Szegő asymptotics for matrix-valued measures with countably many bound states

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Abstract

Let μ be a matrix-valued measure with the essential spectrum a single interval and countably many point masses outside of it. Under the assumption that the absolutely continuous part of μ satisfies Szegő's condition and the point masses satisfy a Blaschke-type condition, we obtain the asymptotic behavior of the orthonormal polynomials on and off the support of the measure.

The result generalizes the scalar analogue of Peherstorfer–Yuditskii (2001) [12] and the matrix-valued result of Aptekarev–Nikishin (1983) [1], which handles only a finite number of mass points.
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1. Introduction

Let μ be an $l \times l$ matrix-valued Hermitian positive semi-definite finite measure on $\mathbb{R}$ of compact support, normalized by $\mu(\mathbb{R}) = \mathbf{1}$, where $\mathbf{1}$ is the $l \times l$ identity matrix. For any $l \times l$ dimensional matrix functions f, g, define

$$\langle\!\langle f, g \rangle\!\rangle_{L^2(\mu)} = \int f(x)^* \mathrm{d}\mu(x) g(x); \tag{1.1}$$

$$\langle\!\langle f \rangle\!\rangle_{L^2(\mu)}^2 = \langle\!\langle f, f \rangle\!\rangle_{L^2(\mu)}. \tag{1.2}$$

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Here we can regard $\langle\!\langle f \rangle\!\rangle_{L^2(\mu)}$ as the square root of the non-negative definite matrix $\langle\!\langle f, f \rangle\!\rangle_{L^2(\mu)}$. By $\langle\!\langle f, g \rangle\!\rangle_{L^2}$, with the index just L^2, we will mean the product with respect to the Lebesgue measure on the real line or the unit circle, depending on the context.

What we have defined here is the right product of f and g, as opposed to the left product $\int f(x)\mathrm{d}\mu(x)g(x)^*$, whose properties are completely analogous.

Measure μ is called non-trivial if $\|\langle\!\langle f \rangle\!\rangle^2_{L^2(\mu)}\| > 0$ for all polynomials f. From now on assume that μ is non-trivial. Then (see [4] for the details) there exist unique monic polynomials P_n of degree n satisfying

$$\langle\!\langle P_n, f \rangle\!\rangle_{L^2(\mu)} = 0 \quad \text{for any polynomial } f \text{ with } \deg f < n.$$

For any choice of unitary $l \times l$ matrices τ_n (we demand $\tau_0 = \mathbf{1}$), the polynomials $p_n = P_n \langle\!\langle P_n \rangle\!\rangle^{-1}_{L^2(\mu)} \tau_n$ are orthonormal:

$$\langle\!\langle p_n, p_m \rangle\!\rangle_{L^2(\mu)} = \delta_{n,m}\mathbf{1}.$$

We are interested in the asymptotic behavior of p_n for the measures μ whose essential support is a single interval. After scaling and translating, we can assume that it is $[-2, 2]$:

$$\text{ess supp}\mu = [-2, 2]. \tag{1.3}$$

Let $\{E_j\}_{j=1}^N$ be the point masses of μ outside $[-2, 2]$ counting multiplicities ($N \leq \infty$). In [1] it was shown that if the absolutely continuous part $f(x) = \frac{\mathrm{d}\mu(x)}{\mathrm{d}x}$ satisfies the Szegő condition

$$\int_{-2}^{2} (4 - x^2)^{-1/2} \log(\det(f(x)))\mathrm{d}x > -\infty, \tag{1.4}$$

and N is finite, then there exists $\lim_{n\to\infty} z^n p_n(z+z^{-1})$ uniformly on the compacts of $\mathbb{D}$, and the limit function was constructed more or less explicitly. The scalar case $l = 1$ (see [12]; another approach is the combination of [9,5]: see [15, Chapter 3]) suggests that $N = \infty$ should not really spoil the picture as long as the condition

$$\sum_{j=1}^{N} \left(|E_j| - 2\right)^{1/2} < \infty \tag{1.5}$$

holds. In fact this condition is necessary if one expects to have the limit $\lim_{n\to\infty} z^n p_n(z + z^{-1})$ to be a Nevanlinna function in $\mathbb{D}$.

Assume from now on that (1.3)–(1.5) hold. We prove in Theorem 1 below that under these assumptions $\lim_{n\to\infty} z^n p_n(z + z^{-1})$ exists uniformly in $\mathbb{D}$ and we give a characterization of the limit function. The results are the exact extension to the matrix-valued case of [12], and include [1, Thm. 2] as its special case ($N < \infty$).

To prove the result, Aptekarev and Nikishin in [1] used an induction on the number of the point masses of μ, which does not work if there are infinitely many of them. The approach used here is similar to the one used in [12] for the scalar case (which in turn is an extension of the original Szegő proof for the no-bound states problem; see [16]). Namely, we first construct a Nevanlinna function $L(z)$ (Section 3), and then consider a certain inner product which, when handled with care, proves that the limit of $z^n p_n(z + z^{-1})$ is indeed L (Section 4).

We start by collecting some auxiliary statements in Section 2.

2. Preliminaries

2.1. Matrix OPRL

For the proofs and additional results on the theory of matrix-valued orthogonal polynomials, see the review article of Damanik–Pushnitski–Simon [4].

Just as in the scalar case, for any choice of unitary $l \times l$ matrices τ_n ($\tau_0 = \mathbf{1}$), the orthonormal polynomials $p_n = P_n \langle\!\langle P_n \rangle\!\rangle_{L^2(\mu)}^{-1} \tau_n$ satisfy the recurrence relation

$$x p_n(x) = p_{n+1}(x) A_{n+1}^* + p_n(x) B_{n+1} + p_{n-1}(x) A_n, \quad n = 1, 2, \ldots,$$

where $A_n = \langle\!\langle p_{n-1}, x p_n \rangle\!\rangle_{L^2(\mu)}$, $B_n = \langle\!\langle p_{n-1}, x p_{n-1} \rangle\!\rangle_{L^2(\mu)}$ (with $p_{-1} = \mathbf{0}$, $A_0 = \mathbf{1}$, the relation holds for $n = 0$ too). The corresponding block Jacobi matrix is

