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Digital homotopy with obstacles☆

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Abstract

In (Ayala et al. (Discrete Appl. Math. 125 (1) (2003) 3) it was introduced the notion of a digital fundamental group $\pi_1^d(O/S;\sigma)$ for a set of pixels O in relation to another set S which plays the role of an "obstacle". This notion intends to be a generalization of the digital fundamental groups of both digital objects and their complements in a digital space. However, the suitability of this group was only checked for digital objects in that paper. As a sequel, we extend here the results in Ayala et al. (2003) for complements of objects. More precisely, we prove that for arbitrary digital spaces the group $\pi_1^d(O/S;\sigma)$ maps onto the usual fundamental group of the difference of continuous analogues $|\mathscr{A}_{O\cup S}| - |\mathscr{A}_S|$. Moreover, this epimorphism turns to be an isomorphism for a large class of digital spaces including most of the examples in digital topology.

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0. Introduction

This paper deals with a notion of digital fundamental group for complements of objects in binary digital pictures. The interest of searching for a good notion of digital fundamental group arises from the theory of 3d image thinning algorithms.

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After applying a 3d thinning algorithm, the "tunnels" in the input and output digital pictures must agree in number and position, and this can be correctly specified by saying that the algorithm preserves the digital fundamental groups of both the object displayed in the picture and its complement (see Criterion 2.3.2 in [8]).

The first notion of a digital fundamental group was the topological approach given by Khalimsky [7] for a particular class of spaces (Khalimsky's spaces) defined on the set $\mathbb{Z}^n$, $n > 0$, and more generally for locally finite topological T_0 spaces. This way, Khalimsky deals with sets of pixels regardless of considering them as digital objects themselves or as complements of other objects. However, depending on how one chooses to represent pixels in a Khalimsky's space, the loops defining this group may not consist entirely of pixels, property which seem not be appropriate for candidates to digital loops. Later on, Kong [8] gave a purely digital notion of fundamental group for a large class of graph-based digital spaces, including the (α,β)-connected spaces defined on the grid $\mathbb{Z}^n$, where $(\alpha,\beta)\in\{(4,8),(8,4)\}$ if n=2 and $(\alpha,\beta)\in\{(6,26),(26,6),(6,18),(18,6)\}$ if $n=3$. As usual in the graph-theoretical approach to Digital Topology, Kong's digital fundamental group involves a different definition for objects and their complements in a given digital space. Namely, if $O \subseteq \mathbb{Z}^n$ is an object in the (α,β)-connected digital space, Kong defines the digital fundamental group of the complement of O in that space as the fundamental group of the object $\mathbb{Z}^n - O$ in the corresponding (β,α)-connected digital space. Nevertheless, this notion is restricted to 2d and 3d digital spaces and seems not generalize to give higher digital homotopy groups.

Recently, the authors [4] have introduced a fairly general notion of digital fundamental group that is close enough to the topological approach of Khalimsky but it is given in quite reasonable digital terms. That notion includes, as particular cases, the corresponding notions for both objects and their complements in a digital space. More precisely, in [4] we define the digital fundamental group $\pi_1^d(O/S,\sigma)$ of a set of pixels O regarding to an object S, which plays the role of an "obstacle" that the loops in O cannot cross; the particular cases $\pi_1^d(O/\emptyset,\sigma)$ and $\pi_1^d(O/(X-O),\sigma)$ correspond to the digital fundamental groups of O when it is respectively considered as the digital object displayed in a picture and as the complement of the digital object $X-O$. As it usually happens to the connectivity and other topological properties of a set of pixels O, that may be different depending on whether O is regarded as a digital object or as the complement of the digital object $X-O$, the groups $\pi_1^d(O/\emptyset,\sigma)$ and $\pi_1^d(O/(X-O),\sigma)$ may be distinct.

This approach presents, at least from a theoretical point of view, several advantages over the notions of Khalimsky and Kong. Firstly, it can be readily generalized to define higher digital homotopy groups (see [4]), as Khalimsky's notion, and, secondly, this group is available on a larger class of arbitrarily dimensional digital spaces than Kong's digital fundamental group.

The group $\pi_1^d(O/S,\sigma)$ was introduced within the framework of the multilevel architecture for Digital Topology given in [3]. That architecture provides a link between the discrete world of digital pictures and Euclidean spaces, where the "continuous perception" that an observer may take on a picture is represented *via* a polyhedron called its *continuous analogue*. In general, this link can be used to obtain new results in Digital Topology, by translating the corresponding continuous results (for instance, we use it in [2] to prove a general Digital Index Theorem for all (α, β)-connected spaces on $\mathbb{Z}^3$ and also for digital spaces defined on the grid $\mathbb{Z}^n$, for $n \geqslant 3$). Moreover, this link can be also used to check that a new digital notion is an accurate counterpart of the usual continuous one. So, we give in [4] an isomorphism from the digital fundamental group $\pi_1^d(O/\emptyset, \sigma)$ of an object O onto the classical fundamental group of its continuous analogue.

As a sequel, we extend in this paper the results in [4] to the more elaborate case of the digital fundamental group $\pi_1^d(O/(X-O), \sigma)$ of an object's complement. More precisely, for an arbitrary obstacle $S \neq \emptyset$, we give in Section 3.1 an epimorphism from the digital fundamental group $\pi_1^d(O/S, \sigma)$ onto the fundamental group of the complement of the obstacle's continuous analogue. Although there is strong evidence that this epimorphism is not injective in general, we show in Section 3.2 that it is actually an isomorphism for a large class of digital spaces, including those most used in image processing. This supports also for complements of objects the suitability of our definition of the digital fundamental group π_1^d in [4].

For the convenience of the reader we review the basic notions of the multilevel architecture quoted above and the definition of the group $\pi_1^d(O/S, \sigma)$ in Sections 1 and 2, respectively.

1. Preliminaries

In this section we briefly summarize the basic notions of the multilevel architecture for digital topology developed in [3] as well as the notation that will be used through all the paper.

In that architecture, the spatial layout of pixels in a digital image is represented by a *device model*, which is a homogeneously n-dimensional locally finite polyhedral complex K. Each n-cell in K is representing a pixel, and so the digital object displayed in a digital image is a subset of the set $\mathrm{cell}_n(K)$ of n-cells in K; while the other lower dimensional cells in K are used to describe how the pixels could be linked to each other. A *digital space* is a pair (K, f), where K is a device model and f is *weak lighting function* defined on K. The function f is used to provide a continuous interpretation, called *continuous analogue*, for each digital object $O \subseteq \mathrm{cell}_n(K)$. Next we describe these two notions in detail.

By a homegeneously n-dimensional locally finite polyhedral complex we mean a set K of polytopes, in some Euclidean space $\mathbb{R}^d$, provided with the natural ordering given by the relationship "to be face of", that in addition satisfies the four following properties:

1. If $\sigma \in K$ and τ is a face of σ then $\tau \in K$.
2. If $\sigma, \tau \in K$ then $\sigma \cap \tau$ is a face of both σ and τ.
3. For each point x in the underlying polyhedron $|K| = \cup\{\sigma; \sigma \in K\}$ of K, there exists a neigbourhood of x which intersects only a finite number of polytopes in K; in particular, each polytope of K is face of a finite number of other polytopes in K.
4. Each polytope $\sigma \in K$ is face of some n-dimensional polytope in K.

Given a device model K and two polytopes $\gamma, \sigma \in K$, we shall write $\gamma \leqslant \sigma$ if γ is a face of σ, and $\gamma < \sigma$ if in addition $\gamma \neq \sigma$. A centroid-map on K is a map $c : K \to |K|$ such that $c(\sigma)$ belongs to the interior of σ; that is, $c(\sigma) \in \overset{\circ}{\sigma} = \sigma - \partial\sigma$, where $\partial\sigma = \cup\{\gamma; \gamma < \sigma\}$ stands for the boundary of σ. We refer to [13,15] for further notions on polyhedral topology.

Remark 1. A homegeneously n-dimensional locally finite polyhedral complex K can be regarded as an *abstract cellular complex* whose cells are the polytopes in K. So, for simplicity, K will be called a polyhedral complex, and its polytopes will be simply referred to as cells in this paper. Moreover, the abstract complex K can be endowed with the structure of a locally finite topological T_0 space (i.e., such that each point has a finite closure and a finite neighbourhood) with base $\mathscr{B} = \{U_\alpha; \alpha \in K\}$, where $U_\alpha = \{\beta \in K; \alpha \leqslant \beta\}$. Actually, this topological space $(K, \mathscr{B})$ is a quotient of the Euclidean polyhedron $|K|$ by the map $q : |K| \to K$ that assigns the cell σ to each point $x \in \overset{\circ}{\sigma}$.

