta_2' \end{bmatrix}. \tag{5.13}$$

Next we evaluate the entries of the matrix $\mathcal{M} = (\mathcal{M}_{ij})$. In view of (2.14) we have

$$\alpha_1\alpha_1' + \beta_1\beta_1' = 1, \qquad \alpha_2\alpha_2' + \beta_2\beta_2' = 1, \qquad \alpha_1\alpha_2' + \beta_1\beta_2' = 0, \qquad \alpha_2\alpha_1' + \beta_2\beta_1' = 0,$$

so $\mathcal{M}_{ii} = 1,\ i = 1, 2, 3, 4$, and $\mathcal{M}_{14} = \mathcal{M}_{23} = \mathcal{M}_{32} = \mathcal{M}_{41} = 0$. In order to find the remaining elements of $\mathcal{M}$ recall that $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ and $\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ are eigenvectors of the matrix $\begin{bmatrix} b & a \\ d & c \end{bmatrix}$ which correspond to its eigenvalues $-z_1$ and $-z_2$ (see the text between (2.13) and (2.14)). Therefore, we have

$$b\alpha_1 + a\alpha_2 = -z_1\alpha_1, \qquad d\alpha_1 + c\alpha_2 = -z_1\alpha_2,$$
$$b\beta_1 + a\beta_2 = -z_2\beta_1, \qquad d\beta_1 + c\beta_2 = -z_2\beta_2.$$

In addition, (2.13) implies that

$$z_1 + z_2 = -(b+c), \qquad z_1 z_2 = bc - ad.$$

Using the above formulas we obtain

$$\begin{aligned}
\mathcal{M}_{12} &= \alpha_1\alpha_1' z_1 + \beta_1\beta_1' z_2 = -\alpha_1'(b\alpha_1 + a\alpha_2) - \beta_1'(b\beta_1 + a\beta_2) \\
&= -b(\alpha_1\alpha_1' + \beta_1\beta_1') - a(\alpha_2\alpha_1' + \beta_2\beta_1') = -b; \\
\mathcal{M}_{13} &= \alpha_1\alpha_2' z_1 + \beta_1\beta_2' z_2 = -\alpha_2'(b\alpha_1 + a\alpha_2) - \beta_2'(b\beta_1 + a\beta_2) \\
&= -b(\alpha_1\alpha_2' + \beta_1\beta_2') - a(\alpha_2\alpha_2' + \beta_2\beta_2') = -a; \\
\mathcal{M}_{21} &= \frac{\alpha_1\alpha_1'}{z_1} + \frac{\beta_1\beta_1'}{z_2} = \frac{1}{z_1 z_2}(\alpha_1\alpha_1' z_2 + \beta_1\beta_1' z_1) \\
&= \frac{1}{bc-ad}\left[\alpha_1\alpha_1'(-b-c-z_1) + \beta_1\beta_1'(-b-c-z_2)\right] \\
&= \frac{1}{bc-ad}\left[-(b+c)(\alpha_1\alpha_1' + \beta_1\beta_1') - \mathcal{M}_{12}\right] = \frac{-c}{bc-ad}; \\
\mathcal{M}_{24} &= \frac{\alpha_1\alpha_2'}{z_1} + \frac{\beta_1\beta_2'}{z_2} = \frac{1}{z_1 z_2}(\alpha_1\alpha_2' z_2 + \beta_1\beta_2' z_1) \\
&= \frac{1}{bc-ad}\left[\alpha_1\alpha_2'(-b-c-z_1) + \beta_1\beta_2'(-b-c-z_2)\right] \\
&= \frac{1}{bc-ad}\left[-(b+c)(\alpha_1\alpha_2' + \beta_1\beta_2') - \mathcal{M}_{13}\right] = \frac{a}{bc-ad}.
\end{aligned}$$

In an analogous way one can find $\mathcal{M}_{31}, \mathcal{M}_{34}, \mathcal{M}_{42}$ and $\mathcal{M}_{43}$; we omit the details and give the final result:

$$\mathcal{M} = \frac{1}{2}\begin{bmatrix} 1 & -b & -a & 0 \\ \dfrac{-c}{bc-ad} & 1 & 0 & \dfrac{a}{bc-ad} \\ \dfrac{d}{bc-ad} & 0 & 1 & \dfrac{-b}{bc-ad} \\ 0 & -d & -c & 1 \end{bmatrix}. \tag{5.14}$$

Hence, (5.4) holds if bc is strictly regular or periodic type.

In the case bc is regular but not strictly regular or periodic type we use the same argument but work with the operators A and A^{-1} defined by (2.29) and (2.30). By (5.7) and (2.29)

$$\begin{pmatrix} \tilde f(0) \\ \tilde f(\pi) \\ \tilde g(0) \\ \tilde g(\pi) \end{pmatrix} = \begin{bmatrix} 0 & \alpha_1 z_* & 0 & \beta_1 z_* \\ \alpha_1 & 0 & \beta_1 - \alpha_1\pi & 0 \\ \alpha_2 & 0 & \beta_2 & 0 \\ 0 & \alpha_2 z_* & 0 & (\beta_2 + \alpha_2\pi) z_* \end{bmatrix} \begin{pmatrix} \tilde F(0) \\ \tilde F(\pi) \\ \tilde G(0) \\ \tilde G(\pi) \end{pmatrix}, \tag{5.15}$$

where $\Delta = \alpha_1\beta_2 - \alpha_2\beta_1 + \pi\alpha_1\alpha_2$ and $z^* = e^{i\pi\tau_*}$.

On the other hand, by (5.6) and (2.30) it follows that

$$\begin{pmatrix} F(0) \\ F(\pi) \\ G(0) \\ G(\pi) \end{pmatrix} = \frac{1}{\Delta}\begin{bmatrix} 0 & \beta_2 & -\beta_1 + \pi\alpha_1 & 0 \\ \dfrac{\beta_2 + \pi\alpha_2}{z_*} & 0 & 0 & \dfrac{-\beta_1}{z_*} \\ 0 & -\alpha_2 & \alpha_1 & 0 \\ \dfrac{-\alpha_2}{z_*} & 0 & 0 & \dfrac{\alpha_1}{z^*} \end{bmatrix} \begin{pmatrix} f(0) \\ f(\pi) \\ g(0) \\ g(\pi) \end{pmatrix}. \tag{5.16}$$

Now (5.15), (5.10) and (5.16) imply that (5.12) holds with

$$\mathcal{M} = \frac{1}{\Delta}\begin{bmatrix} \Delta & (\alpha_1\beta_2 - \beta_1\alpha_2)z_* & \pi\alpha_1^2 z_* & 0 \\ \dfrac{\Delta + \pi\alpha_1\alpha_2}{z_*} & \Delta & 0 & -\dfrac{\pi\alpha_1^2}{z_*} \\ \dfrac{\pi\alpha_2^2}{z_*} & 0 & \Delta & \dfrac{\alpha_1\beta_2 - \beta_1\alpha_2}{z_*} \\ 0 & -\pi\alpha_2^2 z_* & (\Delta + \pi\alpha_1\alpha_2)z_* & \Delta \end{bmatrix}. \tag{5.17}$$

The parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ come from (2.20), where we consider three cases: (i) $a = 0$; (ii) $d = 0$; (iii) $a \neq 0,\ b \neq 0$.

In case (iii), we have

$$\alpha_1 = a, \qquad \alpha_2 = (c-b)/2, \qquad \beta_1 = 0, \qquad \beta_2 = \pi b.$$

Recall also that $z_* = -\frac{b+c}{2}$ and $z_*^2 = bc - ad$ because z_* is a double root of (2.13). Therefore,

$$\Delta = \alpha_1\beta_2 - \alpha_2\beta_1 + \pi\alpha_1\alpha_2 = \pi ab + \pi a\frac{c-b}{2} = \pi a\frac{b+c}{2} = -\pi a z_*.$$

Next we evaluate the entries of $\mathcal{M}$ in case (iii):

$$\mathcal{M}_{12} = \frac{1}{\Delta}(\alpha_1\beta_2 - \alpha_2\beta_1)z_* = \frac{1}{-\pi a z_*}\pi ab z_* = -b;$$

$$\mathcal{M}_{21} = \frac{\Delta + \pi\alpha_1\alpha_2}{\Delta z_*} = \frac{\pi a\frac{b+c}{2} + \pi a\frac{c-b}{2}}{-\pi a z_*^2} = \frac{-c}{bc-ad};$$

$$\mathcal{M}_{13} = \frac{\pi\alpha_1^2 z_*}{\Delta} = \frac{\pi a^2 z_*}{-\pi a z_*} = -a; \qquad \mathcal{M}_{24} = -\frac{\pi\alpha_1^2}{\Delta z_*} = \frac{-\pi a^2}{-\pi a z_*^2} = \frac{a}{bc-ad}.$$

In a similar way we calculate $\mathcal{M}_{31}, \mathcal{M}_{34}, \mathcal{M}_{42}, \mathcal{M}_{43}$ and obtain that in case (iii) the matrix $\mathcal{M}$ is given by (5.14).