$$J = \begin{pmatrix} B_1 & A_1 & \mathbf{0} & \cdots \\ A_1^* & B_2 & A_2 & \cdots \\ \mathbf{0} & A_2^* & B_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Two block Jacobi matrices J and $\widetilde{J}$ are called equivalent if they correspond to the same measure μ (but a different choice of τ_n's). They are equivalent if and only if their Jacobi parameters satisfy

$$\widetilde{A}_n = \sigma_n^* A_n \sigma_{n+1}, \qquad \widetilde{B}_n = \sigma_n^* B_n \sigma_n \tag{2.1}$$

for unitary σ_n's with $\sigma_1 = \mathbf{1}$ (the connection with τ_j's is $\sigma_n = \tau_{n-1}^* \widetilde{\tau}_{n-1}$). It is easy to see that

$$\widetilde{p}_n(x) = p_n(x) \sigma_{n+1}, \tag{2.2}$$

where $\widetilde{p}_n$ are the orthonormal polynomials for $\widetilde{J}$.

We say that a block Jacobi matrix is of type 1 if $A_n > 0$ for all n, of type 2 if $A_1 A_2 \ldots A_n > 0$ for all n, and of type 3 if every A_n is lower triangular with positive entries on the diagonal. Each equivalence class of block Jacobi matrices contains exactly one matrix of type 1, 2 and 3.

Define the m-function of the measure μ to be the meromorphic in $\mathbb{C} \setminus [-2, 2]$ matrix-valued function $m(z) = \int \frac{d\mu(x)}{x-z}$. Define also

$$M(z) = -m(z + z^{-1}), \quad z \in \mathbb{D}.$$

Just as in the scalar case, one easily sees that

$$\operatorname{Im} M(e^{i\theta}) = \pi f(2 \cos \theta), \quad 0 \le \theta \le \pi,$$
$$\operatorname{Im} M(e^{i\theta}) = -\pi f(2 \cos \theta), \quad -\pi \le \theta \le 0,$$

where $\operatorname{Im} T \equiv \frac{T - T^*}{2i}$.

Denote

$$\{z_k\}_{k=1}^N = \left\{ z \in \mathbb{D} \mid z = \frac{1}{2} \left(E_k - \sqrt{E_k^2 - 4} \right) \right\} = \left\{ z \in \mathbb{D} \mid z + z^{-1} = E_k \right\}, \tag{2.3}$$

enumerated in increasing order of their absolute values ($N \leq \infty$). Let us assume that each z_k is different, and let m_k be the multiplicity of $z_k + z_k^{-1}$ as the eigenvalue. Then (1.5) implies

$$\sum_{k=1}^{N} m_k \log |z_k| > -\infty. \tag{2.4}$$

We will be using the so-called C_0 Sum Rule from [3]. In a slightly changed form, it looks as follows.

Proposition 1 (*Damanik–Killip–Simon [3]*). *Suppose* ess supp$\mu = [-2, 2]$ *and* $\{z_k\}_{k=1}^{N}$ *be as in* (2.3). *Let*

$$\mathcal{Z}(J) = -\frac{1}{2} \int_{-\pi}^{\pi} \log \det \frac{\operatorname{Im} M(e^{i\theta})}{\sin \theta} \frac{d\theta}{2\pi},$$

$$\mathcal{E}_0(J) = -\sum_{k=1}^{N} m_k \log |z_k|,$$

$$\mathcal{A}_0(J) = -\lim_{n \to \infty} \sum_{j=1}^{n} \log \det |A_j|.$$

If any two of $\mathcal{Z}, \mathcal{E}_0, \mathcal{A}_0$ *are finite, then so is the third, and*

$$\mathcal{Z}(J) = \mathcal{E}_0(J) + \mathcal{A}_0(J).$$

Remarks. 1. Here $|T| \equiv \sqrt{T^*T}$.

2. The minus in the expression for $\mathcal{E}_0(J)$ comes from the fact that we chose $z \in \mathbb{D}$ in (2.3) as opposed to $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ in [3].

Note that $z^n p_n(z + z^{-1})$ at $z = 0$ is equal to $\kappa_n = \left(A_1^*\right)^{-1} \ldots \left(A_n^*\right)^{-1} > 0$. Since in our case $\mathcal{E}_0(J) < \infty$ (see (2.4)) and $\mathcal{Z}(J) < \infty$ (follows from (1.4)), this proposition allows us to find the limit of the determinants of $z^n p_n(z + z^{-1})$ at $z = 0$. We will see later that this limit is equal to $\det L(0)$. This will be helpful in showing that $z^n p_n(z + z^{-1})$ at $z = 0$ converges to $L(0)$, and this in turn will allow us to show the uniform convergence in $\mathbb{D}$.

2.2. Properties of the matrix product

We will need a couple of facts about the product we defined in (1.1).

Lemma 1. *Let* $L^2(\mathbf{1}\frac{d\theta}{2\pi})$ *be the space of all matrix-valued functions, each entry of which is a scalar* $L^2(\frac{d\theta}{2\pi})$*-function.*

(a) *The following formulae*

$$\|f\|_{L^2,1} \equiv \left(\int_{-\pi}^{\pi} \|f(\theta)\|^2 \frac{d\theta}{2\pi} \right)^{1/2},$$

$$\|f\|_{L^2,2} \equiv \left\| \int_{-\pi}^{\pi} f(\theta)^* f(\theta) \frac{d\theta}{2\pi} \right\|^{1/2} = \|\langle\!\langle f, f \rangle\!\rangle_{L^2}\|^{1/2}$$

define two equivalent (semi)norms on $L^2(\mathbf{1}\frac{d\theta}{2\pi})$:

$$\|f\|_{L^2,2} \leq \|f\|_{L^2,1} \leq l^{1/2} \|f\|_{L^2,2}.$$

(b) *For any* $f, g \in L^2$,

$$\left\| \langle\!\langle f, g \rangle\!\rangle_{L^2} \right\| \le l \|f\|_{L^2,2} \|g\|_{L^2,2}.$$

(c) *If* $f \in L^2(\mathbf{1}\frac{d\theta}{2\pi})$, *then its nth matrix Fourier coefficient* $\langle\!\langle \mathrm{e}^{\mathrm{i}n\theta} I, f \rangle\!\rangle_{L^2} \to \mathbf{0}$ *as* $n \to \infty$.