Example 2. In this paper it will be essential the role played by the archetypical device model R^n, termed the standard cubical decomposition of the Euclidean n-space $\mathbb{R}^n$. The device model R^n is the complex determined by the collection of unit n-cubes in $\mathbb{R}^n$ whose edges are parallel to the coordinate axes and whose centers are in the set $\mathbb{Z}^n$. The centroid-map we will consider in R^n associates to each cube σ its barycenter $c(\sigma)$, which is a point in the set $\mathscr{Z}^n$. Here, $\mathscr{Z} = \frac{1}{2}\mathbb{Z}$ stands for the set of points $\{x \in \mathbb{R}; x = z/2, z \in \mathbb{Z}\}$. In particular, if $\dim \sigma = n$ then $c(\sigma) \in \mathbb{Z}^n$, where $\dim \sigma$ denotes the dimension of σ. So that, every digital object O in R^n can be identified with a subset of points in $\mathbb{Z}^n$. Henceforth we shall use this identification without further comment.

Before to proceed with the definition of weak lighting function, we need some notions, which are illustrated in Fig. 1 for an object O in the device model R^2.

The first two notions formalize two types of "digital neighbourhoods" of a cell $\alpha \in K$ in a given digital object $O \subseteq \mathrm{cell}_n(K)$. Indeed, we call the *star of* α *in* O to the set $\mathrm{st}_n(\alpha; O) = \{\sigma \in O; \alpha \leqslant \sigma\}$ of n-cells (pixels) in O having α as a face. Similarly, the *extended star of* α *in* O is the set $\mathrm{st}_n^*(\alpha; O) = \{\sigma \in O; \alpha \cap \sigma \neq \emptyset\}$ of n-cells (pixels) in O intersecting α.

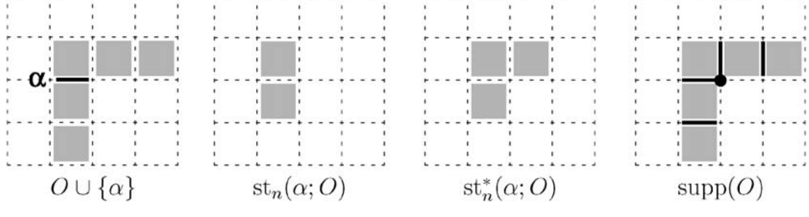

Fig. 1. The support of an object O and two types of digital neighbourhoods in O for a cell α. The cells in O together with the bold edges and dots are the elements in $\mathrm{supp}(O)$.

The third notion is the *support* of a digital object O which is defined as the set $\mathrm{supp}(O)$ of cells of K (not necessarily pixels) that are the intersection of n-cells (pixels) in O. Namely, $\alpha \in \mathrm{supp}(O)$ if and only if $\alpha = \cap\{\sigma; \sigma \in \mathrm{st}_n(\alpha; O)\}$. In particular, if α is a pixel in O then $\alpha \in \mathrm{supp}(O)$. Notice also that, among all the lower dimensional cells of K, only those in $\mathrm{supp}(O)$ are directly joining pixels in O.

To ease the writing, we shall use the following notation: $\mathrm{supp}(K) = \mathrm{supp}(\mathrm{cell}_n(K))$, $\mathrm{st}_n(\alpha; K) = \mathrm{st}_n(\alpha; \mathrm{cell}_n(K))$ and $\mathrm{st}^*_n(\alpha; K) = \mathrm{st}^*_n(\alpha; \mathrm{cell}_n(K))$. Finally, we shall write $\mathscr{P}(A)$ for the family of all subsets of a given set A.

Definition 3. Given a device model K, a *weak lighting function* (w.l.f.) on K is a map $f : \mathscr{P}(\mathrm{cell}_n(K)) \times K \to \{0, 1\}$ satisfying the following five axioms for all $O \in \mathscr{P}(\mathrm{cell}_n(K))$ and $\alpha \in K$:

(1) *object axiom*: if $\alpha \in O$ then $f(O, \alpha) = 1$;
(2) *support axiom*: if $\alpha \notin \mathrm{supp}(O)$ then $f(O, \alpha) = 0$;
(3) *weak monotone axiom*: $f(O, \alpha) \leqslant f(\mathrm{cell}_n(K), \alpha)$;
(4) *weak local axiom*: $f(O, \alpha) = f(\mathrm{st}^*_n(\alpha; O), \alpha)$; and,
(5) *complement connectivity axiom*: if $O' \subseteq O \subseteq \mathrm{cell}_n(K)$ and $\alpha \in K$ are such that $\mathrm{st}_n(\alpha; O) = \mathrm{st}_n(\alpha; O')$, $f(O', \alpha) = 0$ and $f(O, \alpha) = 1$, then: (a) the set of cells $\alpha(O'; O) = \{\omega < \alpha; f(O', \omega) = 0,\ f(O, \omega) = 1\}$ is not empty; (b) the set $\cup\{\overset{\circ}{\omega}; \omega \in \alpha(O'; O)\}$ is connected in $\partial\alpha$; and, (c) if $O \subseteq \bar{O} \subseteq \mathrm{cell}_n(K)$, then $f(\bar{O}, \omega) = 1$ for every $\omega \in \alpha(O'; O)$.

If $f(O, \alpha) = 1$ we say that f *lights* the cell α for the object O.

A w.l.f. f is said to be *strongly local at a cell* $\alpha \in K$ if $f(O, \alpha) = f(\mathrm{st}_n(\alpha; O), \alpha)$ for any digital object $O \subseteq \mathrm{cell}_n(K)$, and f will be simply called a *strongly local lighting function* in case it is strongly local at each cell $\alpha \in K$.

Remark 4. (1) It is readily checked that condition (b) in axiom (5) is equivalent to the connectness of the set $\alpha(O'; O)$ in the topological space $(K, \mathscr{B})$ given in Remark 1.

(2) Notice that the strong local property implies both axioms (4) and (5) above.

(3) In Example 6(2) below we define a w.l.f. h on the device model R^n which is not strongly local. However, it is immediate to check that any w.l.f. on a device model K is strongly local at each vertex and at each top dimensional cell $\alpha \in K$.

A weak lighting function f on a device model K can be regarded as a mapping that assigns a subset $\{\alpha \in K; f(O, \alpha) = 1\}$ of cells of K to each digital object $O \subseteq \mathrm{cell}_n(K)$. In this sense, lighting functions are particular examples of "face membership rules" as introduced by Kovalevsky in [11]. Our contribution in this point are axioms (1)–(5) in Definition 3. These axioms are intended for limiting the set of Kovalevsky's face membership rules to those that do not lead to topological properties which are contradictory with the natural perception of digital objects (see [6]).

Before giving a brief account of the intuitive ideas underlying these axioms, we next derive from the lighting function f an Euclidean polyhedron for any digital object

$O \subseteq \mathrm{cell}_n(K)$, called its continuous analogue, which is used in our approach to define the topological properties of both O and its complement. For this we use a fixed, but arbitrary, centroid-map $c : K \to |K|$ on the device model K to introduce several other intermediate models for O as follows.

The *device level* of O is the subcomplex $K(O) = \{\alpha \in K; \alpha \leqslant \sigma \in O\}$ induced by O. Notice that $K(O)$ can be considered as a device model itself.

The *logical level* of O is an undirected graph, $\mathscr{L}^f_O$, whose vertices are the centroids of n-cells in O and two of them $c(\sigma)$, $c(\tau)$ are adjacent if there exists a common face $\alpha \leqslant \sigma \cap \tau$ such that $f(O, \alpha) = 1$.

The *conceptual level of* O is the directed graph $\mathscr{C}^f_O$ whose vertices are the centroids $c(\alpha)$ of all cells $\alpha \in K$ with $f(O, \alpha) = 1$, and its directed edges are $(c(\alpha), c(\beta))$ with $\alpha < \beta$.

The *simplicial analogue* of O is the order complex $\mathscr{A}^f_O$ associated to the directed graph $\mathscr{C}^f_O$. That is, $\langle c(\alpha_0), c(\alpha_1), \ldots, c(\alpha_m)\rangle$ is an m-simplex of $\mathscr{A}^f_O$ if $c(\alpha_0), c(\alpha_1), \ldots,$ $c(\alpha_m)$ is a directed path in $\mathscr{C}^f_O$; or, equivalently, if $f(O, \alpha_i) = 1$, for $0 \leqslant i \leqslant m$, and $\alpha_0 < \alpha_1 < \cdots < \alpha_m$. That is, $\mathscr{A}^f_O$ is obtained by "filling in" all the triangles, tetrahedra, etc... in the conceptual level $\mathscr{C}^f_O$. Finally, the *continuous analogue* of O is the underlying polyhedron $|\mathscr{A}^f_O|$ of $\mathscr{A}^f_O$.

For the sake of simplicity, we will usually drop "f" from the notation of the levels of an object. Moreover, for the whole object $\mathrm{cell}_n(K)$ we will simply write $\mathscr{L}_K$, $\mathscr{C}_K$ and $\mathscr{A}_K$ for its levels.

Remark 5. (1) The simplicial analogue $\mathscr{A}_O$ of any digital object $O \subseteq \mathrm{cell}_n(K)$ is, by construction, a full subcomplex of the first derived subdivision $K^{(1)}$ of K induced by the chosen centroid-map c. Moreover, axiom (3) in Definition 3 yields that $\mathscr{A}_O \subseteq \mathscr{A}_K$, and so $\mathscr{A}_O$ is also a full subcomplex of $\mathscr{A}_K$.