An elementary calculation (which is omitted) shows that in cases (i) and (ii) the matrix $\mathcal{M}$ is given by (5.14) also, so (5.4) holds if bc is regular but not strictly regular or periodic type as well. This completes the proof. □

6. Generalizations

6.1. Weighted eigenvalue problems and general potential matrices

Suppose $\rho \in L^1([x_1, x_2])$ and $\rho(x) \geq \text{const} > 0$. Let $L^2([x_1, x_2], \rho)$ be the space of all measurable functions $f : [x_1, x_2] \to \mathbb{C}$ such that

$$\|f\|_\rho^2 = \int_{x_1}^{x_2} |f(x)|^2 \rho(x)dx < \infty.$$

Suppose that

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \qquad \frac{1}{\rho}T_{ij} \in L^2([x_1, x_2], \rho).$$

Consider the operator

$$L_{bc}(T, \rho)y := \frac{1}{\rho(x)}\left[i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{dy}{dx} + Ty\right], \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{6.1}$$

subject to the boundary conditions bc

$$\begin{aligned} &y_1(x_1) + b\, y_1(x_2) + a\, y_2(x_1) = 0, \\ &d\, y_1(x_2) + c\, y_2(x_1) + y_2(x_2) = 0, \end{aligned} \tag{6.2}$$

in the domain $Dom\, L_{bc}(T,\rho) \subset \left(L^2([x_1,x_2],\rho)\right)^2$ which consists of all absolutely continuous functions y such that (6.2) holds and $y_1'/\rho, y_2'/\rho \in L^2([x_1,x_2],\rho)$. It is easy to see that $L_{bc}(T,\rho)$ is a densely defined closed operator. A standard computation of the adjoint operators leads to the following.

Lemma 23. *In the above notations,*

$$(L_{bc}(T,\rho))^* = L_{\widetilde{bc}}(T^*,\rho), \quad \textit{where } T^* = \begin{pmatrix} \overline{T}_{11} & \overline{T}_{21} \\ \overline{T}_{12} & \overline{T}_{22} \end{pmatrix} \tag{6.3}$$

and the boundary conditions $\widetilde{bc}$ are defined by

$$\overline{b} y_1(x_1) + y_1(x_2) + \overline{d} y_2(x_2) = 0, \qquad \overline{a} y_1(x_1) + y_2(x_1) + \overline{c} y_2(x_2) = 0. \tag{6.4}$$

The boundary conditions (6.4) are not written in the standard form (6.2) but a multiplication of the system of Eqs. (6.4) from the left by the inverse matrix

$$\begin{pmatrix} \overline{b} & \overline{d} \\ \overline{a} & \overline{c} \end{pmatrix}^{-1} = \begin{pmatrix} \overline{c}/\overline{\Delta} & -\overline{d}/\overline{\Delta} \\ -\overline{a}/\overline{\Delta} & \overline{b}/\overline{\Delta} \end{pmatrix}, \quad \Delta = bc - ad,$$

would bring the boundary conditions $\widetilde{bc}$ to the standard form. This observation leads to the following

Corollary 24. *The operator $L_{bc}(T,\rho)$ is self-adjoint if and only if*

$$b = \overline{c}/\overline{\Delta}, \qquad a = -\overline{d}/\overline{\Delta}, \qquad d = -\overline{a}/\overline{\Delta}, \qquad c = \overline{b}/\overline{\Delta}, \qquad T = T^*.$$

An appropriate change of the variable transforms the operator $L_{bc}(T,\rho)$ into an operator acting in $\left(L^2([0,\pi])\right)^2$. Indeed, let

$$t(x) = K\int_{x_1}^{x} \rho(\xi)d\xi, \quad x_1 \le x \le x_2, \tag{6.5}$$

where the constant $K > 0$ is chosen so that $t(x_2) = K\int_{x_1}^{x_2}\rho(\xi)d\xi = \pi$, and let $x(t) : [0,\pi] \to [x_1,x_2]$ be the inverse function of $t(x)$. The change of variable $x = x(t)$ give rise of an isomorphism

$$W : L^2([x_1,x_2],\rho) \to L^2([0,\pi]), \quad (Wf)(t) = f(x(t))$$

because $\int_0^\pi |f(x(t))|^2 dt = K \cdot \int_{x_1}^{x_2} |f(x)|^2 \rho(x)dx$. Of course, the operator

$$W^{(2)} : \left(L^2([x_1,x_2],\rho)\right)^2 \to \left(L^2([0,\pi])\right)^2, \qquad W^{(2)}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} Wf \\ Wg \end{pmatrix}$$

is also an isomorphism.

Consider the operator

$$L_{bc}(S)u := i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{du}{dt} + Su, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{6.6}$$

with

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}, \qquad S_{ij}(t) = \frac{1}{K\rho(x(t))} T_{ij}(x(t)), \quad i, j \in \{1, 2\},$$

subject to the boundary conditions bc

$$\begin{aligned} &u_1(0) + bu_1(\pi) + au_2(0) = 0, \\ &du_1(\pi) + cu_2(0) + u_2(\pi) = 0, \end{aligned} \tag{6.7}$$

in the domain $D(L_{bc}(S)) \subset \left(L^2([0,\pi])\right)^2$ which consists of all absolutely continuous functions u such that (6.7) holds and $u_1', u_1' \in L^2([0,\pi])$.

Lemma 25. *The operators $L_{bc}(T,\rho)$ and $K \cdot L_{bc}(S)$ are similar.*

Proof. Change the variables in (6.1), (6.2) by

$$x = x(t), \quad u(t) = y(x(t)) = \begin{pmatrix} y_1(x(t)) \\ y_2(x(t)) \end{pmatrix}, \quad 0 \le t \le \pi. \tag{6.8}$$

Then $u'(t) = y'(x(t)) \cdot x'(t) = y'(x(t)) \frac{1}{K\rho(x(t))}$, the boundary conditions (6.2) transform into (6.7), the domain $D(L_{bc}(T,\rho))$ transforms into $D(L_{bc}(S))$, so the operator $L(T_{bc},\rho)$ transforms into the operator $K \cdot L_{bc}(S)$. In other words, we obtain that

$$W^{(2)} L(T_{bc},\rho) = K \cdot L_{bc}(S)\, W^{(2)},$$

which completes the proof. □

Set

$$s_1(t) = \int_0^t S_{11}(\tau) d\tau, \qquad s_2(t) = \int_0^t S_{22}(\tau) d\tau, \quad 0 \le t \le \pi. \tag{6.9}$$

(This Liouville type transformation is often used in analysis of systems of ordinary differential equations; see for example [22].)

Proposition 26. *In the above notations, the Dirac operator*

$$L_{bc}(S)u = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{du}{dt} + \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} u, \quad S_{ij} \in L^2([0,\pi]), \tag{6.10}$$

subject to the boundary conditions bc

$$\begin{aligned} &u_1(0) + bu_1(\pi) + au_2(0) = 0, \\ &du_1(\pi) + cu_2(0) + u_2(\pi) = 0, \end{aligned} \tag{6.11}$$

is similar to the Dirac operator

$$L_{\widetilde{bc}}(v)\tilde{u} = L^0 \tilde{u} + v\tilde{u}, \qquad v = \begin{pmatrix} 0 & S_{12} e^{-i(s_1(t)+s_2(t))} \\ S_{21} e^{i(s_1(t)+s_2(t))} & 0 \end{pmatrix}, \tag{6.12}$$

subject to the boundary conditions $\widetilde{bc}$

$$\tilde{u}_1(0) + \tilde{b}\tilde{u}_1(\pi) + \tilde{a}\tilde{u}_2(0) = 0, \\ \tilde{d}\tilde{u}_1(\pi) + \tilde{c}\tilde{u}_2(0) + \tilde{u}_2(\pi) = 0, \tag{6.13}$$

where

$$\tilde{b} = be^{is_1(\pi)}, \qquad \tilde{a} = a, \qquad \tilde{d} = de^{i(s_1(\pi)+s_2(\pi))}, \qquad \tilde{c} = ce^{is_2(\pi)}. \tag{6.14}$$

Proof. A simple calculation shows that formally

$$AL_{bc}(S) = L_{\widetilde{bc}}(v)A, \quad \text{where } A = \begin{pmatrix} e^{-is_1(t)} & 0 \\ 0 & e^{is_2(t)} \end{pmatrix}. \tag{6.15}$$

The domain $Dom(L_{bc}(S))$ consists of all absolutely continuous functions $u = \binom{u_1}{u_2}$ such that (6.11) holds and $u_1', u_2' \in L^2([0,\pi])$, and the domain $Dom(L_{\widetilde{bc}}(v))$ consists of all absolutely continuous functions $\tilde{u} = \binom{\tilde{u}_1}{\tilde{u}_2}$ such that (6.13) holds and $\tilde{u}_1', \tilde{u}_2' \in L^2([0,\pi])$. Therefore, $u \in Dom(L_{bc}(S))$ if and only if $\tilde{u} = Au \in Dom(L_{\widetilde{bc}}(v))$. This, together with (6.15), means that the operator $L_{bc}(S)$ subject to the boundary conditions (6.11) is similar to the operator $L_{\widetilde{bc}}(v)$ subject to the boundary conditions (6.13). □

In view of Lemma 25 and Proposition 26, now we can extend our results from the previous sections to the case of weighted eigenvalue problems on an arbitrary finite interval $[x_1, x_2]$.