Proof. (a) The first inequality is obvious. The second follows from

$$\|f\|_{L^2,1}^2 = \int_{-\pi}^{\pi} \|f(\theta)\|^2 \frac{d\theta}{2\pi} \le \int_{-\pi}^{\pi} \mathrm{Tr}(f(\theta)^* f(\theta)) \frac{d\theta}{2\pi}$$

$$= \mathrm{Tr}\left(\int_{-\pi}^{\pi} f(\theta)^* f(\theta) \frac{d\theta}{2\pi} \right) \le l \|f\|_{L^2,2}^2.$$

(b) Using Hölder, and the equivalence from (a), we get

$$\left\| \langle\!\langle f, g \rangle\!\rangle_{L^2} \right\| \le \int_{-\pi}^{\pi} \|g(\theta)\| \, \|f(\theta)\| \frac{d\theta}{2\pi} \le \|f\|_{L^2,1} \|g\|_{L^2,1} \le l \|f\|_{L^2,2} \|g\|_{L^2,2}.$$

(c) Follows by looking at each entry separately. $\square$

Just as in the scalar case (see [12]) we will be expecting the limit function to have the factorized form of a product of a matrix outer factor and a matrix Blaschke product.

2.3. Matrix outer functions

Recall that a scalar analytic function G on $\mathbb{D}$ is called outer if it can be recovered from its boundary values $G(\mathrm{e}^{\mathrm{i}\theta}) \equiv \lim_{r \nearrow 1} G(r\mathrm{e}^{\mathrm{i}\theta})$ by the formula

$$G(z) = c \exp \left\{ \int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}\theta} + z}{\mathrm{e}^{\mathrm{i}\theta} - z} \log |G(\mathrm{e}^{\mathrm{i}\theta})| \frac{d\theta}{2\pi} \right\} \tag{2.5}$$

for some constant $|c| = 1$. Note that it is necessary and sufficient $\log |G(\mathrm{e}^{\mathrm{i}\theta})|$ to be integrable.

Lemma 2 (*Wiener–Masani [17]*)**.** *Suppose* $w(\theta)$ *is a non-negative matrix-valued function on the unit circle satisfying*

$$\int_{-\pi}^{\pi} \log \det w(\theta) \frac{d\theta}{2\pi} > -\infty.$$

Then there exists a unique matrix-valued $H_2(\mathbb{D})$ *function* $G(z)$ *satisfying*

$$G(\mathrm{e}^{\mathrm{i}\theta})^* G(\mathrm{e}^{\mathrm{i}\theta}) = w(\theta), \tag{2.6}$$

$$G(0)^* = G(0) > 0, \tag{2.7}$$

$$\log |\det G(0)| = \int_{-\pi}^{\pi} \log |\det G(\mathrm{e}^{\mathrm{i}\theta})| \frac{d\theta}{2\pi}. \tag{2.8}$$

This is a well-known result of Wiener–Masani [17]. The proof of the uniqueness part can be found, e.g., in [6].

Equality (2.8) implies (see [14, Section 17.17]) that $\det G(z)$ is a scalar outer function, which implies (by definition) that $G(z)$ is a matrix-valued outer function. It follows from [8, Thm. 2] that there exists a Hermitian matrix-valued integrable function $M(\theta)$ such that

$$\mathrm{Tr}\, M(\theta) = \log |\det G(\mathrm{e}^{\mathrm{i}\theta})| \tag{2.9}$$

and

$$G(z) = \rho \int_{-\pi}^{\overset{\frown}{\pi}} \exp\left\{ \frac{e^{i\theta} + z}{e^{i\theta} - z} M(\theta) \frac{d\theta}{2\pi} \right\}, \tag{2.10}$$

where $\int_{-\pi}^{\overset{\frown}{\pi}}$ is the Potapov multiplicative integral (see [13])

$$\int_{-\pi}^{\overset{\frown}{\pi}} \exp\left\{ F(\theta) \frac{d\theta}{2\pi} \right\} = \lim_{\Delta\theta_j \to 0} \overset{\overset{\frown}{n-1}}{\prod_{j=0}} e^{F(\phi_j)\Delta\theta_j},$$

$$-\pi = \theta_0 \leq \phi_0 \leq \theta_1 \leq \phi_1 \leq \cdots \leq \theta_{n-1} \leq \phi_{n-1} \leq \theta_n = \pi.$$

The arrow above the product sign simply defines the order of the multiplication in the matrix-valued product. ρ in (2.10) is a constant unitary matrix which makes the right-hand side of (2.10) positive definite.

Clearly (2.9) and (2.10) becomes (2.5) if $l = 1$.

2.4. Blaschke–Potapov products

The Blaschke–Potapov elementary factor is a generalization of scalar Blaschke factors (for those familiar with the Potapov theory of J-contractive matrix functions: we are considering the signature matrix J to be just the identity matrix $\mathbf{1}$):

$$B_{z_j, s, U}(z) = U^* \begin{pmatrix} \frac{|z_j|}{z_j} \frac{z_j - z}{1 - z_j z} & 0 & 0 & & 0 & \cdots & 0 \\ 0 & \ddots & 0 & & 0 & \cdots & 0 \\ 0 & 0 & \frac{|z_j|}{z_j} \frac{z_j - z}{1 - z_j z} & 0 & \cdots & 0 \\ 0 & 0 & 0 & & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \ddots & \\ 0 & 0 & 0 & & 0 & \cdots & 1 \end{pmatrix} U, \quad z \in \mathbb{D},$$

where $z_j \in \mathbb{D}$, s is the number of the scalar Blaschke factors on the diagonal ($0 \leq s \leq l$), and U is a unitary constant matrix. Clearly $B_{z_j, s, U}$ is an analytic in $\mathbb{D}$ function with unitary values on the unit circle.

The well-known result for the convergence of the scalar Blaschke products is still valid for the matrix-valued case: if

$$\sum_{k=1}^{\infty} (1 - |z_k|) < \infty,$$

then the product

$$\overset{\overset{\frown}{\infty}}{\prod_{j=1}} B_{z_j, s_j, U_j}(z)$$

converges uniformly on the compacts of the unit disk (see [13,7] where this is proven even more generally for the operator-valued setting). The limit function is holomorphic in $\mathbb{D}$ with unitary boundary values (see [2]).