(2) Given a locally finite topological T_0 space X, Kong and Khalimsky construct in [9] a *polyhedral analogue* $|K(X)|$ for X. It can be checked that, for any digital object O in a digital space (K, f), our continuous analogue $|\mathscr{A}_O|$ coincides with the polyhedral analogue $|K(X_O)|$ of the set $X_O = \{\alpha \in K; f(O, \alpha) = 1\}$ of cells which are lighted for O endowed with the relative topology of $(K, \mathscr{B})$ in Remark 1.

Example 6. (1) Every device model $K \neq \emptyset$ admits the weak lighting functions $f_{\max}$ and g given respectively by:

(a) $f_{\max}(O, \alpha) = 1$ if and only if $\alpha \in \mathrm{supp}(O)$
(b) $g(O, \alpha) = 1$ if and only if $\alpha \in \mathrm{supp}(O)$ and $\mathrm{st}_n(\alpha; K) \subseteq O$

In Fig. 2 are shown two objects, O and $\mathrm{cell}_2(R^2)$, in the device model R^2, and their levels for these lighting functions. More precisely, Figs. 2(a) and (b) show the 2-cells (grey squares) of the object O and the low-dimensional cells (bold edges and vertices) that the w.l.f.'s $f_{\max}$ and g light, respectively, for O. As these sets, $\{\alpha \in R^2; f_{\max}(O, \alpha) = 1\}$ and $\{\alpha \in R^2; g(O, \alpha) = 1\}$, do not agree, all the levels of O in the digital spaces $(R^2, f_{\max})$ and (R^2, g) are distinct, in particular $|\mathscr{A}^{f_{\max}}_O| \neq |\mathscr{A}^g_O|$. On

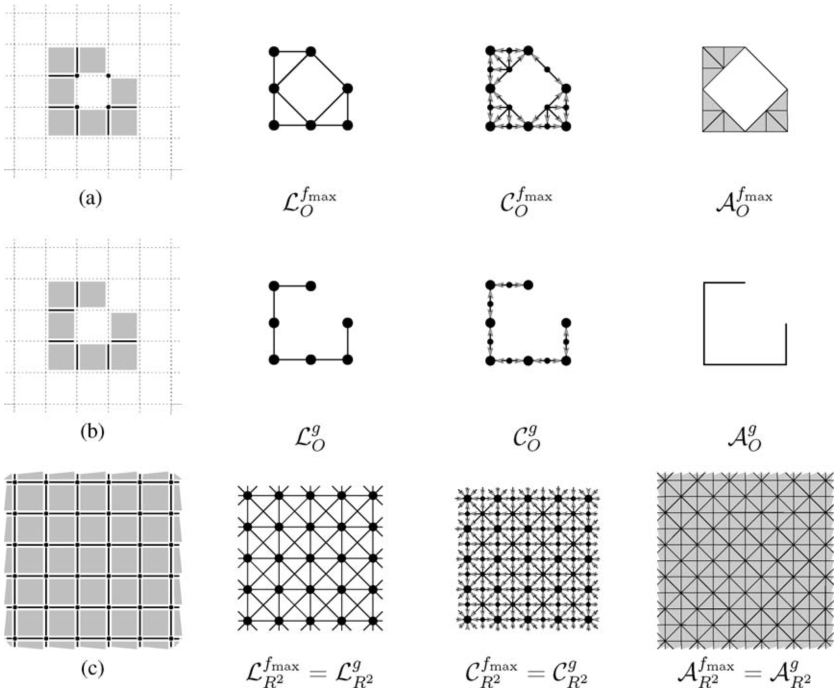

Fig. 2. Levels of the objects O and $\mathrm{cell}_2(R^2)$ for the w.l.f.'s $f_{\max}$ and g in Example 6(1).

the other hand,

$$\{\alpha \in R^2; f_{\max}(\mathrm{cell}_2(R^2), \alpha) = 1\} = \{\alpha \in R^2; g(\mathrm{cell}_2(R^2), \alpha) = 1\}$$

(see Fig. 2(c)), and so all the levels of the object $\mathrm{cell}_2(R^2)$ are the same in these two digital spaces.

At this point, it is worth to point out that g induces in R^n the $(2n, 3^n - 1)$–connectivity (see [1, Definition 11]); that is, the generalization to arbitrary dimension of the $(4, 8)$-connectivity on $\mathbb{Z}^2$. On the other hand, $f_{\max}$ induces in R^n the $(3^n - 1, 2n)$-connectivity (see Fig. 2).

(2) Both the w.l.f.'s $f_{\max}$ and g given above satisfy the strong local axiom in Definition 3. Next we give an example of a non-strongly local digital space (R^n, h). For any integer $n > 0$, the w.l.f. h is defined on the device model R^n by $h(O, \alpha) = 1$ if and only if: (a) $\dim \alpha = n$ and $\alpha \in O$; (b) $\dim \alpha \leqslant n - 2$ and $\mathrm{st}_n(\alpha; R^n) \subseteq O$; and, (c) $\dim \alpha = n - 1$, $\alpha \in \mathrm{supp}(O)$, and either $\mathrm{st}_n^*(\alpha; R^n) \subseteq O$ or there exist $\sigma, \tau \in \mathrm{st}_n^*(\alpha; R^n) - O$ such that $\sigma \cap \tau = \emptyset$.

We devote the rest of this section to give some intuitive ideas underlying axioms (1) –(5) in Definition 3.

Axiom (1) says that to display a given digital object O on a computer screen all its pixels (n-cells in O) must be lighted. Axiom (2) implies that, in addition to these

n-cells, only the lower dimensional cells in supp(O), but not necessarily all of them, may be lighted in order to connect immediately adjacent pixels in O. In particular, and according to our usual perception, this prevent two isolated pixels $\sigma, \tau \in O$, with $\sigma \cap \tau = \emptyset$, from being connected in $|\mathscr{A}_O|$ by a sequence of lower dimensional cells of K that are not faces of pixels in O. Axiom (4) as well as the strong local axiom say that our perception of a digital object is local, and so the lighting of a cell α depends on a "digital neighbourhood" of α in O. And axiom (3) states that a cell which is lighted for a digital object must be also lighted for the object $\mathrm{cell}_n(K)$ consisting of all the pixels in the digital space. This way, the continuous analogue $|\mathscr{A}_O|$ of any digital object is a subspace of $|\mathscr{A}_K|$; see Remark 5(1).

At this point it is worth pointing out that, for any two digital objects $O_1 \subseteq O_2$, axioms (1)–(5) in Definition 3 do not imply $|\mathscr{A}_{O_1}| \subseteq |\mathscr{A}_{O_2}|$. This property is equivalent to the following stronger version of axiom (3): a w.l.f. f is said to be *strongly monotone* if $f(O_1,\alpha) \leqslant f(O_2,\alpha)$ for any cell $\alpha \in K$ and any pair of objects $O_1 \subseteq O_2$. This apparently natural property is not, however, always desirable. For example, if one wants to deal with a digital space (R^3, f) such that each 18-connected digital object has a connected continuous analogue and, moreover, the continuous analogue of each $(18,6)$-surface, as introduced by Kong and Roscoe in [10], is a surface, it is not difficult to check that the lighting function f cannot be strongly monotone (see [1]).

We start our comments about axiom (5) by showing in the next example that in case any of its parts (a) or (b) fails, then the complement $|\mathscr{A}_K| - |\mathscr{A}_O|$ of an object's continuous analogue may not appropriately represent the connectivity of the complement $\mathrm{cell}_n(K) - O$ of O.

Example 7. Let us consider the functions f_a and f_b defined on the device model R^2 by

- $f_a(O,\alpha) = 1$ iff $\alpha \in \mathrm{supp}(O)$ for $\dim \alpha \in \{0,2\}$ and $\mathrm{st}_2^*(\alpha; R^2) \subseteq O$ for $\dim \alpha = 1$
- $f_b(O,\alpha) = 1$ iff $\alpha \in O$ or $\mathrm{st}_2^*(\alpha; R^2) \subseteq O$.

Fig. 3(b) and (c) show the cells lighted by f_a and f_b, respectively, for the object O_1 in Fig. 3(a). Notice that axioms (1)–(4) in Definition 3 hold for both f_a and f_b. However, it is not difficult to check that parts (a) and (b) of axiom (5) fail for f_a and f_b, respectively, by using the objects O_1 and $\mathrm{cell}_2(R^2)$ and the 1-cell α. The continuous analogues $|\mathscr{A}^{f_a}_{O_1}|$ and $|\mathscr{A}^{f_b}_{O_1}|$ are the polyhedra depicted (in grey colour) in Fig. 3(d) and (e), while $|\mathscr{A}^{f_a}_{R^2}| = |\mathscr{A}^{f_b}_{R^2}| = \mathbb{R}^2$. Hence, the complements $|\mathscr{A}^{f_i}_{R^2}| - |\mathscr{A}^{f_i}_{O_1}|$, $i \in \{a,b\}$, have not exactly two components. However, we intuitively perceive just two components in the complement of O_1, each containing one of the two isolated pixels in $\mathrm{cell}_2(R^2) - O_1 = \{\sigma_1, \sigma_2\}$.