Definition 27. We say that the Eqs. (6.2) give regular, strictly regular or periodic type boundary conditions for the operator (6.1) if (6.13) are regular, strictly regular or periodic type boundary conditions for the operator (6.12).

By (2.11) and (6.14), the boundary conditions (6.13) are regular if

$$\tilde{b}\tilde{c} - \tilde{a}\tilde{d} = (bc - ad)e^{i(s_1(\pi)+s_2(\pi))} \neq 0,$$

so (6.11) are *regular* boundary conditions for the operator (6.10) if and only if

$$bc - ad \neq 0. \tag{6.16}$$

From (2.12) and (6.14) it follows that (6.13) are *strictly regular* boundary conditions for the operator (6.10) if and only if $(\tilde{b} - \tilde{c})^2 + 4\tilde{a}\tilde{d} \neq 0$ which is equivalent to

$$\left(be^{is_1(\pi)} - ce^{is_2(\pi)}\right)^2 + 4ade^{i(s_1(\pi)+s_2(\pi))} \neq 0. \tag{6.17}$$

Finally, by (2.18) and (6.14), the Eqs. (6.11) give *periodic type* boundary conditions for the operator (6.10) if

$$be^{is_1(\pi)} = ce^{is_2(\pi)}, \quad a = 0, \ d = 0. \tag{6.18}$$

The next theorem generalizes the results in Sections 3–5 (see Theorems 10, 20 and 22).

Theorem 28. *Suppose* $\rho \in L^1([x_1, x_2]), \ \rho(x) \geq \text{const} > 0$, *and*

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \quad \frac{1}{\rho} T_{ij} \in L^2([x_1, x_2], \rho). \tag{6.19}$$

Consider the Dirac the operator

$$L_{bc}(T,\rho)y := \frac{1}{\rho(x)}\left[i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{dy}{dx} + Ty\right], \quad y = \begin{pmatrix}y_1\\ y_2\end{pmatrix}, \tag{6.20}$$

subject to regular bc (in the sense of Definition 27)

$$y_1(x_1) + by_1(x_2) + ay_2(x_1) = 0, \qquad dy_1(x_2) + cy_2(x_1) + y_2(x_2) = 0. \tag{6.21}$$

(A) *If bc are strictly regular (i.e., (6.16) and (6.17) hold), then in* $(L^2([x_1,x_2],\rho))^2$ *there is a basis of Riesz projections* $\{S_N,\ P_n^\alpha,\ \alpha = 1,2,\ N,n \in 2\mathbb{Z},\ |n| > N\}$ *of the operator* $L_{bc}(T,\rho)$ *such that* $\dim S_N = 2N+2$, $\dim P_n^\alpha = 1$, *and*

$$\mathbf{f} = S_N\mathbf{f} + \sum_{|n|>N}\sum_{\alpha=1}^{2} P_n^\alpha \mathbf{f} \quad \forall \mathbf{f} = \begin{pmatrix}f_1\\ f_2\end{pmatrix} \in (L^2([x_1,x_2],\rho))^2, \tag{6.22}$$

where the series converge unconditionally in $(L^2([x_1,x_2],\rho))^2$.

(B) *If bc are regular but not strictly regular (i.e., (6.16) holds but (6.17) fails), then in* $(L^2([x_1,x_2],\rho))^2$ *there is a bases of Riesz projections* $\{S_N,\ P_n,\ N,n \in 2\mathbb{Z},\ |n| > N\}$ *of the operator* $L_{bc}(T,\rho)$ *such that* $\dim S_N = 2N+2$, $\dim P_n = 2$, *and*

$$\mathbf{f} = S_N\mathbf{f} + \sum_{|n|>N} P_n \mathbf{f} \quad \forall \mathbf{f} = \begin{pmatrix}f_1\\ f_2\end{pmatrix} \in (L^2([x_1,x_2],\rho))^2, \tag{6.23}$$

where the series converge unconditionally in $(L^2([x_1,x_2],\rho))^2$.

(C) *If* $\mathbf{f} = \begin{pmatrix}f_1\\ f_2\end{pmatrix}$, *where* f_1 *and* f_2 *are functions of bounded variation on* $[x_1,x_2]$, *then the series (6.22) and (6.23) converge point-wise to a function* $\tilde{\mathbf{f}}(x) = \begin{pmatrix}\tilde f_1(x)\\ \tilde f_2(x)\end{pmatrix}$ *in the sense that*

$$(S_N\mathbf{f})(x) + \lim_{M\to\infty}\sum_{N<|n|\le M}\sum_{\alpha=1}^{2}\left(P_n^\alpha\mathbf{f}\right)(x) = \tilde{\mathbf{f}}(x) \tag{6.24}$$

in the strictly regular case, and

$$(S_N\mathbf{f})(x) + \lim_{M\to\infty}\sum_{N<|n|\le M}(P_n\mathbf{f})(x) = \tilde{\mathbf{f}}(x) \tag{6.25}$$

if bc is regular but not strictly regular. Moreover,

$$\tilde{\mathbf{f}}(x) = \frac{1}{2}(\mathbf{f}(x-0) + \mathbf{f}(x+0)) \quad \text{if } x \in (x_1,x_2), \tag{6.26}$$

and

$$\tilde{\mathbf{f}}(x_1) = \frac{1}{2}\begin{pmatrix} f_1(x_1+0) - bf_1(x_2-0) - af_2(x_1+0) \\ \dfrac{d}{bc-ad}f_1(x_1+0) + f_2(x_1+0) - \dfrac{b}{bc-ad}f_2(x_2-0)\end{pmatrix}, \tag{6.27}$$

$$\tilde{\mathbf{f}}(x_2) = \frac{1}{2}\begin{pmatrix} \dfrac{-c}{bc-ad}f_1(x_1+0) + f_1(x_2-0) + \dfrac{a}{bc-ad}f_2(x_2-0) \\ -df_1(x_2-0) - cf_2(x_1+0) + f_2(x_2-0)\end{pmatrix}. \tag{6.28}$$

If, in addition, both $f_1(x_2 - x)$ *and* $f_2(x)$ *are continuous on some closed subinterval of* (x_1, x_2) *then the convergence in (6.24) and (6.25) is uniform on that interval. The convergence is uniform on the closed interval* $[x_1, x_2]$ *if and only if* f_1 *and* f_2 *are continuous on* $[0, \pi]$ *and* $\binom{f_1}{f_2}$ *satisfies the boundary condition bc given by (6.21).*

Remark. One can easily see by (6.21), (6.27) and (6.28) that if the function $\mathbf{f}$ is continuous at x_1 and x_2 then

$$\tilde{\mathbf{f}}(x_1) = \mathbf{f}(x_1), \qquad \tilde{\mathbf{f}}(x_2) = \mathbf{f}(x_2)$$

if and only if $\mathbf{f}$ satisfies the boundary conditions (6.21).

Proof. In view of Lemma 25 and Proposition 26, (A) and (B) follow from Theorem 10. Below, we show that (C) follows from Theorem 22.

By Lemma 25, a suitable change of variable $t = t(x)$ transforms the operator $L_{bc}(T, \rho)$ subject to the boundary conditions given by the matrix $\begin{pmatrix} 1 & b & a & 0 \\ 0 & d & c & 1 \end{pmatrix}$ on $[x_1, x_2]$ into the operator $L_{bc}(S)$ subject to bc given by the same matrix on $[0, \pi]$. Therefore, it is enough to prove (C) in the case where $x_1 = 0,\ x_2 = \pi,\ \rho \equiv 1$ and $T \equiv S$.

By Proposition 26, the operator $L_{bc}(S)$ is similar to the operator $L_{\widetilde{bc}}(v)$ defined in (6.12) and subject to the boundary conditions $\widetilde{bc}$ given by the matrix $\begin{pmatrix} 1 & \tilde{b} & \tilde{a} & 0 \\ 0 & \tilde{d} & \tilde{c} & 1 \end{pmatrix}$, where $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ are defined by (6.14). By (6.15),

$$L_{bc}(S) = A^{-1} L_{\widetilde{bc}}(v) A \quad \text{with } A = \begin{pmatrix} e^{-is_1(t)} & 0 \\ 0 & e^{is_2(t)} \end{pmatrix},$$

where $s_1(t)$ and $s_2(t)$ come from (6.9).

Since the operators A and A^{-1} act on vector-functions by multiplying their components by exponential functions, the point-wise convergence of the spectral decompositions of the operator $L_{\widetilde{bc}}(v)$ yields a point-wise convergence of the spectral decompositions of the operator $L_{bc}(S)$. Therefore, under the assumptions in (C), (6.26) holds, and the convergence is uniform on a closed subinterval $I \subset (0, \pi)$ provided that $f_1(\pi - t)$ and $f_2(t)$ are continuous on I. Moreover, if f_1 and f_2 are continuous and $\binom{f_1}{f_2}$ satisfies the boundary condition (6.21), then $A\binom{f_1}{f_2}$ satisfies the boundary conditions $\widetilde{bc}$, so the uniform convergence on $[0, \pi]$ follows from Theorem 22.