We have freedom here in the choice of the unitary matrices U_j and numbers s_j. We will make use of it in the following lemma.

Lemma 3. *Let $\{z_k\}_{k=1}^{\infty}$ with $\sum_{k=1}^{\infty}(1 - |z_k|) < \infty$ be given, with all z_k pairwise different. For any sequence of subspaces $V_k \subset \mathbb{C}^l$, there exists a unique product $B(z) = \prod_{j=1}^{\infty} B_{z_j,s_j,U_j}(z)$ for some choice of numbers s_k, $0 \le s_k \le l$, and unitary matrices U_k, that satisfies*

$$\ker \operatorname*{res}_{z=z_k} B(z)^{-1} = V_k \quad \text{for all } k. \tag{2.11}$$

Proof. Easy induction does the job. Let I_m ($0 \le m \le l$) be the diagonal $l \times l$ matrix with first m diagonal elements 1 and the rest 0, and $B_n(z) = \prod_{j=1}^{n} B_{z_j,s_j,U_j}(z)$ be the partial finite product. Assume that we already chose $\{s_k\}_{k=1}^{n-1}$ and $\{U_k\}_{k=1}^{n-1}$ so that $B_{n-1}(z)$ satisfies

$$\ker \operatorname*{res}_{z=z_k} B_{n-1}(z)^{-1} = V_k, \quad 1 \le k \le n-1.$$

Observe that this implies (2.11) holds for $1 \le k \le n-1$ as well. Put $s_n = l - \dim V_n$. Note that

$$B_{z_n,s_n,U_n}(z) = U_n^* \left(\frac{|z_n|}{z_n} \frac{z_n - z}{1 - zz_n} I_{s_n} + (I - I_{s_n}) \right) U_n \tag{2.12}$$

and

$$\ker \operatorname*{res}_{z=z_n} B(z)^{-1} = \ker \operatorname*{res}_{z=z_n} B_n(z)^{-1} = \ker I_{s_n} U_n B_{n-1}(z_n)^{-1}.$$

Note that $B_{n-1}(z_n)$ is invertible (as $z_n \notin \{z_1, \ldots, z_{n-1}\}$), so we can put U_n to be any unitary matrix taking the subspace $B_{n-1}(z_n)^{-1} V_n$ to $\langle \delta_{s_n+1} \cdots \delta_l \rangle$. Note that the choice of U_n is not unique, but the factor B_{z_n,s_n,U_n} is uniquely defined. $\square$

3. Construction of the limit function

Let J be the type 2 Jacobi matrix corresponding to μ, and let p_n be the orthonormal polynomials for J.

Let $\nu = \operatorname{Sz}(\mu|_{[-2,2]})$ be the image measure on $\partial\mathbb{D}$ of $\mu|_{[-2,2]}$ under $\theta \mapsto 2\cos\theta$: $\int_{-\pi}^{\pi} g(2\cos\theta)\mathrm{d}\nu(\theta) = \int_{-2}^{2} g(x)\mathrm{d}\mu(x)$ for measurable g's. This is what is called the Szegő mapping. Let the Lebesgue decomposition of ν be

$$\mathrm{d}\nu(\theta) = w(\theta)\frac{\mathrm{d}\theta}{2\pi} + \mathrm{d}\nu_s.$$

Then

$$w(\theta) = 2\pi|\sin\theta| f(2\cos\theta), \tag{3.1}$$

and so (1.4) implies

$$\int_{-\pi}^{\pi} \log \det w(\theta)\frac{\mathrm{d}\theta}{2\pi} > -\infty.$$

Therefore Lemma 2 applies, so there exists a matrix-valued outer $H_2(\mathbb{D})$-function $G(z)$ such that

$$w(\theta) = G(\mathrm{e}^{\mathrm{i}\theta})^* G(\mathrm{e}^{\mathrm{i}\theta}), \tag{3.2}$$

$$G(0)^* = G(0) > 0, \tag{3.3}$$

$$\log|\det G(0)| = \int_{-\pi}^{\pi} \log|\det G(e^{i\theta})|\frac{d\theta}{2\pi}. \tag{3.4}$$

Denote w_k to be the weight of μ at $z_k + z_k^{-1}$:

$$w_k = \mu(z_k + z_k^{-1}).$$

Now apply Lemma 3 to obtain the Blaschke–Potapov product $B(z) = \prod_{j=1}^{\infty} B_{z_j,s_j,U_j}(z)$ (by (2.4) it converges) satisfying

$$\ker \operatorname*{res}_{z=z_k} \left(B(z)^{-1} G(z) \right) = \ker w_k \quad \text{for all } k. \tag{3.5}$$

Indeed, note that $G(z_k)$ is invertible for any k (since $\det G$ is outer, it cannot vanish in $\mathbb{D}$), so we can apply Lemma 3 with $V_k = G(z_k)^{-1} \ker w_k$.

Define for $z \in \mathbb{D}$,

$$L(z) = \frac{1}{\sqrt{2}} G(z)^{-1} B(z) V, \tag{3.6}$$

where V is a constant unitary such that $L(0) > 0$.

Now we can formulate the main result of the paper.

Theorem 1. *Let μ satisfy (1.3)–(1.5). Assume that J is of type 2, and let $\widetilde{J}$ be any equivalent to it matrix with Jacobi parameters (2.1) and orthonormal polynomials $\widetilde{p}_n$ (2.2). Assume that $\sigma = \lim_{n\to\infty} \sigma_n$ exists. Then*

$$z^n \widetilde{p}_n \left(z + z^{-1} \right) \to L(z)\sigma \quad \text{uniformly on compacts of } \mathbb{D}; \tag{3.7}$$

$$\widetilde{p}_n(2\cos\theta) = \frac{1}{\sqrt{2}} \left(e^{-in\theta} L(e^{i\theta}) + e^{in\theta} L(e^{-i\theta}) \right) \sigma + o(1) \quad \text{in } L^2 \left(w(\theta)\frac{d\theta}{2\pi} \right) \text{ sense; } \tag{3.8}$$

$$\langle\!\langle \widetilde{p}_n(x) \rangle\!\rangle_{L^2(\mu_s)} \to \mathbf{0}, \tag{3.9}$$

where w is defined in (3.1).