In the general case, given a digital object $O \subseteq \mathrm{cell}_n(K)$, part (a) of axiom (5) is a sufficient condition to prove that each component of $|\mathscr{A}_K| - |\mathscr{A}_O|$ contains at least the centroid $c(\sigma)$ of a pixel $\sigma \in \mathrm{cell}_n(K) - O$; while part (b) is used to show that two pixels $\sigma_1, \sigma_2 \in \mathrm{cell}_n(K) - O$ are represented in the same component of $|\mathscr{A}_K| - |\mathscr{A}_O|$ if and only if σ_1 and σ_2 can be connected by a sequence of cells that are faces of pixels

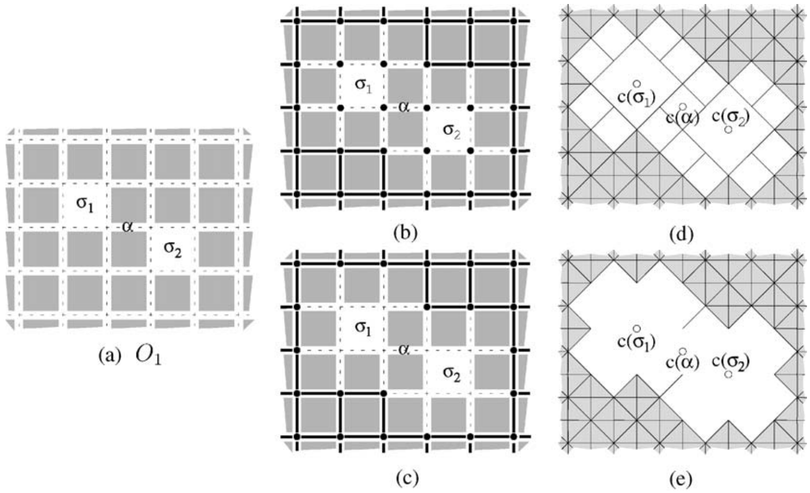

Fig. 3. Axiom (5) in Definition 3 is required to obtain a right representation of the connectivity of the complement of any object.

in the complement of O. On the other hand, part (c) of axiom (5) provides us with a natural notion of digital subspace, as it is stated in the next straightforward result.

Lemma 8. *For any digital object O in a digital space (K, f), the map*

$$f_O : \mathscr{P}(\mathrm{cell}_n(K(O))) \times K(O) = \mathscr{P}(O) \times K(O) \to \{0, 1\}$$

given by $f_O(O', \alpha) = f(O, \alpha) f(O', \alpha)$, for $O' \subseteq O$ and $\alpha \in K(O)$, is a w.l.f. on the device model $K(O)$. So, we call the pair $(K(O), f_O)$ the digital subspace of (K, f) induced by O.

Remark 9. Let $(K(O), f_O)$ be the digital subspace induced by a digital object O in a digital space (K, f). If $O' \subseteq O$, one easily checks the equality $\mathscr{A}^{f_O}_{O'} = \mathscr{A}^{f}_{O} \cap \mathscr{A}^{f}_{O'}$, since all these are full subcomplexes of $K^{(1)}$ by Remark 5(1). In particular $\mathscr{A}^{f_O}_{O} = \mathscr{A}^{f}_{O}$; that is, the continuous analogue of an object does not change when it is considered as the ambient digital space.

Actually, a particular class of digital subspaces (called *windows*) of the digital space (R^n, g) given in Example 6 are the key that will allow us to introduce a notion of digital fundamental group in next Section.

2. A digital fundamental group

The fundamental group of a topological space X, $\pi_1(X, x_0)$, is usually defined to be the set of homotopy classes of loops based at fixed point x_0 (i.e., maps $\xi : I = [0, 1] \to X$

with $\xi(0)=\xi(1)=x_0$), where an homotopy between two loops ξ_1,ξ_2 is a continuous map $H: I\times I\to X$ such that $H(x,0)=\xi_1(x)$, $H(x,1)=\xi_2(x)$ and $H(0,t)=H(1,t)=x_0$ (see [14,16]).

In this section we collect the definitions and basic facts involved in the notion of digital fundamental group as introduced in [4]. In particular we need suitable digital analogues for loops and homotopies in ordinary topology. We give these notions in a more general setting provided by *digital maps*, which allow us to define higher dimensional digital groups in a straightforward way (see Remark 16 in [4]).

Since the device model of a digital space is a polyhedral complex, one may define a digital map from a digital space (K_1,f_1) into another (K_2,f_2) as a cellular map between the device models K_1 and K_2, satisfying certain restrictions. However, this kind of definition is not convenient for our purposes as then the domain of such a digital map would be the whole set of cells in K_1, and not only those cells lighted by the w.l.f. f_1; that is, the cells in K_1 which are relevant in the digital space. Due to this, we define first a more suitable family of domains for our digital maps.

Definition 10. Let $S\subseteq \mathrm{cell}_n(K)$ be a digital object in a digital space (K,f). The *light body of* (K,f) *shaded with* S is the set of cells $\mathrm{Lb}(K/S)$ not lighted for the object S but lighted for $\mathrm{cell}_n(K)$; that is

$$\begin{aligned}\mathrm{Lb}(K/S) &= \{\alpha\in K;\, f(\mathrm{cell}_n(K),\alpha)=1,\, f(S,\alpha)=0\}\\ &= \{\alpha\in K;\, c(\alpha)\in|\mathscr{A}_K|-|\mathscr{A}_S|\}.\end{aligned}$$

An additional requirement for digital maps between two digital spaces (K_1,f_1) and (K_2,f_2) is that each digital object in (K_1,f_1) should be naturally assigned to a digital object in (K_2,f_2). These observations leads to the following:

Definition 11. Let $(K_1,f_1),(K_2,f_2)$ be two digital spaces, with $\dim K_i=n_i$ $(i=1,2)$, and let $S_1\subset \mathrm{cell}_{n_1}(K_1)$ and $S_2\subset \mathrm{cell}_{n_2}(K_2)$ be two digital objects. A *digital* (S_1,S_2)*-map* (or, simply, a d*-map*) $\Phi_{S_1,S_2}:(K_1,f_1)\to(K_2,f_2)$ from (K_1,f_1) into (K_2,f_2) is a map $\phi:\mathrm{Lb}(K_1/S_1)\to\mathrm{Lb}(K_2/S_2)$ satisfying the two following properties:

1. $\phi(\mathrm{cell}_{n_1}(K_1)-S_1)\subseteq \mathrm{cell}_{n_2}(K_2)-S_2$; and,
2. for $\alpha,\beta\in\mathrm{Lb}(K_1/S_1)$ with $\alpha<\beta$ then $\phi(\alpha)\leqslant\phi(\beta)$.

That is, ϕ carries top dimensional cells in $\mathrm{Lb}(K_1/S_1)$ to top dimensional cells in $\mathrm{Lb}(K_2/S_2)$ and preserves the face relations (although ϕ needs not be dimension preserving).

Notice that, for a given d-map $\Phi_{S_1,S_2}:(K_1,f_1)\to(K_2,f_2)$, property (1) in the previous definition is a necessary and sufficient condition to check that the image $\Phi_{S_1,S_2}(O)$ of any digital object $O\subseteq\mathrm{cell}_{n_1}(K_1)-S_1$ is a digital object in (K_2,f_2). Moreover, by property (2), Φ_{S_1,S_2} is a continuous map if we consider $\mathrm{Lb}(K_1/S_1)$ and $\mathrm{Lb}(K_2/S_2)$ as subspaces of the abstract complexes K_1 and K_2 topologized as in Remark 1. The following result also holds.

Proposition 12. *Any d-map* $\Phi_{S_1,S_2} : (K_1, f_1) \to (K_2, f_2)$ *induces a simplicial map* $\mathscr{A}(\Phi_{S_1,S_2}) : \mathscr{A}_{K_1} \setminus \mathscr{A}_{S_1} \to \mathscr{A}_{K_2} \setminus \mathscr{A}_{S_2}$, *which is defined on the vertices* $c_1(\alpha)$ *of* $\mathscr{A}_{K_1} \setminus \mathscr{A}_{S_1}$ *by* $\mathscr{A}(\Phi_{S_1,S_2})(c_1(\alpha)) = c_2(\Phi_{S_1,S_2}(\alpha))$. *Here* c_i *is a centroid-map on the device model* K_i, *for* $i = 1, 2$.

In the previous proposition $L_1 \setminus L_2 = \{\alpha \in L_1; \alpha \cap |L_2| = \emptyset\}$ denotes the simplicial complement of the subcomplex $L_2 \subseteq L_1$.