Next we consider the convergence at the points $x_1 = 0, x_2 = \pi$ and show that (6.27) and (6.28) hold. Let $\{\hat{S}_N(v),\ \hat{P}_n(v),\ |n| > N\}$ be a basis of Riesz projections of the operator $L_{\widetilde{bc}}(v)$, and let

$$\{S_N = A^{-1}\hat{S}_N(v)A,\ P_n = A^{-1}\hat{P}_n(v)A,\ |n| > N\}$$

be the corresponding basis of Riesz projections of the operator $L_{bc}(S)$. Then we have, for every $t \in [0, \pi]$,

$$\begin{aligned} \tilde{\mathbf{f}}(t) &= (S_N\mathbf{f})(t) + \lim_{M\to\infty} \sum_{N<|n|\le M} (P_n\mathbf{f})(t) \\ &= (A^{-1}\hat{S}_N(v)A\mathbf{f})(t) + \lim_{M\to\infty} \sum_{N<|n|\le M} \left(A^{-1}\hat{P}_n(v)A\mathbf{f}\right)(t). \end{aligned}$$

Therefore,

$$\tilde{\mathbf{f}}(t) = A^{-1} \begin{pmatrix} \widehat{f_1} \\ \widehat{f_1} \end{pmatrix}(t), \quad t \in [0, \pi], \tag{6.29}$$

where

$$\begin{pmatrix} \widehat{f_1} \\ \widehat{f_2} \end{pmatrix}(t) = S_N(v)A\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(t) + \lim_{M\to\infty} \sum_{N<|n|\le M} P_n(v)A\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(t). \tag{6.30}$$

Since

$$S\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}(t) = \begin{pmatrix} e^{-is_1(t)} f_1(t) \\ e^{is_2(t)} f_2(t) \end{pmatrix} \quad \text{and} \quad A^{-1}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}(t) = \begin{pmatrix} e^{is_1(t)} g_1(t) \\ e^{-is_2(t)} g_2(t) \end{pmatrix},$$

we obtain

$$\begin{pmatrix} \widehat{f_1}(0) \\ \widehat{f_1}(\pi) \\ \widehat{f_2}(0) \\ \widehat{f_2}(\pi) \end{pmatrix} = \frac{1}{2} M_{\widetilde{bc}} \begin{pmatrix} f_1(0) \\ e^{-is_1(\pi)} f_1(\pi) \\ f_2(0) \\ e^{is_2(\pi)} f_2(\pi) \end{pmatrix}, \qquad \begin{pmatrix} \widetilde{f_1}(0) \\ \widetilde{f_1}(\pi) \\ \widetilde{f_2}(0) \\ \widetilde{f_2}(\pi) \end{pmatrix} = \begin{pmatrix} \widehat{f_1}(0) \\ e^{is_1(\pi)} \widehat{f_1}(\pi) \\ \widehat{f_2}(0) \\ e^{-is_2(\pi)} \widehat{f_2}(\pi) \end{pmatrix},$$

where $\mathcal{M}_{\widetilde{bc}}$ is the transition matrix (5.14) corresponding to the boundary conditions $\widetilde{bc}$. Since

$$\mathcal{M}_{\widetilde{bc}} = \begin{bmatrix} 1 & -\tilde{b} & -\tilde{a} & 0 \\ \dfrac{-\tilde{c}}{\tilde{b}\tilde{c} - \tilde{a}\tilde{d}} & 1 & 0 & \dfrac{\tilde{a}}{\tilde{b}\tilde{c} - \tilde{a}\tilde{d}} \\ \dfrac{\tilde{d}}{\tilde{b}\tilde{c} - \tilde{a}\tilde{d}} & 0 & 1 & \dfrac{-\tilde{b}}{\tilde{b}\tilde{c} - \tilde{a}\tilde{d}} \\ 0 & -\tilde{d} & -\tilde{c} & 1 \end{bmatrix},$$

an easy calculation (which is omitted) shows that the formulas (6.27) and (6.28) hold with $x_1 = 0,\ x_2 = \pi$. This completes the proof. $\square$

7. Self-adjoint separated boundary conditions

A boundary condition bc given by a matrix $\begin{pmatrix} 1 & b & a & 0 \\ 0 & d & c & 1 \end{pmatrix}$ is called *separated* if $b = c = 0$; such bc has the form

$$y_1(x_1) + ay_2(x_1) = 0, \qquad dy_1(x_2) + y_2(x_2) = 0. \tag{7.1}$$

In the case

$$a = e^{2i\alpha_1}, \qquad d = e^{-2i\alpha_2}, \quad \alpha_1, \alpha_2 \in [0, \pi), \tag{7.2}$$

we have a *self-adjoint separated bc* which could be written in the form

$$e^{-i\alpha_1} y_1(x_1) + e^{i\alpha_1} y_2(x_1) = 0, \qquad e^{-i\alpha_2} y_1(x_2) + e^{i\alpha_2} y_2(x_2) = 0. \tag{7.3}$$

In view of Corollary 24, (7.3) gives the general form of self-adjoint separated boundary conditions.

Theorem 29. *Consider on* $[x_1, x_2]$ *the Dirac operator*

$$L_{bc}(D)y = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \frac{dy}{dx} + Dy, \qquad D = \begin{pmatrix} A_1 + iA_2 & P_1 + iP_2 \\ P_1 - iP_2 & A_1 - iA_2 \end{pmatrix}, \tag{7.4}$$

where A_1, A_2, P_1, P_2 *are real* L^2*-functions, and bc is given by* (7.3).

(a) *The spectrum of $L_{bc}(D)$ is discrete; each eigenvalue is real and has equal geometric and algebraic multiplicities. Moreover, there are numbers $N = N(D, bc) \in \mathbb{N}$ and $\tau = \tau(D, bc) \in \mathbb{R}$ such that, with $\ell = x_2 - x_1$, the interval $\left(\left(\tau - N - \frac{1}{4}\right)\frac{\pi}{\ell}, \left(\tau + N + \frac{1}{4}\right)\frac{\pi}{\ell}\right)$ contains exactly $2N + 1$ eigenvalues (counted with multiplicity), and for $n \in \mathbb{Z}$ with $|n| > N$ there is exactly one (simple!) eigenvalue $\lambda_n \in \left(\left(\tau + n - \frac{1}{4}\right)\frac{\pi}{\ell}, \left(\tau + n + \frac{1}{4}\right)\frac{\pi}{\ell}\right)$.*

(b) *There is a Riesz basis in $L^2([x_1, x_2], \mathbb{C}^2)$ which elements are eigenfunctions of the operator $(L_{bc}(D))$ of the form*

$$\Phi = \left\{ \begin{pmatrix} \varphi_k \\ \overline{\varphi}_k \end{pmatrix} = \begin{pmatrix} u_k + i v_k \\ u_k - i v_k \end{pmatrix}, \ u_k, v_k \in L^2([x_1, x_2], \mathbb{R}), \ k \in \mathbb{Z} \right\}, \tag{7.5}$$

and its adjoint biorthogonal system has the form

$$\Psi = \left\{ \begin{pmatrix} \psi_k \\ \overline{\psi}_k \end{pmatrix} = \begin{pmatrix} a_k + i b_k \\ a_k - i b_k \end{pmatrix}, \ a_k, b_k \in L^2([x_1, x_2], \mathbb{R}), \ k \in \mathbb{Z} \right\}. \tag{7.6}$$

(c) *If $F = \binom{F_1}{F_2}$ is a function of bounded variation on $[x_1, x_2]$, then its expansion about the basis Φ converges point-wise to a function $\tilde{F}(x)$,*

$$\tilde{F}(x) = \begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{pmatrix}(x) = \lim_{M \to \infty} \sum_{|k| \le M} c_k(F) \begin{pmatrix} u_k + i v_k \\ u_k - i v_k \end{pmatrix}(x), \tag{7.7}$$

where

$$c_k(F) = \left\langle \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \begin{pmatrix} a_k + i b_k \\ a_k - i b_k \end{pmatrix} \right\rangle, \quad k \in \mathbb{Z}, \tag{7.8}$$

and

$$\tilde{F}(x) = \frac{1}{2}(F(x-0) + F(x+0)), \quad x_1 < x < x_2, \tag{7.9}$$

$$\tilde{F}(x_1) = \begin{pmatrix} F_1(x_1+0) - \exp(2i\alpha_1) F_2(x_1+0) \\ -\exp(-2i\alpha_1) F_1(x_1+0) + F_2(x_1+0) \end{pmatrix}, \tag{7.10}$$

$$\tilde{F}(x_2) = \begin{pmatrix} F_1(x_2-0) - \exp(2i\alpha_2) F_2(x_2-0) \\ -\exp(-2i\alpha_2) F_1(x_2-0) + F_2(x_2-0) \end{pmatrix}. \tag{7.11}$$

Moreover, if $F_1(x_2 - x)$ and $F_2(x)$ are continuous on some closed subinterval of (x_1, x_2) then the convergence in (7.7) is uniform on that interval. The convergence is uniform on the closed interval $[x_1, x_2]$ if and only if F_1 and F_2 are continuous on $[0, \pi]$ and $\binom{F_1}{F_2}$ satisfies the boundary condition bc given by (7.3).