The limit function L has a factorization (3.6), where G is the unique $H_2(\mathbb{D})$-function satisfying (3.2)–(3.4) (and thus has the form (2.9)–(2.10)), B is a Blaschke–Potapov product, V is a unitary matrix. We have

$$\ker \operatorname*{res}_{z=z_k} L(z)^{-1} = \ker w_k \quad \text{for all } k;$$

$$L(0) > 0.$$

Remarks. 1. We will show that the asymptotics holds for type 2 Jacobi matrix. Thus by (2.2), the polynomials $\widetilde{p}_n$ obey Szegő asymptotics if and only if the limit $\lim_{n\to\infty} \sigma_n$ exists, so this condition is also necessary.

2. The equivalent way of writing (3.8) is

$$G(e^{i\theta})\widetilde{p}_n(2\cos\theta) = \frac{1}{\sqrt{2}} \left(e^{-in\theta} B(e^{i\theta}) + e^{in\theta} G(e^{i\theta}) G(e^{-i\theta})^{-1} B(e^{-i\theta}) \right) V\sigma + o(1)$$

$$\text{in } L^2 \left(\mathbf{1}\frac{d\theta}{2\pi} \right) \text{ sense.}$$

Using results from Section 14 of [3] we immediately obtain

Corollary 1. *Assume that the Jacobi parameters of J satisfy*

$$\sum_{n=1}^{\infty} \left[\|1 - A_n A_n^*\| + \|B_n\| \right] < \infty. \tag{3.10}$$

Then the associated measure μ satisfies (1.3)–(1.5)*, and so the conclusions of Theorem 1 hold.*

Remarks. 1. As in Theorem 1 this establishes Szegő asymptotics for the type 2 Jacobi matrix, and for any equivalent to it matrix for which the limit $\lim_{n\to\infty} \sigma_n$ exists. In [10] it is shown that under (3.10) this limit does exist for matrices of type 1 and 3 (or more generally, for any $\widetilde{J}$ the $\widetilde{A}_n$-coefficients of which have eventually only real eigenvalues).

2. See also [11] for another proof of Corollary 1.

4. Proof

The beginning of the proof follows closely the proof of the lemma in [12]. Denote

$$s(e^{i\theta}) = G(e^{i\theta})G(e^{-i\theta})^{-1},$$

and consider the following expression. Expanding the product and using (3.2), we get

$$
\begin{aligned}
\mathbf{0} &\leq \left\langle\!\!\left\langle G(e^{i\theta})\widetilde{p}_n(2\cos\theta) - \frac{1}{\sqrt{2}}\left(e^{-in\theta}B(e^{i\theta}) + e^{in\theta}s(e^{i\theta})B(e^{-i\theta}) \right) \right\rangle\!\!\right\rangle_{L^2}^2 + \langle\!\langle \widetilde{p}_n(x) \rangle\!\rangle_{L^2(\mu_s)}^2 \\
&= \int_{-\pi}^{\pi} \widetilde{p}_n(2\cos\theta)^* w(\theta) \widetilde{p}_n(2\cos\theta) \frac{d\theta}{2\pi} \\
&\quad + \langle\!\langle \widetilde{p}_n(x) \rangle\!\rangle_{L^2(\mu_s)}^2 + \frac{1}{2}\langle\!\langle e^{-in\theta}B(e^{i\theta}) + e^{in\theta}s(e^{i\theta})B(e^{-i\theta}) \rangle\!\rangle_{L^2}^2 \\
&\quad - \sqrt{2}\,\mathrm{Re}\,\langle\!\langle G(e^{i\theta})\widetilde{p}_n(2\cos\theta), e^{-in\theta}B(e^{i\theta}) + e^{in\theta}s(e^{i\theta})B(e^{-i\theta}) \rangle\!\rangle_{L^2}, \tag{4.1}
\end{aligned}
$$

where by $\mathrm{Re}\,T$ we mean $\frac{T+T^*}{2}$.

First of all,

$$\int_{-\pi}^{\pi} \widetilde{p}_n(2\cos\theta)^* w(\theta) \widetilde{p}_n(2\cos\theta) \frac{d\theta}{2\pi} + \langle\!\langle \widetilde{p}_n(x) \rangle\!\rangle_{L^2(\mu_s)}^2 = \langle\!\langle \widetilde{p}_n(x) \rangle\!\rangle_{L^2(\mu)}^2 = \mathbf{1}. \tag{4.2}$$

Now, observe that

$$
\begin{aligned}
s(e^{i\theta})^* s(e^{i\theta}) &= G(e^{-i\theta})^{-*} G(e^{i\theta})^* G(e^{i\theta}) G(e^{-i\theta})^{-1} \\
&= G(e^{-i\theta})^{-*} w(\theta) G(e^{-i\theta})^{-1} = G(e^{-i\theta})^{-*} w(-\theta) G(e^{-i\theta})^{-1} = \mathbf{1}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{1}{2}\langle\!\langle e^{-in\theta}B(e^{i\theta}) + e^{in\theta}s(e^{i\theta})B(e^{-i\theta}) \rangle\!\rangle_{L^2}^2 &= \mathbf{1} + \mathrm{Re}\,\langle\!\langle e^{-in\theta}B(e^{i\theta}), e^{in\theta}s(e^{i\theta})B(e^{-i\theta}) \rangle\!\rangle_{L^2} \\
&= \mathbf{1} + \int_{-\pi}^{\pi} e^{2in\theta} B(e^{i\theta})^* s(e^{i\theta}) B(e^{-i\theta}) \frac{d\theta}{2\pi} \\
&= \mathbf{1} + o(1) \tag{4.3}
\end{aligned}
$$

since the function $k(\theta) = B(e^{i\theta})^* s(e^{i\theta}) B(e^{-i\theta})$ satisfies $\int_{-\pi}^{\pi} k(\theta)^* k(\theta) \frac{d\theta}{2\pi} = \mathbf{1}$, so by parts (a) and (c) of Lemma 1, its Fourier coefficients converge to the zero matrix.