In order to define digital loops and digital homotopies as particular types of digital maps, next definition provides us with a particular class of digital spaces, called *windows*, that will play the same role as the unit interval, I, and the unit square, $I \times I$, in continuous homotopy. For this, we will use the following notation. Given two points $x = (x_1, \dots, x_m), y = (y_1, \dots, y_m) \in \mathbb{R}^m$, we write $x \preceq y$ if $x_i \leqslant y_i$ for all $1 \leqslant i \leqslant m$, while $x + y$ will stand for the usual vector addition $x + y = (x_1 + y_1, \dots, x_m + y_m) \in \mathbb{R}^m$.

Definition 13. Given two points $r, x \in \mathbb{Z}^m$, with $r_i \geqslant 0$ for $1 \leqslant i \leqslant m$, we call a *window of size* r (or r-*window*) *of* R^m *based at* x to the digital subspace V_r^x of (R^m, g) induced by the digital object $O_r^x = \{\sigma \in \mathrm{cell}_m(R^m); x \preceq c(\sigma) \preceq x + r\}$, where (R^m, g) is the digital space defined in Example 6. For the sake of simplicity, we shall write V_r to denote the r-window of R^m based at the point $x = (0, \dots, 0) \in \mathbb{Z}^m$.

Remark 14. To ease the writing, given an r-window V_r of R^m, we will identify each cell $\alpha \in \mathrm{Lb}(V_r/\emptyset)$ with its centroid $c(\alpha) \in \mathscr{Z}^m$ (see Example 2). In particular, if V_r is an r-window of R^1, then $\mathrm{Lb}(V_r/\emptyset) = \{\sigma_0, \sigma_1, \dots, \sigma_{2r-1}, \sigma_{2r}\}$ consists of $2r + 1$ cells (points and segments) which will be identified with the numbers $c(\sigma_i) = i/2$, for $0 \leqslant i \leqslant 2r$. And, similarly, for a window $V_{(r_1,r_2)}$ of R^2, we identify each cell $\alpha \in \mathrm{Lb}(V_{(r_1,r_2)}/\emptyset)$ with a pair $c(\alpha) = (i/2, j/2)$, where $0 \leqslant i \leqslant 2r_1$ and $0 \leqslant j \leqslant 2r_2$.

We are now ready to define digital loops and digital homotopies as follows.

Definition 15. Let $S, O \subseteq \mathrm{cell}_n(K)$ be two disjoint digital objects in a digital space (K, f), and σ, τ two n-cells in O. A S-*walk in* O *of length* $r \in \mathbb{Z}$ *from* σ *to* τ is a digital $(\emptyset, S)$-map $\phi_r : \mathrm{Lb}(V_r/\emptyset) \to \mathrm{Lb}(K(O \cup S)/S)$ such that $\phi_r(0) = \sigma$ and $\phi_r(r) = \tau$. A S-*loop in* O *based at* σ is a S-walk ϕ_r such that $\phi_r(0) = \phi_r(r) = \sigma$.

The *juxtaposition* of two given S-walks ϕ_r, ϕ_s in O, with $\phi_r(r) = \phi_s(0)$, is the S-walk $\phi_r * \phi_s : \mathrm{Lb}(V_{r+s}/\emptyset) \to \mathrm{Lb}(K(O \cup S)/S)$ of length $r + s$ given by

$$\phi_r * \phi_s(i/2) = \begin{cases} \phi_r(i/2) & \text{if } 0 \leqslant i \leqslant 2r, \\ \phi_s(i/2 - r) & \text{if } 2r \leqslant i \leqslant 2(r+s). \end{cases}$$

Notice that a S–loop ϕ_r in O is actually a sequence $(\phi_r(i))_{i=0}^r$ of adjacent pixels in O such that each pair $\phi_r(i-1), \phi_r(i)$ of successive pixels have a common face $\phi_r(i - \frac{1}{2})$ which is not lighted for the object S. In this sense ϕ_r does not cross the obstacle S. Similarly, a digital homotopy, as defined below, transforms a S-loop ϕ^1 to ϕ^2 using adjacent pixels but, in the same way, it is not allowed to cross the obstacle S.

Definition 16. Let ϕ_r^1, ϕ_r^2 two S-walks in O of the same length $r \in \mathbb{Z}$ from σ to τ.

We say that ϕ_r^1, ϕ_r^2 are *digitally homotopic* (or, simply, *d-homotopic*) *relative* $\{\sigma,\tau\}$, and we write $\phi_r^1 \simeq_d \phi_r^2$ rel. $\{\sigma,\tau\}$, if there exists an (r,s)-window $V_{(r,s)}$ in R^2 and a $(\emptyset, S)$-map $H : \mathrm{Lb}(V_{(r,s)}/\emptyset) \to \mathrm{Lb}(K(O \cup S)/S)$, called a *d-homotopy*, such that $H(i/2,0)=\phi_r^1(i/2)$ and $H(i/2,s)=\phi_r^2(i/2)$, for $0 \leqslant i \leqslant 2r$, and moreover $H(0,j/2)=\sigma$ and $H(r,j/2)=\tau$, for $0 \leqslant j \leqslant 2s$. Here we use the identification $H(a_1,a_2)=H(\alpha)$, where $c(\alpha)=(a_1,a_2)\in\mathscr{Z}^2$ is the centroid of a cell $\alpha \in \mathrm{Lb}(V_{(r,s)}/\emptyset)$; see Remark 14.

Definition 17. Let ϕ_r, ϕ_s two S-walks in O from σ to τ of lengths r and s respectively. We say that ϕ_r is d-homotopic to ϕ_s relative $\{\sigma,\tau\}$, $\phi_r \simeq_d \phi_s$ rel. $\{\sigma,\tau\}$, if there exist constant S-loops $\phi_{r'}^\tau$ and $\phi_{s'}^\tau$ such that $r+r'=s+s'$ and $\phi_r * \phi_{r'}^\tau \simeq_d \phi_s * \phi_{s'}^\tau$ rel. $\{\sigma,\tau\}$.

The following result, whose proof can be found in [4], will be needed in the sequel.

Proposition 18. *Let ϕ_r be a S-walk in O from σ to τ, and ϕ_s^σ, ϕ_s^τ two constant S-loops of the same length $s\in\mathbb{Z}$. Then, $\phi_s^\sigma * \phi_r \simeq_d \phi_r * \phi_s^\tau$ rel. $\{\sigma,\tau\}$.*

Notice that d-homotopies induce an equivalence relation in the set of S-walks in O from σ to τ. Moreover, from Proposition 18 it is not difficult to check that the juxtaposition is compatible with d-homotopies between S-walks. Thus, the juxtaposition of S-loops naturally induces a product operation that endows the set of classes of S-loops in O based at an n-cell $\sigma \in O$ with a group structure, for which the trivial element is the class of constant S-loops at σ, and the inverse of the class $[\phi_r]$ is represented by the S-loop ϕ_r^{-1} obtained by traversing ϕ_r backwards; that is, $\phi_r^{-1}(i/2)=\phi_r(r-i/2)$ for all $0 \leqslant i \leqslant 2r$. So, we next introduce the notion of digital fundamental group as follows.

Definition 19. Let S, O be two disjoint digital objects in a digital space (K, f), and σ an n-cell in O. The *digital fundamental group of O at σ with obstacle at S* is the set $\pi_1^d(O/S,\sigma)$ of d-homotopy classes of S-loops in O based at σ with the product operation $[\phi_r]\cdot[\psi_s]=[\phi_r * \psi_s]$. In case $S=\emptyset$, we will simply call $\pi_1^d(O/\emptyset,\sigma)=\pi_1^d(O,\sigma)$ the *digital fundamental group of O at σ*.

Remark 20. Definition 19 provides an entire family of digital fundamental groups for a given set of pixels O when the object S is allowed to range over the family of all subsets of $\mathrm{cell}_n(K)-O$. Particularly interesting are the groups $\pi_1^d(O/\emptyset,\sigma)=\pi_1^d(O,\sigma)$ and $\pi_1^d(O/(\mathrm{cell}_n(K)-O),\sigma)$ that, respectively, represents the digital fundamental group of the object O itself and the digital fundamental group of O as the complement of the object $\mathrm{cell}_n(K)-O$.

These groups $\pi_1^d(O,\sigma)$ and $\pi_1^d(O/(\mathrm{cell}_n(K)-O),\sigma)$ may be distinct, as it usually happens to connectivity and other topological properties that depend on how a set of pixels O is regarded. For example, let us consider the digital space $(R^2, f_{\max})$ given in Example 6(1), and let O be a set of pixels whose complement $S=\mathrm{cell}_2(R^2)-O$ consists of two pixels σ_1,σ_2 with $\dim(\sigma_1\cap\sigma_2)=0$. For these objects it can be readily checked, by using Corollary 42, that $\pi_1^d(O,\sigma)=\mathbb{Z}\times\mathbb{Z}$ while $\pi_1^d(O/S,\sigma)=\mathbb{Z}$.