Proof. (a) Set

$$s_1(x) = \int_{x_1}^{x} (A_1(\xi) + i A_2(\xi)) d\xi, \qquad s_2(x) = \int_{x_1}^{x} (A_1(\xi) - i A_2(\xi)) d\xi;$$

then $\overline{s_1(x)} = s_2(x)$, so the sum $s_1(x) + s_2(x)$ is real-valued. As in Proposition 26 (see (6.12)–(6.14)), one can easily see that the operator $L_{bc}(D)$ is similar to the Dirac operator $L_{\widetilde{bc}}(v)$, with

$$v = \begin{pmatrix} 0 & (P_1 + i P_2) e^{-i(s_1(x) + s_2(x))} \\ (P_1 - i P_2) e^{i(s_1(x) + s_2(x))} & 0 \end{pmatrix},$$

$$\widetilde{bc}: \ \tilde{a} = a = e^{2i\alpha_1}, \quad \tilde{b} = b = 0, \ \tilde{c} = 0, \ \tilde{d} = e^{-2i\alpha_2} e^{i(s_1(x_2)+s_2(x_2))},$$

and

$$M L_{bc}(D) = L_{\widetilde{bc}}(v) M, \quad M = \begin{pmatrix} e^{-is_1(x)} & 0 \\ 0 & e^{is_2(x)} \end{pmatrix}. \tag{7.12}$$

The matrix v is hermitian because the sum $s_1(x) + s_2(x)$ is real-valued. Since $\widetilde{bc}$ is a self-adjoint boundary condition of the form (7.3), it follows that the operator $L_{\widetilde{bc}}(v)$ is self-adjoint. Therefore, its spectrum is real, and moreover, discrete by Part (A) of Theorem 28.

In the case $x_1 = 0$, $x_2 = \pi$ a localization of the spectrum of $L_{\widetilde{bc}}(v)$ can be obtained by the general scheme from Section 2 and Lemma 9. Indeed, now the characteristic equation (2.13) becomes

$$z^2 = \tilde{a}\tilde{d} = e^{2i(\alpha_1-\alpha_2)} e^{i(s_1(\pi)+s_2(\pi))},$$

so its solutions z_1, z_2 can be written as

$$z_1 = e^{i\pi\tau}, \qquad z_2 = e^{i\pi(\tau+1)}, \quad \text{where } \tau = \frac{1}{2\pi}[2\alpha_1 - 2\alpha_2 + s_1(\pi) + s_2(\pi)].$$

Therefore, it follows that $Sp(L^0_{\widetilde{bc}}) = \{\lambda = \tau + n, \ n \in \mathbb{Z}\}$, so Lemma 9 implies (a) in this case.

The general case of an arbitrary interval $[x_1, x_2]$ could be reduced to the case of $[0, \pi]$ by the change of variable $x = x_1 + \frac{\ell}{\pi} t, \ \ell = x_2 - x_1$.

(b) In view of Part (B) of Theorem 28 there is a Riesz basis in $L^2([x_1, x_2], \mathbb{C}^2)$ which consists of eigenfunctions of $L_{\widetilde{bc}}(v)$. Moreover, since it is a self-adjoint operator there is an orthonormal basis which consists of eigenfunctions of $L_{\widetilde{bc}}(v)$.

One can easily see that the real vector subspace of $L^2([x_1, x_2], \mathbb{C}^2)$

$$H = \left\{ \begin{pmatrix} \varphi \\ \overline{\varphi} \end{pmatrix} : \varphi \in L^2([x_1, x_2], \mathbb{C}) \right\}$$

is invariant subspace for both $L_{bc}(D)$ and $L_{\widetilde{bc}}(v)$. Moreover, since

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} \varphi \\ \overline{\varphi} \end{pmatrix} + i \begin{pmatrix} \psi \\ \overline{\psi} \end{pmatrix} \quad \text{with } \varphi = \frac{g_1 + \overline{g_2}}{2} \text{ and } \psi = \frac{g_1 - \overline{g_2}}{2i},$$

we have $L^2([x_1, x_2], \mathbb{C}^2) = H \oplus iH$.

Suppose $\lambda \in Sp\,(L_{\widetilde{bc}}(v))$, and let $E_\lambda = \{y : \ L_{\widetilde{bc}}(v) y = \lambda y\}$ be the space of eigenvectors corresponding to λ. By (a) we know that λ is real, and $\dim E_\lambda < \infty$. Since λ is real, one can easily see by taking the conjugates that

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in E_\lambda \Rightarrow \begin{pmatrix} \overline{g_2} \\ \overline{g_1} \end{pmatrix} \in E_\lambda. \tag{7.13}$$

Suppose that $\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} \varphi \\ \overline{\varphi} \end{pmatrix} + i \begin{pmatrix} \psi \\ \overline{\psi} \end{pmatrix} \in E_\lambda$. Then (7.13) implies $\begin{pmatrix} \overline{g_2} \\ \overline{g_1} \end{pmatrix} = \begin{pmatrix} \varphi \\ \overline{\varphi} \end{pmatrix} - i \begin{pmatrix} \psi \\ \overline{\psi} \end{pmatrix} \in E_\lambda$, which yields $\begin{pmatrix} \varphi \\ \overline{\varphi} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \overline{g_2} \\ \overline{g_1} \end{pmatrix} \in E_\lambda$ and $\begin{pmatrix} \psi \\ \overline{\psi} \end{pmatrix} = \frac{1}{2i}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} - \frac{1}{2i}\begin{pmatrix} \overline{g_2} \\ \overline{g_1} \end{pmatrix} \in E_\lambda$. Hence,

$$E_\lambda = (E_\lambda \cap H) \oplus i(E_\lambda \cap H),$$

which implies that every basis in $(E_\lambda \cap H)$ (regarded as a real vector space) is a basis in E_λ (regarded as a complex vector space) as well. Therefore, one may choose in each of the spaces E_λ a basis consisting of mutually orthogonal normalized vectors from $E_\lambda \cap H$. Since all but finitely many of these spaces are one-dimensional, it follows that there is an orthonormal basis $f_k,\ k \in \mathbb{Z}$, of the form (7.5) which consists of eigenfunctions of $L_{\widetilde{bc}}(v)$.

By (7.12), the system

$$\Phi = \{Mf_k,\ k \in \mathbb{Z}\}$$

is a Riesz basis in $L^2([x_1, x_2], \mathbb{C}^2)$ which consists of eigenfunctions of $L_{bc}(D)$, and the corresponding biorthogonal system is

$$\Psi = \{(M^{-1})^* f_k,\ k \in \mathbb{Z}\}, \qquad (A^{-1})^* = \begin{pmatrix} e^{-i\overline{s_1}} & 0 \\ 0 & e^{i\overline{s_2}}. \end{pmatrix}$$

Since $\overline{s_1(x)} = s_2(x)$, one can easily verify that the system Φ has the form (7.5) and Ψ has the form (7.6).

Finally, in view of (7.2), (c) follows from part (C) of Theorem 28. $\square$

Next we provide a version of Theorem 29 for real-valued functions.

Theorem 30. *Let $\rho \in L^1([x_1, x_2])$, $\rho(x) \geq \text{const} > 0$ for $x \in [x_1, x_2]$, and*

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \quad \frac{1}{\rho} T_{ij} \in L^2([x_1, x_2], \rho),\ T_{ij} - \textit{real-valued}. \tag{7.14}$$

Consider the operator

$$R_{\widehat{bc}}(T) \begin{pmatrix} u \\ v \end{pmatrix} := \frac{1}{\rho} \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx} \begin{pmatrix} u \\ v \end{pmatrix} + T \begin{pmatrix} u \\ v \end{pmatrix} \right], \tag{7.15}$$

subject to the boundary conditions

$$\widehat{bc}:\ u(x_j) \cos \alpha_j + v(x_j) \sin \alpha_j = 0, \quad \alpha_1 \neq \alpha_2,\ j = 1, 2. \tag{7.16}$$

(A) *The spectrum of the operator $R_{\widehat{bc}}$ is discrete; each eigenvalue is real and has equal geometric and algebraic multiplicities. Moreover, there are numbers $N = N(T, bc)$, $\tau = \tau(T, bc)$ and $\ell = \ell(\rho, x_2 - x_1)$ such that the interval $\left(\left(\tau - N - \frac{1}{4}\right)\frac{\pi}{\ell}, \left(\tau + N + \frac{1}{4}\right)\frac{\pi}{\ell}\right)$ contains $2N + 1$ eigenvalues λ_k, $-N \leq k \leq N$ (counted with multiplicity), and for $n \in \mathbb{Z}$, $|n| > N$, there is only one (simple!) eigenvalue λ_n in the interval $\left(\left(\tau + n - \frac{1}{4}\right)\frac{\pi}{\ell}, \left(\tau + n + \frac{1}{4}\right)\frac{\pi}{\ell}\right)$.*

(B) *Let $\mathcal{B} = \left\{ \begin{pmatrix} u_n \\ v_n \end{pmatrix},\ n \in \mathbb{Z} \right\}$ be a system of normalized real-valued eigenfunctions corresponding to the sequence of eigenvalues $(\lambda_n)_{n \in \mathbb{Z}}$. Then the system $\mathcal{B}$ is a Riesz basis in $\left(L^2([x_1, x_2], \rho)\right)^2$, i.e.,*

$$\begin{pmatrix} f \\ g \end{pmatrix} = \sum_{n \in \mathbb{Z}} C_n(f, g) \begin{pmatrix} u_n \\ v_n \end{pmatrix} \quad \forall \begin{pmatrix} f \\ g \end{pmatrix} \in \left(L^2([x_1, x_2], \rho)\right)^2, \tag{7.17}$$

where the series converge unconditionally.