Note that for any function g on the unit circle we have

$$
\langle\!\langle G(\mathrm{e}^{\mathrm{i}\theta})\widetilde{p}_n(2\cos\theta), s(\mathrm{e}^{\mathrm{i}\theta})g(\mathrm{e}^{-\mathrm{i}\theta})\rangle\!\rangle_{L^2}
$$
$$
=\int_{-\pi}^{\pi}\widetilde{p}_n(2\cos\theta)^* G(\mathrm{e}^{\mathrm{i}\theta})^* G(\mathrm{e}^{\mathrm{i}\theta})G(\mathrm{e}^{-\mathrm{i}\theta})^{-1}g(\mathrm{e}^{-\mathrm{i}\theta})\frac{\mathrm{d}\theta}{2\pi}
$$
$$
=\int_{-\pi}^{\pi}\widetilde{p}_n(2\cos\theta)^* w(\theta)G(\mathrm{e}^{-\mathrm{i}\theta})^{-1}g(\mathrm{e}^{-\mathrm{i}\theta})\frac{\mathrm{d}\theta}{2\pi}
$$
$$
=\int_{-\pi}^{\pi}\widetilde{p}_n(2\cos\theta)^* w(-\theta)G(\mathrm{e}^{-\mathrm{i}\theta})^{-1}g(\mathrm{e}^{-\mathrm{i}\theta})\frac{\mathrm{d}\theta}{2\pi}
$$
$$
=\int_{-\pi}^{\pi}\widetilde{p}_n(2\cos\theta)^* G(\mathrm{e}^{-\mathrm{i}\theta})^* g(\mathrm{e}^{-\mathrm{i}\theta})\frac{\mathrm{d}\theta}{2\pi}
$$
$$
=\int_{-\pi}^{\pi}\widetilde{p}_n(2\cos\theta)^* G(\mathrm{e}^{\mathrm{i}\theta})^* g(\mathrm{e}^{\mathrm{i}\theta})\frac{\mathrm{d}\theta}{2\pi}
$$
$$
=\langle\!\langle G(\mathrm{e}^{\mathrm{i}\theta})\widetilde{p}_n(2\cos\theta), g(\mathrm{e}^{\mathrm{i}\theta})\rangle\!\rangle_{L^2},
$$

so the third term on the right-hand side of (4.1) becomes

$$
\mathrm{Re}\,\langle\!\langle G(\mathrm{e}^{\mathrm{i}\theta})\widetilde{p}_n(2\cos\theta), \mathrm{e}^{-\mathrm{i}n\theta}B(\mathrm{e}^{\mathrm{i}\theta})+\mathrm{e}^{\mathrm{i}n\theta}s(\mathrm{e}^{\mathrm{i}\theta})B(\mathrm{e}^{-\mathrm{i}\theta})\rangle\!\rangle_{L^2}
$$
$$
=2\,\mathrm{Re}\,\langle\!\langle G(\mathrm{e}^{\mathrm{i}\theta})\widetilde{p}_n(2\cos\theta), \mathrm{e}^{-\mathrm{i}n\theta}B(\mathrm{e}^{\mathrm{i}\theta})\rangle\!\rangle_{L^2}. \tag{4.4}
$$

Lemma 4. *Let* $\widetilde{p}_n(x)=r_n x^n+\cdots$ *(in other words,* $r_n=(\widetilde{A}_1^*)^{-1}\cdots(\widetilde{A}_n^*)^{-1}$*). Then* r_n *are uniformly bounded (with respect to the operator norm).*

Proof. On the one hand, by (4.2),

$$
\left\|G(\mathrm{e}^{\mathrm{i}\theta})\widetilde{p}_n(2\cos\theta)\right\|_{L^2,2}\le\|\langle\!\langle\widetilde{p}_n(x)\rangle\!\rangle_{L^2(\mu)}^2\|^{1/2}=1. \tag{4.5}
$$

On the other hand, by Lemma 1(a) and subharmonicity of $\|h(\cdot)\|^2$,

$$
\left\|G(\mathrm{e}^{\mathrm{i}\theta})\widetilde{p}_n(2\cos\theta)\right\|_{L^2,2}\ge l^{-1/2}\left\|G(\mathrm{e}^{\mathrm{i}\theta})\widetilde{p}_n(2\cos\theta)\right\|_{L^2,1}
$$
$$
=l^{-1/2}\left(\int_{-\pi}^{\pi}\left\|h(\mathrm{e}^{\mathrm{i}\theta})\right\|^2\frac{\mathrm{d}\theta}{2\pi}\right)^{1/2}
$$
$$
\ge l^{-1/2}\|h(0)\|=l^{-1/2}\|G(0)r_n\|,
$$

where $h(z)\equiv G(z)\left[z^n\widetilde{p}_n\left(z+\frac{1}{z}\right)\right]$ is analytic in $\mathbb{D}$. $G(0)$ is invertible, so r_n are uniformly bounded. $\square$

The next lemma will allow us to compute the right-hand side of (4.4).

Lemma 5. *Let* r_n *be as in the previous lemma. Then*

$$
\langle\!\langle\mathrm{e}^{-\mathrm{i}n\theta}B(\mathrm{e}^{\mathrm{i}\theta}), G(\mathrm{e}^{\mathrm{i}\theta})\widetilde{p}_n(2\cos\theta)\rangle\!\rangle_{L^2}=B(0)^{-1}G(0)r_n+o(1). \tag{4.6}
$$

Proof. The partial products $B_N(z)$ converge to $B(z)$ uniformly on compacts of $\mathbb{D}$. This implies that each Fourier coefficient of $B(\mathrm{e}^{\mathrm{i}\theta})-B_N(\mathrm{e}^{\mathrm{i}\theta})$ goes to 0 as $N\to\infty$. Since $\|B\|_{L^2,2}=\|B_N\|_{L^2,2}=1$, weak convergence implies the norm convergence $\|B(\mathrm{e}^{\mathrm{i}\theta})-B_N(\mathrm{e}^{\mathrm{i}\theta})\|_{L^2,2}\to 0$.