3. The relationship with the continuous fundamental group

In [4] we show that the digital fundamental group $\pi_1^d(O,\sigma)$ of a digital object coincides with the classical fundamental group of its continuous analogue $|\mathscr{A}_O|$. In this Section we tackle the problem of computing the digital fundamental group $\pi_1^d(O/S,\sigma)$ of O with a disjoint object S acting as an "obstacle" for the loops in O. The section is divided into two parts, in Section 3.1 we deal with the general case and we produce an epimorphism

$$h:\pi_1^d(O/S,\sigma)\to\pi_1(|\mathscr{A}_{O\cup S}|-|\mathscr{A}_S|,c(\sigma))$$

onto the classical fundamental group of the complement of the obstacle's continuous analogue. The second part, Section 3.2, provides us with a large class of digital spaces for which the above homomorphism yields an isomorphism.

We recall that, for a triangulated polyhedron $|L|$, there is an alternative definition of the fundamental group $\pi_1(|L|,x_0)$ that will be more convenient for our purposes. So we next explain it briefly. For this, we call an *edge-walk* in L from a vertex v_0 to a vertex v_n to a sequence α of vertices $(v_0,v_1,\dots,v_n)$, such that for each $k=1,2,\dots,n$ the vertices v_{k-1},v_k span a simplex in L (possibly $v_{k-1}=v_k$). If $v_0=v_n$, α is called an *edge-loop based at* v_0.

Given another edge-walk $\beta=(v_j)_{j=n}^{m+n}$ whose first vertex is the same as the last vertex of α, the *juxtaposition* $\beta=(v_i)_{i=0}^{m+n}$ is defined in the obvious way. The *inverse* of α is $\alpha^{-1}=(v_n,v_{n-1},\dots,v_0)$.

Two edge-walks α and β are said to be *equivalent* if one can be obtained from the other by a finite sequence of operations of the form:

(a) if $v_{k-1}=v_k$, replace $\dots,v_{k-1},v_k,\dots$ by $\dots,v_k,\dots$, or conversely replace $\dots,v_k,\dots$ by $\dots,v_{k-1},v_k,\dots$; or
(b) if v_{k-1},v_k,v_{k+1} span a simplex of L (not necessarily 2-dimensional), replace $\dots,v_{k-1},v_k,v_{k+1},\dots$ by $\dots,v_{k-1},v_{k+1},\dots$, or conversely.

This clearly sets up an equivalence relation between edge-walks, and the set $\pi_1(L,v_0)$ of equivalence classes $[\alpha]$ of edge-loops α in L, based at a vertex v_0, is given the structure of group by the operation $[\alpha]\cdot[\beta]=[\alpha*\beta]$. This group is called the *edge-group* of L.

Each edge–walk α in L defines in an obvious way a continuous path $\theta(\alpha)$ in the underlying polyhedron $|L|$; and so, we will identify henceforth the edge-walk α with the continuous path $\theta(\alpha)$. Actually this correspondence yields an isomorphism $\pi_1(|L|,v_0)\cong\pi_1(L,v_0)$. More precisely,

Theorem 21 (Maunder [13; 3.3.9]). *There exists an isomorphism* $\Theta:\pi_1(L,v_0)\to\pi_1(|L|,v_0)$ *which carries the class* $[\alpha]$ *to the class* $[\theta(\alpha)]$.

Corollary 22. *Let* O,S *be two disjoint digital objects in a digital space* (K,f). *Then* $\pi_1(\mathscr{A}_{O\cup S}\setminus\mathscr{A}_S,c(\sigma))\cong\pi_1(|\mathscr{A}_{O\cup S}|-|\mathscr{A}_S|,c(\sigma))$ *for any* $\sigma\in O$.

Proof. By Remark 5(1) we know that both $\mathscr{A}_{O\cup S}$ and $\mathscr{A}_S$ are full subcomplexes of $\mathscr{A}_K$. Then Lemma 72.2 in [14] yields that $|\mathscr{A}_{O\cup S}\setminus\mathscr{A}_S| = |\mathscr{A}_{O\cup S}\setminus(\mathscr{A}_{O\cup S}\cap\mathscr{A}_S)|$ is a strong deformation retract of $|\mathscr{A}_{O\cup S}| - |\mathscr{A}_S| = |\mathscr{A}_{O\cup S}| - |\mathscr{A}_{O\cup S}\cap\mathscr{A}_S|$ and the result follows by Theorem 21. □

Let (K,f) be an arbitrary digital space. Given two disjoint digital objects $O,S\subseteq \mathrm{cell}_n(K)$ and any n-cell $\sigma\in O$ we next define a natural homomorphism,

$$h:\pi_1^d(O/S,\sigma)\to\pi_1(\mathscr{A}_{O\cup S}\setminus\mathscr{A}_S,c(\sigma)),\tag{1}$$

from the digital fundamental group of O based at σ and with obstacle at the object S into the edge–group of the simplicial complex $\mathscr{A}_{O\cup S}\setminus\mathscr{A}_S$ based at the centroid $c(\sigma)$, as follows. Let ϕ_r be any S-loop in O. Then, we just observe that the sequence $c(\phi_r)=(c(\phi_r(i/2)))_{i=0}^{2r}$ defines an edge-loop in $\mathscr{A}_{O\cup S}\setminus\mathscr{A}_S$. So that, we simply set $h([\phi_r])=[c(\phi_r)]$. Notice that h is the generalization to the case $S\neq\emptyset$ of the homomorphism used in [4] to show the isomorphism $\pi_1^d(O,\sigma)=\pi_1^d(O/\emptyset,\sigma)\cong\pi_1(\mathscr{A}_O,c(\sigma))$.

Remark 23. The following properties are easily checked

(1) If ϕ_r and ϕ_s are two S-loops at σ, then $c(\phi_r)*c(\phi_s)=c(\phi_r*\phi_s)$.
(2) If ϕ_r is a constant S-loop at σ then $c(\phi_r)$ is a constant edge–loop at $c(\sigma)$.

Lemma 24. *The correspondence h, given in (1) above, is well defined. Moreover h is a group homomorphism.*

Proof. Assume $\phi_r\simeq_d\phi_s$ rel. σ are two equivalent S-loops in O. Then there exist two constant S-loops $\phi_{r'}^\sigma$ and $\phi_{s'}^\sigma$ such that $r+r'=s+s'$ and a d-homotopy $H:\phi_r*\phi_{r'}^\sigma\simeq_d\phi_s*\phi_{s'}^\sigma$ rel. σ. That is, H is an $(\emptyset,S)$-map $H:(V_{(r+r',t)},g)\to(K(O\cup S),f_{O\cup S})$, where $V_{(r+r',t)}$ is a window in Def. 13 and $(K(O\cup S),f_{O\cup S})$ is the digital subspace of (K,f) induced by $O\cup S$; see Lemma 8. Therefore, by Proposition 12 and Remark 9 we get a simplicial map $\mathscr{A}(H):\mathscr{A}_{V_{(r+r',t)}}\to\mathscr{A}_{O\cup S}\setminus\mathscr{A}_S$. Notice that from the definition of the w.l.f. g in Example 6(1) it readily follows that $\mathscr{A}_{V_{(r+r',t)}}$ is simplicially isomorphic to a triangulation of the unit square $I\times I$, and hence $\mathscr{A}(H)$ yields a homotopy between the edge-loops $c(\phi_r*\phi_{r'}^\sigma)$ and $c(\phi_s*\phi_{s'}^\sigma)$. Finally, the properties in Remark 23 and suitable equivalence transformations of type (a) yield that $c(\phi_r*\phi_{r'}^\sigma)=c(\phi_r)*c(\phi_{r'}^\sigma)$ is equivalent to $c(\phi_r)$, and similarly $c(\phi_s*\phi_{s'}^\sigma)$ is also equivalent to $c(\phi_s)$. Notice also that h is an homomorphism of groups as an immediate consequence of property (1) in Remark 23. □

3.1. The general case: epimorphism onto the classical fundamental group

This section is aimed to show that, for arbitrary disjoint digital objects $O,S\subseteq\mathrm{cell}_n(K)$ in a digital space (K,f), the homomorphism of groups

$$h:\pi_1^d(O/S,\sigma)\to\pi_1(\mathscr{A}_{O\cup S}\setminus\mathscr{A}_S,c(\sigma))$$

is always an epimorphism. For $S=\emptyset$, we proved in [4] that the homomorphism h above is actually an isomorphism of groups $\pi_1^d(O/\emptyset,\sigma)\cong\pi_1(\mathscr{A}_O,c(\sigma))$. For this we associate to each edge-loop Γ in $\mathscr{A}_O$ a family of *digital representatives* $F(\Gamma)$ such that for each digital $\emptyset$-loop $\phi_r \in F(\Gamma)$ the edge-loop $c(\phi_r)$ is equivalent to Γ. In this section we show that this procedure can be generalized to get a non-empty family $\mathscr{D}(\Gamma)$ of S-loops in O of digital representatives for any edge-loop Γ in $\mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$. This immediately yields that the homomorphism h is onto with full generality. However, the construction of the family $\mathscr{D}(\Gamma)$ suggests that the homomorphism h need not to be injective in general. In any case, Section 3.2 provides a large class of digital spaces, including those often used in image processing, for which h is in fact an isomorphism.