(C) *If f and g are real-valued functions of bounded variation on $[x_1, x_2]$, then the series in (7.17) converges point-wise in the sense that*

$$\lim_{M \to \infty} \sum_{|n| \leq M} C_n(f, g) \begin{pmatrix} u_n(x) \\ v_n(x) \end{pmatrix} := \begin{pmatrix} \tilde{f}(x) \\ \tilde{g}(x) \end{pmatrix} \tag{7.18}$$

where

$$\begin{pmatrix}\tilde f(x)\\ \tilde g(x)\end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix}f(x-0)\\ g(x-0)\end{pmatrix} + \begin{pmatrix}f(x+0)\\ g(x+0)\end{pmatrix}\right] \quad \textit{for } x \in (x_1, x_2), \tag{7.19}$$

and

$$\begin{pmatrix}\tilde f(x_1)\\ \tilde g(x_1)\end{pmatrix} = \frac{1}{2}\begin{pmatrix} f(x_1+0)(1-\cos 2\alpha_1) - g(x_1+0)\sin 2\alpha_1\\ -f(x_1+0)\sin 2\alpha_1 + g(x_1+0)(1+\cos 2\alpha_1)\end{pmatrix}, \tag{7.20}$$

$$\begin{pmatrix}\tilde f(x_2)\\ \tilde g(x_2)\end{pmatrix} = \frac{1}{2}\begin{pmatrix} f(x_2-0)(1-\cos 2\alpha_2) - g(x_2-0)\sin 2\alpha_2\\ -f(x_2-0)\sin 2\alpha_2 + g(x_2-0)(1+\cos 2\alpha_2)\end{pmatrix}. \tag{7.21}$$

In addition, if the functions f and g are continuous on a closed subinterval $[x_1+\delta, x_2-\delta] \subset (x_1, x_2)$, then the convergence in (7.18) is uniform on $[x_1+\delta, x_2-\delta]$. The convergence is uniform on the closed interval $[x_1, x_2]$ if and only if f and g are continuous on $[x_1, x_2]$ and $\binom{f}{g}$ satisfies the boundary condition $\widehat{bc}$ given in (7.16).

Proof. (A) In view Lemma 25, we may assume that $\rho \equiv 1$. Set

$$A_1 = \frac{1}{2}(T_{11}+T_{22}), \qquad P_1 = \frac{1}{2}(T_{11}-T_{22}), \qquad A_2 = \frac{1}{2}(T_{21}-T_{12}),$$
$$P_2 = \frac{1}{2}(T_{21}+T_{12})$$

and consider the operator $L_{bc}(D)$ in (7.4) with $D = \begin{pmatrix} A_1+iA_2 & P_1+iP_2\\ P_1-iP_2 & A_1-iA_2\end{pmatrix}$ and bc given by (7.3).

A simple calculation shows that $\binom{u}{v}$ satisfies (7.16) if and only if $\binom{u+iv}{u-iv}$ satisfies (7.3), and

$$L_{bc}(D)\begin{pmatrix}u+iv\\ u-iv\end{pmatrix} = \lambda\begin{pmatrix}u+iv\\ u-iv\end{pmatrix} \Leftrightarrow R_{\widehat{bc}}(T)\begin{pmatrix}u\\ v\end{pmatrix} = \lambda\begin{pmatrix}u\\ v\end{pmatrix}. \tag{7.22}$$

Therefore, λ is an eigenvalue of $R_{\widehat{bc}}(T)$ if and only if it is an eigenvalue of $L_{bc}(D)$ with the same geometric multiplicity, and the localization of eigenvalues of $R_{\widehat{bc}}(T)$ given in (A) follows from part (a) of Theorem 29. But it remains to explain that the spectrum of $R_{\widehat{bc}}(T)$ is discrete — see below.

(B) By Theorem 29, the system Φ in (7.5) is a Riesz basis in $L^2([x_1, x_2], \mathbb{C}^2)$ consisting of eigenfunctions of the operator $L_{bc}(D)$, and its biorthogonal system is given by (7.6). Fix $\binom{f}{g} \in L^2([x_1, x_2], \mathbb{R}^2)$ and consider the expansion of $\binom{f+ig}{f-ig}$ about the Riesz basis Φ. Then

$$\begin{pmatrix}f+ig\\ f-ig\end{pmatrix} = \sum_k C_k(f,g)\begin{pmatrix}u_k+iv_k\\ u_k-iv_k\end{pmatrix}, \tag{7.23}$$

where the series converges in unconditionally in $L^2([x_1, x_2], \mathbb{C}^2)$ and

$$C(f,g) = \left\langle \begin{pmatrix}f+ig\\ f-ig\end{pmatrix}, \begin{pmatrix}a_k+ib_k\\ a_k-ib_k\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix}f\\ g\end{pmatrix}, 2\begin{pmatrix}a_k\\ b_k\end{pmatrix}\right\rangle. \tag{7.24}$$

By taking the first components in (7.23) and separating the real and imaginary parts we obtain

$$\begin{pmatrix}f\\ g\end{pmatrix} = \sum_k C_k(f,g)\begin{pmatrix}u_k\\ v_k\end{pmatrix} \quad \forall \begin{pmatrix}f\\ g\end{pmatrix} \in L^2([x_1, x_2], \mathbb{R}^2), \tag{7.25}$$

where the series converge unconditionally in $L^2([x_1, x_2], \mathbb{R}^2)$.

In view of (7.22), (7.24) and (7.25), the system

$$\mathcal{B} = \left\{ \begin{pmatrix} u_k \\ v_k \end{pmatrix} : \begin{pmatrix} u_k + i v_k \\ u_k - i v_k \end{pmatrix} \in \Phi,\ k \in \mathbb{Z} \right\} \tag{7.26}$$

is a Riesz basis in $L^2([x_1, x_2], \mathbb{R}^2)$ (and therefore, in $L^2([x_1, x_2], \mathbb{C}^2)$) which consists of eigenfunctions of the operator $R_{\widehat{bc}}(T)$, and its biorthogonal system is

$$\mathcal{B}^* = \left\{ 2 \begin{pmatrix} a_k \\ b_k \end{pmatrix} : \begin{pmatrix} a_k + i b_k \\ a_k - i b_k \end{pmatrix} \in \Psi,\ k \in \mathbb{Z} \right\}. \tag{7.27}$$

Now one can use the Riesz basis (7.26) in order to construct the resolvent $(\lambda - R_{\widehat{bc}}(T))^{-1}$ for any $\lambda \neq \lambda_k,\ k \in \mathbb{Z}$, which shows that the spectrum of $R_{\widehat{bc}}(T)$ is discrete and consists of the eigenvalues $\lambda_k,\ k \in \mathbb{Z}$.

(C) Let f and g be functions of bounded variation on $[x_1, x_2]$. Set

$$F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} f + ig \\ f - ig \end{pmatrix}; \quad \text{then } f = Re\, F_1,\ g = Im F_1$$

In view of Part (c) of Theorem 29, we have

$$\lim_{M \to \infty} \sum_{|n| \le M} C_n(f, g) \begin{pmatrix} u_n(x) + i v_n(x) \\ u_n(x) - i v_n(x) \end{pmatrix} := \tilde{F}(x), \tag{7.28}$$

where $\tilde{F}(x) = \binom{\tilde{F}_1(x)}{\tilde{F}_2(x)}$ is given by Formulas (7.9)–(7.11) in terms of F. Obviously, we have $\tilde{F}_2(x) = \overline{\tilde{F}_1(x)}$. Taking the first components in (7.28) and separating the real and imaginary parts we obtain

$$\lim_{M \to \infty} \sum_{|n| \le M} C_n(f, g) \begin{pmatrix} u_n(x) \\ v_n(x) \end{pmatrix} := \begin{pmatrix} \tilde{f}(x) \\ \tilde{g}(x) \end{pmatrix} \tag{7.29}$$

with $\tilde{f}(x) = Re\, \tilde{F}_1(x)$ and $\tilde{g}(x) = Im\, \tilde{F}_1(x)$. Now (C) follows immediately from Part (c) of Theorem 29. $\square$

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Appendix. Discrete Hilbert transform and multipliers

The aim of this Appendix is to prove Lemma 19. In fact, we explain that if $f \in H(\Omega)$ and $g \in C^1([0,\pi])$, then $f \cdot g \in H(\Omega)$ for a wider class of weights then we need in Lemma 19 — see below Proposition 32.