Using (4.5) and Lemma 1(b), we can find $N \in \mathbb{N}$ such that

$$\left\| \langle\!\langle \mathrm{e}^{-\mathrm{i}n\theta}(B(\mathrm{e}^{\mathrm{i}n\theta}) - B_N(\mathrm{e}^{-\mathrm{i}n\theta})), G(\mathrm{e}^{\mathrm{i}\theta})\widetilde{p}_n(2\cos\theta)\rangle\!\rangle_{L^2} \right\|$$

$$\leq l\|B(\mathrm{e}^{\mathrm{i}\theta}) - B_N(\mathrm{e}^{\mathrm{i}\theta})\|_{L^2,2}\|G(\mathrm{e}^{\mathrm{i}\theta})\widetilde{p}_n(2\cos\theta)\|_{L^2,2} < \epsilon \tag{4.7}$$

holds for any $n \in \mathbb{N}$. By Lemma 4, we can also assume that for this N,

$$\|B(0)^{-1}G(0)r_n - B_N(0)^{-1}G(0)r_n\| < \epsilon \tag{4.8}$$

also holds for any n. Now, $B_N(\mathrm{e}^{\mathrm{i}\theta})^* = B_N(\mathrm{e}^{\mathrm{i}\theta})^{-1}$, so

$$\langle\!\langle \mathrm{e}^{-\mathrm{i}n\theta}B_N(\mathrm{e}^{\mathrm{i}\theta}), G(\mathrm{e}^{\mathrm{i}\theta})\widetilde{p}_n(2\cos\theta)\rangle\!\rangle_{L^2} = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}n\theta}B_N(\mathrm{e}^{\mathrm{i}\theta})^{-1}G(\mathrm{e}^{\mathrm{i}\theta})\widetilde{p}_n(2\cos\theta)\frac{\mathrm{d}\theta}{2\pi}$$

$$= \int_{\partial\mathbb{D}} B_N(z)^{-1}G(z)\widetilde{p}_n\left(z + \frac{1}{z}\right)z^n\frac{\mathrm{d}z}{2\pi\mathrm{i}z}$$

$$= B_N(0)^{-1}G(0)r_n + \sum_{k=1}^{N}\operatorname*{res}_{z=z_k}(B_N(z)^{-1}G(z))\widetilde{p}_n(E_k)z_k^{n-1} \tag{4.9}$$

By the construction, (3.5) holds, which implies $\ker \operatorname*{res}_{z=z_k}(B_N(z)^{-1}G(z)) = \ker w_k = \ker w_k^{1/2}$, which allows us to write $\operatorname*{res}_{z=z_k}(B_N(z)^{-1}G(z)) = S_k w_k^{1/2}$ for some matrix S_k. Thus,

$$\left\| \sum_{k=1}^{N}\operatorname*{res}_{z=z_k}(B_N(z)^{-1}G(z))\widetilde{p}_n(E_k)z_k^{n-1} \right\| \leq \sup_{1\leq k\leq N}\|S_k\|\sum_{k=1}^{N}\|w_k^{1/2}\widetilde{p}_n(E_k)\|\,|z_k|^{n-1}. \tag{4.10}$$

But $\|w_k^{1/2}\widetilde{p}_n(E_k)\| = (\|\widetilde{p}_n(E_k)^* w_k \widetilde{p}_n(E_k)\|)^{1/2} \leq \|\langle\!\langle \widetilde{p}_n(x)\rangle\!\rangle_{L^2(\mu)}\| = 1$. Since N was fixed, this proves that the right-hand side of (4.10) goes to 0 when $n \to \infty$. Combining (4.7)–(4.10), we obtain (4.6). $\square$

Now, plugging (4.2)–(4.4) and (4.6) into (4.1), we obtain

$$\mathbf{0} \leq 2\mathbf{1} - 2\sqrt{2}\operatorname{Re}\left(B(0)^{-1}G(0)r_n\right) + o(1). \tag{4.11}$$

Observe that (4.11) holds for any initial choice of unitaries σ_n in (2.1). Let $p_n(x) = \kappa_n x^n + \cdots$ (in other words, $\kappa_n = (A_1^*)^{-1}\cdots(A_n^*)^{-1} > 0$). Then (2.2) gives $r_n = \kappa_n\sigma_{n+1}$. For each n, pick unitary σ_{n+1} such that $B(0)^{-1}G(0)r_n = B(0)^{-1}G(0)\kappa_n\sigma_{n+1}$ is positive definite. Then (4.11) gives

$$\sqrt{2}\,B(0)^{-1}G(0)\kappa_n\sigma_{n+1} \leq \mathbf{1} + o(1). \tag{4.12}$$

Denote $H_n \equiv \sqrt{2}\,B(0)^{-1}G(0)r_n > \mathbf{0}$. Let $\{\eta_s^{(n)}\}_{s=1}^{l}$ be the eigenvalues of H_n in non-increasing order. $\eta_s^{(n)} > 0$ for any n, s. Then (4.12) implies

$$\limsup_{n\to\infty}\eta_s^{(n)} \leq 1 \tag{4.13}$$

for each $s = 1, \ldots, l$.

On the other hand, let us compute the determinant of H_n. By (2.12) and (2.8),

$$\log\det B(0)^{-1}G(0) = -\sum_{k}m_k\log|z_k| + \int_{-\pi}^{\pi}\log|\det G(\mathrm{e}^{\mathrm{i}\theta})|\frac{\mathrm{d}\theta}{2\pi},$$

and by Proposition 1,

$$
\lim_{n\to\infty}\sum_{j=1}^{n}\log\det|\widetilde{A}_j| + \sum_{k}m_k\log|z_k|
$$
$$
= \frac{1}{2}\int_{-\pi}^{\pi}\log\det\frac{\operatorname{Im}M(e^{i\theta})}{\sin\theta}\frac{d\theta}{2\pi} = \frac{1}{2}\int_{-\pi}^{\pi}\log\det\frac{\pi f(2\cos\theta)}{|\sin\theta|}\frac{d\theta}{2\pi}
$$
$$
= \frac{1}{2}\int_{-\pi}^{\pi}\log\det\frac{w(\theta)}{2\sin^2\theta}\frac{d\theta}{2\pi} = \int_{-\pi}^{\pi}\log|\det G(e^{i\theta})|\frac{d\theta}{2\pi} - \frac{l}{4\pi}\int_{-\pi}^{\pi}\log\left(2\sin^2\theta\right)d\theta
$$
$$
= \int_{-\pi}^{\pi}\log|\det G(e^{i\theta})|\frac{d\theta}{2\pi} + \frac{l}{2}\log 2. \qquad (4.14)
$$