In order to define the family $\mathscr{D}(\Gamma)$ we start generalizing the notion of irreducible edge-loop introduced in [4].

Definition 25. A vertex $c(\gamma_i)$, of and edge-walk $\Gamma = (c(\gamma_i))_{i=0}^t$ in $\mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$, is said to be *reducible in* Γ if $i > 0$ and one of the following properties holds

(1) $\gamma_{i-1} = \gamma_i$
(2) there exists a vertex $c(\gamma_k)$ with $i < k \leqslant t$ such that $\gamma_k \neq \gamma_i$ and either $\gamma_{i-1} < \gamma_i < \gamma_j$ or $\gamma_{i-1} > \gamma_i > \gamma_j$, where $j = \min\{k; i < k \leqslant t, \gamma_i \neq \gamma_k\}$.

An edge-walk is said to be *reducible* if it contains a reducible vertex; otherwise we say that Γ is *irreducible*.

The proof of the next lemma is similar to that of Lemma 4.7 in [4] with the obvious changes.

Lemma 26. *Any edge-walk Γ in $\mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$ is equivalent to an irreducible edge-walk, $\bar{\Gamma} = (c(\bar{\gamma}_i))_{i=0}^k$, obtained by deleting all reducible vertices in Γ.*

Remark 27. (a) If $\Gamma = (c(\gamma_i))_{i=0}^t$ is an irreducible edge-walk in $\mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$ then either $\gamma_{i-1} < \gamma_i > \gamma_{i+1}$ or $\gamma_{i-1} > \gamma_i < \gamma_{i+1}$ for all $0 < i < t$. Moreover, in case both γ_0 and γ_t are n-cells in O then the length of Γ is an even number, $t = 2r$, and so $\gamma_{2i-2} > \gamma_{2i-1} < \gamma_{2i}$, for $1 \leqslant i \leqslant r$. In particular, this property holds for any edge-loop Γ in $\mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$ which is based at a vertex $c(\sigma)$ with $\sigma \in O$.

(b) Notice also that for an arbitrary edge-walk $\Gamma = (c(\gamma_i))_{i=0}^t$ in $\mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$ the vertex $c(\gamma_0)$ is never reducible. And, if $\bar{\Gamma} = (c(\bar{\gamma}_i))_{i=0}^k$ is the irreducible edge-walk obtained from Γ by deleting all its reducible vertices, then $\gamma_t = \bar{\gamma}_k$.

(c) Let $c(\phi_r) = (c(\phi_r(i/2)))_{i=0}^{2r}$ be the edge-loop defined by a given S-loop ϕ_r in O. It is not difficult to show that the irreducible edge-loop $\overline{c(\phi_r)}$ is, in fact, $c(\psi_s)$ for some S-loop ψ_s $(s \leqslant r)$ d-homotopic to ϕ_r.

For arbitrary digital spaces it may happen, for a cell $\alpha \in K$ with $c(\alpha) \in \mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$, that the set $\mathrm{st}_n(\alpha; O) = \emptyset$ is empty. This fact makes the search of digital representatives for an arbitrary edge-loop Γ in $\mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$ much more intricate than the case $S = \emptyset$ in [4]. In order to obtain such digital representatives for the edge-loop Γ we first set the following.

Definition 28. Let $O, S \subseteq \mathrm{cell}_n(K)$ be two disjoint digital objects in a digital space (K, f). We say that a cell $\alpha \in K$ is a *singular cell* for the pair (O, S), or simply an (O, S)-*singular cell*, if $c(\alpha) \in \mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$ but $\mathrm{st}_n(\alpha; O) = \emptyset$ (or, equivalently, $\mathrm{st}_n(\alpha; S) = \mathrm{st}_n(\alpha; O \cup S)$). Otherwise, if $c(\alpha) \in \mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$ and $\mathrm{st}_n(\alpha; O) \neq \emptyset$, α is called an (O, S)-*regular cell*.

We will also call (O, S)-regular to any edge-loop $\Omega = (c(\omega_i))_{i=0}^t$ in $\mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$ whose vertices correspond to (O, S)-regular cells; that is, ω_i is (O, S)-regular for $0 \leqslant i \leqslant t$.

Remark 29. (a) Notice that all cells $\alpha \in O$ are (O, S)-regular for any digital object S such that $O \cap S = \emptyset$. And, similarly, if α is a vertex of K such $c(\alpha) \in \mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$, then α is (O, S)-regular by Remark 4(3).

(b) If α is an (O, S)-singular cell then axiom (5) in the definition of w.l.f. applies. So, the set $\alpha(S; O \cup S) = \{\beta < \alpha; c(\beta) \in \mathscr{A}_{O\cup S} \setminus \mathscr{A}_S\}$ is not empty and connected in $\partial\alpha$. Moreover, from Lemma 4.5 in [3] it is derived the existence of (O, S)-regular cells in the set $\alpha(S; O \cup S)$.

Despite the difficulties above, it is still not hard to define the digital representatives for the family of irreducible (O, S)-regular edge-loops in $\mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$. We proceed as follows.

Definition 30. Let $\Omega = (c(\omega_i))_{i=0}^{2r}$ be an irreducible (O, S)-regular edge-loop in $\mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$ based at $c(\sigma)$, with $\sigma \in O$. The set $\mathscr{D}(\Omega)$ of *digital representatives of* Ω consists of all S-loops ϕ_r in O for which $\phi_r(0) = \phi_r(r) = \sigma$, $\phi_r(i - \frac{1}{2}) = \omega_{2i-1}$, and $\phi_r(i) \in \mathrm{st}_n(\omega_{2i}; O)$, for $1 \leqslant i \leqslant r$.

Remark 31. For any S-loop ϕ_r in O, the edge-loop $c(\phi_r) = (c(\phi_r(i/2)))_{i=0}^{2r}$ is (O, S)-regular since $\phi_r(i) \in O$, for $0 \leqslant i \leqslant r$, and $\phi_r(i/2 - 1) \leqslant \phi_r(i)$, for $1 \leqslant i \leqslant r$. In addition, $c(\phi_r)$ is irreducible in case $\phi_r(i - \frac{1}{2}) \neq \phi_r(i)$, for $1 \leqslant i \leqslant 2r$, and thus $\mathscr{D}(c(\phi_r)) = \{\phi_r\}$.

Next we state the crucial property of the digital representatives of an irreducible (O, S)-regular edge-loop in relation with the homomorphism $h : \pi_1^d(O/S, \sigma) \to \pi_1(\mathscr{A}_{O\cup S} \setminus \mathscr{A}_S, c(\sigma))$ above; compare with Proposition 4.12 in [4].

Proposition 32. *Let* $\Omega = (c(\omega_i))_{i=0}^{2t}$ *be any irreducible* (O, S)*-regular edge-loop in* $\mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$ *based at* $c(\sigma)$, *with* $\sigma \in O$. *For any S-loop* $\phi_t \in \mathscr{D}(\Omega)$ *the equality* $h([\phi_t]) = [\Omega]$ *holds. Moreover, any two S-loops in* $\mathscr{D}(\Omega)$ *are d-homotopic.*

Proof. First we show that the edge-loop $c(\phi_t) = (c(\phi_t(i/2)))_{i=0}^{2t}$ defined by ϕ_t is equivalent to Ω. For this, let $\bar{\mathscr{D}}(\Omega)$ be the set of edge-loops $\Lambda = (c(\lambda_i))_{i=0}^{2t}$ at $c(\sigma)$ such that $\lambda_0 = \lambda_{2t} = \sigma$, $\lambda_{2i-1} = \omega_{2i-1}$ for $1 \leqslant i \leqslant t$, and $\lambda_{2i} \in \mathrm{st}_n(\omega_{2i}; O) \cup \{\omega_{2i}\}$ for $1 < i < t$. Notice that $\bar{\mathscr{D}}(\Omega)$ contains the set of edge-loops $\{c(\phi_t); \phi_t \in \mathscr{D}(\Omega)\} \cup \{\Omega\}$. Moreover, any $\Lambda \in \bar{\mathscr{D}}(\Omega)$ is equivalent to Ω. This will be proved by induction on the number $k(\Lambda)$ of vertices $c(\lambda_{2i})$ in Λ for which $\lambda_{2i} \neq \omega_{2i}$. For $k(\Lambda) = 0$ we get $\Lambda = \Omega$. Assume that all $\Lambda \in \bar{\mathscr{D}}(\Omega)$ are equivalent to Ω for $k(\Lambda) \leqslant k - 1$, and let Λ be any edge-loop with

$k=k(\Lambda)$. Given any vertex $c(\lambda_{2i})$ in Λ with $\lambda_{2i}\neq\omega_{2i}$ (notice that $0\neq i\neq t$) we get $\omega_{2i-1},\omega_{2i+1}<\omega_{2i}<\lambda_{2i}$ since Ω is irreducible. Therefore we obtain a new edge–loop $\tilde{\Lambda}\in\bar{\mathscr{D}}(\Omega)$, with $k(\tilde{\Lambda})=k-1$, by setting $c(\tilde{\lambda}_j)=c(\lambda_j)$ for $j\neq 2i$, and $c(\tilde{\lambda}_{2i})=c(\omega_{2i})$. Moreover, $\tilde{\Lambda}$ is equivalent to Λ (by two equivalence transformations of type (b)) and hence Λ is equivalent to Ω by the induction hypothesis.