Recall that if $\xi = (\xi_k) \in \ell^2(\mathbb{Z})$ then its discrete Hilbert transform is defined by

$$(\mathcal{H}\xi)_n = \sum_{k\neq n} \frac{\xi_k}{n-k}, \quad n \in \mathbb{Z}.$$

It is well known that the operator $\mathcal{H} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is bounded. Moreover, let $\Omega = (\Omega(k))_{k\in\mathbb{Z}}$ be a weight sequence such that

$$\Omega(0) \geq 1, \qquad \Omega(-k) = \Omega(k), \qquad \Omega(k) \leq \Omega(k+1) \quad \text{for } k \geq 0. \tag{A.1}$$

Then it is known by [17, Theorem 10] that the discrete Hilbert transform $\mathcal{H}$ acts continuously in the weighted space

$$\ell^2(\Omega) = \left\{\xi : \ \|\xi\|^2_\Omega = \sum |\xi_k|^2 \Omega^2(k) < \infty\right\}$$

if and only if the weight Ω satisfies the condition

$$\sup_{k,n} \left(\frac{1}{n+1} \sum_{m=k}^{k+n} \Omega^2(m) \times \frac{1}{n+1} \sum_{m=k}^{k+n} \frac{1}{\Omega^2(m)} \right) < \infty. \tag{A.2}$$

See [1] for the proof of a particular version of this criterion which is good enough for Lemma 31.

Lemma 31. *Suppose the weight Ω satisfies (A.1) and*

$$\exists C > 0 : \ \Omega(2k) \leq C\,\Omega(k) \quad \forall k \in 2\mathbb{Z}; \tag{A.3}$$

$$\exists C > 0 : \ \Omega(k) \leq C\sqrt{1+|k|} \quad \forall k \in 2\mathbb{Z}. \tag{A.4}$$

If $f \in H(\Omega)$ and $g \in H^1_{per}$, then $f \cdot g \in H(\Omega)$.

Proof. Recall that $H^1_{per} = H\left(\sqrt{1+k^2}\right)$. Let $(\hat{f}(k))$ and $(\hat{g}(k))$ be the Fourier coefficients of f and g with respect to the system e^{ikx}, $k \in 2\mathbb{Z}$. It is enough to show that $\hat{f} * \hat{g} \in \ell^2(\Omega)$. To this end we consider, for $b \in \left(\ell^2(\Omega)\right)^* = \ell^2(\Omega^{-1})$, the ternary form

$$T = \sum_m \sum_k \hat{f}(k)\hat{g}(m-k)b(m)$$

and show that it is bounded.

Set

$$\xi(k) = \hat{f}(k)\Omega(k), \qquad \eta(k) = \hat{g}(k)\sqrt{1+k^2}, \qquad \beta(k) = b(k)/\Omega(k), \quad k \in 2\mathbb{Z}.$$

Then we have $\xi, \eta, \beta \in \ell^2(2\mathbb{Z})$ and

$$\|\xi\| = \|\hat{f}\|_{\ell^2(\Omega)}, \qquad \|\eta\| = \|\hat{g}\|_{\ell^2(\sqrt{1+k^2})}, \qquad \|\beta\| = \|b\|_{\ell^2(\Omega^{-1})}.$$

Now the ternary form T can be written as

$$T = \sum_{k,m} \frac{\xi(k)}{\Omega(k)} \cdot \frac{\eta(m-k)}{\sqrt{1+(m-k)^2}} \cdot \beta(m)\Omega(m),$$

and the Cauchy inequality implies

$$|T|^2 \le \left(\sum_{k,m} |\xi(k)|^2 |\eta(m-k)|^2\right)\left(\sum_{k,m} \Omega^2(m) \frac{|\beta(m)|^2}{\Omega^2(k)[1+(m-k)^2]}\right) \le S\|\xi\|^2\|\eta\|^2\|\beta\|^2,$$

where

$$S = \sup_m \sum_k \frac{\Omega^2(m)}{\Omega^2(k)[1+(m-k)^2]}.$$

Next we explain that $S < \infty$. Indeed, in view of (A.3), if $|k| \ge |m|/2$ then $\Omega(m) \le \Omega(2k) \le C\Omega(k)$. Therefore,

$$\sum_{|k|\ge|m|/2} \frac{\Omega^2(m)}{\Omega^2(k)[1+(m-k)^2]} \le C^2 \sum_{|k|\ge|m|/2} \frac{1}{[1+(m-k)^2]} \le C^2(1+\pi).$$

On the other hand, if $|k| < |m|/2$ then $|m-k| > |m|/2$. Thus,

$$\sum_{|k|<|m|/2} \frac{\Omega^2(m)}{\Omega^2(k)[1+(m-k)^2]} \le \sum_{|j|>|m|/2} \frac{\Omega^2(m)}{1+j^2} \le \frac{4\Omega^2(m)}{1+|m|}.$$

Now (A.4) implies that

$$S \le \sup_m \left(C^2(1+\pi) + \frac{4\Omega^2(m)}{1+|m|}\right) < \infty,$$

which completes the proof. $\square$

Proposition 32. *If a weight sequence Ω satisfies (A.2)–(A.4), $f \in H(\Omega)$ and $g \in C^1([0,\pi])$, then $f \cdot g \in H(\Omega)$.*

Proof. The C^1-function g could be written as a sum of a linear function and a periodic C^1-function as

$$g(x) = \ell(x) + g_1(x) \quad \text{with } \ell(x) = m\,x,\ m = (g(\pi) - g(0))/\pi.$$

Since $g_1 \in H_{per}$, Lemma 31 implies that $f \cdot g_1 \in H(\Omega)$. So, it remains to prove that $x\,f(x) \in H(\Omega)$.

Since

$$x = \sum_{k\in 2\mathbb{Z}} c(k)e^{ikx}, \quad \text{with } c(0) = \pi/2,\ c(k) = \frac{i}{k} \text{ for } k \ne 0,$$

we obtain that

$$c * \hat{f}(k) = \frac{\pi}{2}\hat{f}(k) + i\sum_{j\ne k} \frac{\hat{f}(j)}{k-j}, \quad \text{that is } c * \hat{f} = \frac{\pi}{2}\hat{f} + i\mathcal{H}(\hat{f}).$$

Since (A.2) holds, $\mathcal{H}(\hat{f}) \in \ell^2(\Omega)$ (due to the results of [17]), so $c * \hat{f} \in \ell^2(\Omega)$. Thus, $x\,f(x) \in H(\Omega)$, which completes the proof. $\square$

Proposition 32 would imply Lemma 19 if we check that the conditions (A.2)–(A.4) for the weights Ω given by

$$\Omega(k) = (1 + |k|)^{\alpha}, \quad 0 \le \alpha < 1/2, \tag{A.5}$$

or

$$\Omega(k) = (\log(e + |k|))^{\delta}, \quad 0 \le \delta < \infty. \tag{A.6}$$

Lemma 33. *Weights Ω given by (A.5) and (A.6) satisfy the conditions (A.2)–(A.4).*

Proof. Elementary inequalities show that (A.3) and (A.4) hold. To check (A.2) we have to show that there is $M > 0$ such that

$$\frac{1}{n+1} s(k,n) \times \frac{1}{n+1} S(k,n) \le M \quad \forall k \in \mathbb{Z},\ n \in \mathbb{Z}_+, \tag{A.7}$$

where

$$s(k,n) = \sum_{m=k}^{k+n} \frac{1}{\Omega^2(m)}, \qquad S(k,n) = \sum_{m=k}^{k+n} \Omega^2(m).$$

There are three cases:

$$\text{(a) } k < -2n; \qquad \text{(b) } -2n \le k \le n; \qquad \text{(c) } k > n.$$

By (A.1), $\Omega(-m) = \Omega(m)$; therefore, with $k_1 = -k - n$ we have

$$s(k,n) = s(k_1,n), \qquad S(k,n) = S(k_1,n),$$

so the case (a) reduces to (c). If $k > n$, then by (A.1) and (A.3)

$$\Omega(k) \le \Omega(m) \le \Omega(2k) \le C\,\Omega(k), \quad k \le m \le k + n.$$

Therefore, it follows that

$$\frac{1}{n+1} s(k,n) \le \frac{1}{\Omega^2(k)} \quad \text{and} \quad \frac{1}{n+1} S(k,n) \le \Omega^2(2k) \le C^2 \Omega^2(k),$$

so the product in (A.7) does not exceed the constant C^2 from (A.3).