Now note that

$$
\log\det r_n = -\sum_{j=1}^{n}\log\det\widetilde{A}_j^* = -\sum_{j=1}^{n}\log\det|\widetilde{A}_j| + \log\det\rho_n
$$

for some unitary matrix ρ_n. However, $H_n > 0$, so $\det\rho_n$ must be 1 as otherwise $\log\det H_n = \frac{l}{2}\log 2 + \log\det B(0)^{-1}G(0) + \log\det r_n$ cannot be real. Thus we obtain

$$
\log\det H_n = \frac{l}{2}\log 2 + \log\det B(0)^{-1}G(0) + \log\det r_n
$$
$$
= \frac{l}{2}\log 2 - \sum_{k}m_k\log|z_k| + \int_{-\pi}^{\pi}\log|\det G(e^{i\theta})|\frac{d\theta}{2\pi} - \sum_{j=1}^{n}\log\det|\widetilde{A}_j| \to 0
$$

by (4.14). Thus $\lim_{n\to\infty}\det H_n = 1$. Together with (4.13) this implies $\lim_{n\to\infty}\eta_s^{(n)} = 1$ for each s, and so $H_n \to \mathbf{1}$. This proves $\kappa_n\sigma_{n+1} \to 2^{-1/2}G(0)^{-1}B(0)$. But $|\kappa_n\sigma_{n+1}| = \kappa_n$ (here temporarily $|T| \equiv \sqrt{TT^*}$ instead of $\sqrt{T^*T}$), so

$$
\kappa_n \to 2^{-1/2}\left|G(0)^{-1}B(0)\right| = L(0).
$$

Also, $\sigma_n \to V^*$.

Thus for the chosen σ's, the right-hand side of (4.11) goes to the zero matrix. This implies

$$
\left\|G(e^{i\theta})\widetilde{p}_n(2\cos\theta) - \frac{1}{\sqrt{2}}\left(e^{-in\theta}B(e^{i\theta}) + e^{in\theta}s(e^{i\theta})B(e^{-i\theta})\right)\right\|_{L^2,2} \to 0
$$

and

$$
\langle\!\langle\widetilde{p}_n(x)\rangle\!\rangle^2_{L^2(\mu_s)} \to \mathbf{0}.
$$

Taking into account that $p_n(x) = \widetilde{p}_n(x)\sigma_{n+1}^*$ and $\sigma_n \to V^*$, we get

$$
\left\|G(e^{i\theta})p_n(2\cos\theta) - \frac{1}{\sqrt{2}}\left(e^{-in\theta}B(e^{i\theta}) + e^{in\theta}s(e^{i\theta})B(e^{-i\theta})\right)V\right\|_{L^2,2} \to 0
$$

and

$$
\langle\!\langle p_n(x)\rangle\!\rangle^2_{L^2(\mu_s)} \to \mathbf{0}.
$$

This proves (3.8) and (3.9) for the type 2 case. To prove (3.7), by Lemma 1(b),

$$
\left\| \left\langle\!\!\!\left\langle \frac{e^{-in\theta}}{1 - e^{i\theta}\bar{z}}\mathbf{1}, \, G(e^{i\theta})p_n(2\cos\theta) - \frac{1}{\sqrt{2}}\left(e^{-in\theta}B(e^{i\theta}) + e^{in\theta}s(e^{i\theta})B(e^{-i\theta})\right)V \right\rangle\!\!\!\right\rangle_{L^2} \right\|
$$
$$
\leq l\,\frac{1}{\sqrt{1-|z|^2}}\left\| G(e^{i\theta})p_n(2\cos\theta) - \frac{1}{\sqrt{2}}\left(e^{-in\theta}B(e^{i\theta}) + e^{in\theta}s(e^{i\theta})B(e^{-i\theta})\right)V \right\|_{L^2,2}
$$
$$
\to 0 \tag{4.15}
$$

uniformly on compacts of $\mathbb{D}$. On the other hand,

$$
\left\langle\!\!\!\left\langle \frac{e^{-in\theta}}{1 - e^{i\theta}\bar{z}}\mathbf{1}, \, G(e^{i\theta})p_n(2\cos\theta) - \frac{1}{\sqrt{2}}e^{-in\theta}B(e^{i\theta})V \right\rangle\!\!\!\right\rangle_{L^2}
$$
$$
= \int_{-\pi}^{\pi} \frac{e^{in\theta}}{1 - e^{-i\theta}z}\left(G(e^{i\theta})p_n(2\cos\theta) - \frac{1}{\sqrt{2}}e^{-in\theta}B(e^{i\theta})V \right)\frac{d\theta}{2\pi}
$$
$$
= z^n G(z)p_n(z + z^{-1}) - \frac{1}{\sqrt{2}}B(z)V, \tag{4.16}
$$

and

$$
\left\langle\!\!\!\left\langle \frac{e^{-in\theta}}{1 - e^{i\theta}\bar{z}}\mathbf{1}, \, \frac{1}{\sqrt{2}}\left(e^{in\theta}s(e^{i\theta})B(e^{-i\theta})\right)V \right\rangle\!\!\!\right\rangle_{L^2} \to \mathbf{0} \quad \text{uniformly on compacts of } \mathbb{D} \tag{4.17}
$$

by Lemma 1(c). Together, (4.15)–(4.17) give

$$
z^n p_n\left(z + z^{-1}\right) \to L(z) \quad \text{uniformly on compacts of } \mathbb{D}.
$$

Thus we proved (3.7)–(3.9) for the type 2 case. The result for any equivalent $\widetilde{J}$ with Jacobi parameters (2.1) for which the limit $\sigma = \lim_{n\to\infty}\sigma_n$ exists, follows immediately from $\widetilde{p}_n(x) = p_n(x)\sigma_{n+1}$.

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