Next, we consider the case (b) where $-2n \le k \le n$. Then, by (A.1), it follows

$$\frac{s(k,n)}{n+1} = \frac{1}{n+1} \sum_{k}^{k+n} \frac{1}{\Omega^2(m)} \le \frac{2}{1+2n} 2 \sum_{0}^{2n} \frac{1}{\Omega^2(m)} = 4 \frac{s(0,2n)}{2n+1} \tag{A.8}$$

and

$$S(k,n) \le 2S(0,2n) \le 2(2n+1)\Omega^2(2n). \tag{A.9}$$

If Ω is of the form (A.5), then

$$s(0,2n) = \sum_{j=0}^{2n} \frac{1}{(1+j)^{2\alpha}} \le 1 + \int_0^{2n} \frac{1}{(1+x)^{2\alpha}} dx \le \frac{2}{1-2\alpha} (1+2n)^{1-2\alpha},$$

so (A.8) and (A.9) show that the product in (A.7) does not exceed

$$\left(\frac{4}{2n+1} \cdot \frac{2}{1-2\alpha} (2n+1)^{1-2\alpha} \right) \cdot 4(2n+1)^{2\alpha} \le \frac{32}{1-2\alpha}.$$

If Ω is of the form (A.6), then

$$\frac{s(0,2n)}{2n+1} = \frac{1}{2n+1}\left(\sum_{0\le j\le\sqrt{n}} \frac{1}{(\log(e+j))^{2\delta}} + \sum_{\sqrt{n}<j\le 2n} \frac{1}{(\log(e+j))^{2\delta}}\right)$$
$$\le \frac{1+\sqrt{n}}{1+2n} + \frac{1+2n-\sqrt{n}}{1+2n}\frac{1}{\left(\log\left(e+\sqrt{n}\right)\right)^{2\delta}} \le \frac{M}{\left(\log\left(e+\sqrt{n}\right)\right)^{2\delta}}$$

with

$$M = 2\max_{n\ge 0}\frac{\left(\log\left(e+\sqrt{n}\right)\right)^{2\delta}}{1+\sqrt{n}} + 1.$$

Since (A.9) holds for every monotone weight Ω it follows that the product (A.7) does not exceed $M\cdot\tilde{M}^{2\delta}$ with $\tilde{M} = \max_{n\ge 0}\frac{\log(e+2n)}{\log(e+\sqrt{n})}$. This completes the proof of Lemma 33. □

References

[1] J. Adduci, B. Mityagin, Eigensystem of an L^2-perturbed harmonic oscillator is an unconditional basis, Cent. Eur. J. Math. 10 (2012) 569–589.

[2] G.D. Birkhoff, On the asymptotic character of the solutions of certain linear differential equations containing a parameter, Trans. Amer. Math. Soc. 9 (1908) 21–231.

[3] G.D. Birkhoff, Boundary value and expansion problems of ordinary linear differential equations, Trans. Amer. Math. Soc. 9 (1908) 373–395.

[4] G.D. Birkhoff, R.E. Langer, The boundary problems and and developments associated with a system of ordinary differential equations of the first order, Proc. Am. Acad. Arts Sci. 58 (1923) 49–128.

[5] L. Carleson, Selected problems on exceptional sets, in: Van Nostrand Mathematical Studies, No. 13, D. Van Nostrand Co., Inc, Princeton, N.J.-Toronto, Ont.-London, 1967.

[6] P. Djakov, B. Mityagin, Smoothness of Schrödinger operator potential in the case of Gevrey type asymptotics of the gaps, J. Funct. Anal. 195 (2002) 89–128.

[7] P. Djakov, B. Mityagin, Spectral triangles of Schrödinger operators with complex potentials, Selecta Math. (N.S.) 9 (2003) 495–528.

[8] P. Djakov, B. Mityagin, Spectra of 1-D periodic Dirac operators and smoothness of potentials, C. R. Math. Acad. Sci. Soc. R. Can. 25 (2003) 121–125.

[9] P. Djakov, B. Mityagin, Instability zones of a periodic 1D Dirac operator and smoothness of its potential, Comm. Math. Phys. 259 (2005) 139–183.

[10] P. Djakov, B. Mityagin, Instability zones of periodic 1D Schrödinger and Dirac operators, Uspehi Mat. Nauk 61 (4) (2006) 77–182 (in Russian). English: Russian Math. Surveys 61 (4) (2006) 663–766.

[11] P. Djakov, B. Mityagin, Bari–Markus property for Riesz projections of 1D periodic Dirac operators, Math. Nachr. 283 (3) (2010) 443–462.

[12] P. Djakov, B. Mityagin, Unconditional convergence of spectral decompositions of 1D Dirac operators with regular boundary conditions, Indiana Univ. Math. J. (in press), arXiv:1008.4095.

[13] N. Dunford, J. Schwartz, Linear Operators, Part III, Spectral Operators, Wiley, New York, 1971.

[14] I.C. Gohberg, M.G. Krein, Introduction to The Theory of Linear Non-Self-Adjoint Operators, in: Translation of Mathematical Monographs, vol. 18, American Mathematical Society, Providence, Rhode Island, 1969.

[15] S. Hassi, L. Oridoroga, Theorem of completeness for a Dirac-type operator with generalized λ-depending boundary conditions, Integral Equations Operator Theory 64 (2009) 357–379.

[16] M. Horvath, Eigenfunction expansions for one-dimensional Dirac operators, Acta Sci. Math. (Szeged) 61 (1995) 225–240.

[17] R. Hunt, B. Muckenhoupt, R. Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform, Trans. Amer. Math. Soc. 176 (1973) 227–251.

[18] V.A. Il'in, Necessary and sufficient conditions for being a basis in L_p and equiconvergence with a trigonometric series of spectral expansions and expansions in systems of exponentials, Dokl. Akad. Nauk SSSR 273 (4) (1983) 789–793 (in Russian).

[19] V.A. Il'in, Equiconvergence, with a trigonometric series, of expansions in root functions of the one-dimensional Schrödinger operator with complex potential in the class L_1, Differ. Uravn. 27 (4) (1991) 577–597. 732 (in Russian); Translation in Differ. Equ. 27 (4) (1991) 401–416.
[20] V.A. Il'in, Componentwise equiconvergence with a trigonometric series of expansions in root vector functions of the Schrödinger operator with a matrix non-Hermitian potential, all elements of which are only summable, Differ. Uravn. 27 (11) (1991) 1862–1879. 2020–2021 (in Russian); Translation in Differ. Equ. 27 (11) (1991) 1307–1321. (1992).
[21] G.M. Keselman, On the unconditional convergence of eigenfunction expansions of certain differential operators, Izv. Vyssh. Uchebn. Zaved. Mat. 39 (2) (1964) 82–93 (in Russian).
[22] M.M. Malamud, Questions of uniqueness in inverse problems for systems of differential equations on a finite interval, Trans. Moscow Math. Soc. 60 (1999) 173–224.
[23] M.M. Malamud, L.L. Oridoroga, Theorems of the completeness for the systems of ordinary differential equations, Funct. Anal. Appl. 34 (3) (2000) 88–90.
[24] M.M. Malamud, L.L. Oridoroga, On the completeness of root vectors of first order systems, Dokl. Math. 82 (3) (2010) 899–905.
[25] M.M. Malamud, L.L. Oridoroga, On the completeness of the system of root vectors for first-order systems, arXiv:1109.3683.
[26] V. Mazya, T. Shaposhnikova, Theory of Sobolev Multipliers. With Applications to Differential and Integral Operators, in: Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences]), vol. 337, Springer-Verlag, Berlin, 2009.
[27] V.P. Mikhailov, On Riesz bases in $L^2(0, 1)$, Dokl. Akad. Nauk SSSR 144 (1962) 981–984 (in Russian).
[28] A.M. Minkin, Equiconvergence theorems for differential operators, in: Functional Analysis, 4, J. Math. Sci. (New York) 96 (6) (1999) 3631–3715.
[29] B. Mityagin, Convergence of expansions in eigenfunctions of the Dirac operator, Dokl. Akad. Nauk 393 (4) (2003) 456–459 (in Russian).
[30] B. Mityagin, Spectral expansions of one-dimensional periodic Dirac operators, Dyn. Partial Differ. Equ. 1 (2004) 125–191.
[31] M.A. Naimark, Linear Differential Operators, Nauka, Moscow, 1969 (in Russian) (English transl.: Part 1, Elementary Theory of Linear Differential Operators, Ungar, New York, 1967; Part 2: Linear Differential Operators in Hilbert Space, Ungar, New York, 1968).
[32] A.A. Shkalikov, The basis property of eigenfunctions of an ordinary differential operator, Uspekhi Mat, Nauk 34 (1979) 235–236 (in Russian).
[33] A.A. Shkalikov, On the basisness property of eigenfunctions of ordinary differential operators with integral boundary conditions, Vestnik Mosk. Univ., Ser. 1, Math. & Mech. 6 (1982) 12–21.
[34] A.A. Shkalikov, Boundary value problems for ordinary differential equations with a parameter in the boundary conditions, Tr. Semin. im. I. G. Petrovskogo 9 (1983) 190–229 (in Russian); English transl.: J. Sov. Math. 33 (6) (1986) 1311–1342.
[35] R. Szmytkowski, Discontinuities in Dirac eigenfunction expansions, J. Math. Phys. 42 4606–4617.
[36] J.D. Tamarkin, Sur quelques points de la theorie des equations differentielles lineaires ordinaires et sur la generalisation de la serie de Fourier, Rend. Circ. Mat. Palermo (2) 34 (1912) 345–382.
[37] J.D. Tamarkin, On some general problems of the theory of ordinary linear differential operators and on expansion of arbitrary function into series, Petrograd, 1917, p. 308.
[38] J.D. Tamarkin, Some general problems of the theory of linear differential equations and expansions of an arbitrary functions in series of fundamental functions, Math. Z. 27 (1928) 1–54.
[39] I. Trooshin, M. Yamamoto, Riesz basis of root vectors of a nonsymmetric system of first-order ordinary differential operators and application to inverse eigenvalue problems, Appl. Anal. 80 (2001) 19–51.
[40] I. Trooshin, M. Yamamoto, Spectral properties and an inverse eigenvalue problem for nonsymmetric systems of ordinary differential operators, J. Inverse Ill–Posed Probl. 10 (6) (2002) 643–658.
[41] A. Zygmund, Trigonometric Series, Cambridge University Press, Cambridge, 1968.