For the second property, we simply observe that the S-loops $\phi_t^1,\phi_t^2\in\mathscr{D}(\Omega)$ are d-homotopic rel. σ by the $(\emptyset,S)$-map $H:\mathrm{Lb}(V_{(r,1)}/\emptyset)\to\mathrm{Lb}(K(O\cup S)/S)$ given by $H(i/2,0)=\phi_t^1(i/2)$, $H(i/2,1)=\phi_t^2(i/2)$ and $H(i/2,1/2)=\omega_i$, for $0\leqslant i\leqslant 2t$. Here, we are using the identification of a cell $\alpha\in\mathrm{Lb}(V_{(r,1)}/\emptyset)$ with its centroid $c(\alpha)=(a_1,a_2)\in\mathscr{Z}^2$ (see Remark 14). □

In order to obtain a family $\mathscr{D}(\Gamma)$ of digital representatives for an arbitrary edge-loop Γ, we construct an auxiliary family $\mathrm{pre}^2\mathscr{D}(\Gamma)$ of irreducible (O,S)-regular edge-loops. For this we shall use of another family of edge-loops $\mathrm{pre}\mathscr{D}(\Gamma)$. This two-step process starts at the irreducible edge-loop $\bar{\Gamma}=(c(\bar{\gamma}_i))_{i=0}^{2r}$ obtained from Γ by deleting all its reducible vertices; see Lemma 26. Then, the edge-loops in $\mathrm{pre}\mathscr{D}(\Gamma)$ are chosen by diverting $\bar{\Gamma}$ off the vertices $c(\bar{\gamma}_{2i-1})$, with an odd index, corresponding to (O,S)-singular cells. And $\mathrm{pre}^2\mathscr{D}(\Gamma)$ consists of the edge-loops obtained from each $\Delta=(c(\delta_{2i}))_{i=0}^{2t}\in\mathrm{pre}\mathscr{D}(\Gamma)$ bypassing each vertex $c(\delta_{2i})$, with δ_{2i} an (O,S)-singular cell, along a new edge-walk whose vertices correspond to (O,S)-regular cells in $\partial\delta_{2i}$.

Indeed, the elements in $\mathrm{pre}\mathscr{D}(\Gamma)$ are the family of edge-loops $\Delta=(c(\delta_i))_{i=0}^{2r}$ in $\mathscr{A}_{O\cup S}\setminus\mathscr{A}_S$ with the same length as $\bar{\Gamma}$ and such that $\delta_{2i}=\bar{\gamma}_{2i}$, for $0\leqslant i\leqslant r$. Moreover, $\delta_{2i-1}=\bar{\gamma}_{2i-1}$ whenever $\bar{\gamma}_{2i-1}$ is an (O,S)-regular cell; and, otherwise, we choose $\delta_{2i-1}\in\{\alpha<\bar{\gamma}_{2i-1};\alpha$ is an (O,S)-regular cell$\}$, which is a non-empty set by Remark 29(b).

Notice that any $\Delta\in\mathrm{pre}\mathscr{D}(\Gamma)$ is irreducible. Moreover, the following lemma is immediate

Lemma 33. *Any $\Delta\in\mathrm{pre}\mathscr{D}(\Gamma)$ is equivalent to $\bar{\Gamma}$, and hence to Γ.*

Proof. Just observe that the substitution of any cell $\bar{\gamma}_{2i-1}$ by one of its faces induces two equivalence transformations of type (b) between $\bar{\Gamma}$ and Δ. □

If we write $\mathrm{pre}\mathscr{D}(\Gamma)=\{\Delta_k\}_{k\in J_\Gamma}$, a new family of irreducible edge-loops $\mathrm{pre}^2\mathscr{D}(\Delta_k)$ is defined for each $\Delta_k=(c(\delta_i))_{i=0}^{2r}$ as follows. An irreducible edge-loop $\Omega\in\mathrm{pre}^2\mathscr{D}(\Delta_k)$ is obtained by removing the reducible vertices from the juxtaposition of edge-walks $\Omega=\Omega_0*\Omega_1*\cdots*\Omega_r$, where $\Omega_0=(c(\delta_0),c(\delta_1))$, $\Omega_r=(c(\delta_{2r-1}),c(\delta_{2r}))$ and the component Ω_j, for $1\leqslant j\leqslant r-1$, is the constant edge-loop $\Omega_j=(c(\delta_{2j-1}))$ if $\delta_{2j-1}=\delta_{2j+1}$. Otherwise, if δ_{2j} is an (O,S)-regular cell, in particular if $\delta_{2j}\in O$ (see Remark 29(a)), then $\Omega_j=(c(\delta_{2j-1}),c(\delta_{2j}),c(\delta_{2j+1}))$. Finally, if δ_{2j} is an (O,S)-singular cell we pick Ω_j out the edge-walks obtained from the following lemma for the (O,S)-regular cells $\beta_1=\delta_{2j-1}$ and $\beta_2=\delta_{2j+1}$.

Lemma 34 (cf. Lemma 4.8 in Ayala et al. [3]). *Let $O,S\subseteq\mathrm{cell}_n(K)$ be two disjoint digital objects in a digital space (K,f), and let $\alpha\in K$ be an (O,S)-singular cell. Then,* *for any two distinct (O,S)-regular cells*

$$\beta_1,\beta_2 \in \alpha(S;O\cup S)=\{\beta<\alpha; f(O\cup S,\beta)=1, f(S,\beta)=0\}$$

there exist irreducible edge-walks $\Theta=(c(\theta_i))_{i=0}^m$ in $\mathscr{A}_{O\cup S}\setminus\mathscr{A}_S$ from $c(\beta_1)$ to $c(\beta_2)$ such that

(1) *for $0\leqslant i\leqslant m$, $\theta_i<\alpha$ and it is an (O,S)-regular cell; and,*
(2) *Θ is equivalent to the edge-walk $(c(\beta_1),c(\alpha),c(\beta_2))$.*

Proof. By axiom (5) in Definition 3 we know that the set $\alpha(S;O\cup S)$ is connected and so we can choose an edge-walk $\Phi=(c(\phi_i))_{i=0}^t$ in $\alpha(S;O\cup S)$ from $c(\beta_1)$ to $c(\beta_2)$. By deleting the reducible vertices we can assume that Φ is irreducible (see Lemma 26). Notice that Φ need not have an even length since β_1 and β_2 may have arbitrary dimensions. In any case, it is obvious that Φ is equivalent to the edge-walk $(c(\beta_1),c(\alpha),c(\beta_2))$.

We derive the walk Θ from Φ as follows. First we observe that $\dim\alpha\geqslant 2$ by Remark 4(3) and axiom (5) in Definition 3. Then we argue inductively on $l=\dim\alpha$. For $l=2$ we have necessarily $\dim\phi_i\leqslant 1$. Moreover, if $\dim\phi_i=0$ then ϕ_i is an (O,S)-regular cell by Remark 29(a). If $\dim\phi_i=1$ with $0<i<t$, the cells ϕ_{i-1} and ϕ_{i+1} are necessarily vertices of the edge $\phi_i\in K$. If, in addition, $\phi_{i-1}\neq\phi_{i+1}$ it follows that ϕ_i is also an (O,S)-regular cell by axiom (5) in Definition 3. Otherwise, if $\phi_{i-1}=\phi_{i+1}$, then we can delete the vertices $c(\phi_i)$ and $c(\phi_{i+1})$ from Φ to get a new irreducible edge-walk from $c(\beta_1)$ to $c(\beta_2)$ which is equivalent to Φ by two equivalence transformations of type (b). By deleting all the pairs $(c(\phi_i),c(\phi_{i+1}))$, for which ϕ_i is an edge in K and $\phi_{i-1}=\phi_{i+1}$, we obtain the desired edge-walk Θ.

Assume now that Θ can be derived from Φ for any cell α with $\dim\alpha<l$, and let $\dim\alpha=l$. If $\phi_0=\beta_1<\phi_1$ we proceed as in the construction of the family pre$\mathscr{D}(\Gamma)$ to define an auxiliary edge-walk $\Phi'=(c(\phi_i'))_{i=0}^t$, and then Θ, as follows (for $\phi_0=\beta_1>\phi_1$ the construction is similar but interchanging the roles played by cells with odd and even indices). The edge-walk Φ' is defined by $\phi'_{2j-1}=\phi_{2j-1}$ and also $\phi'_{2j}=\phi_{2j}$, for $0\leqslant 2j\leqslant t$, if ϕ_{2j} is an (O,S)-regular cell. Otherwise we choose $\phi'_{2j}\in\{\alpha<\phi_{2j};\alpha \text{ is } (O,S)\text{-regular}\}$. It is easily checked that Φ' is an irreducible edge-walk equivalent to Φ with its same length. Moreover, $\dim\phi'_{2j-1}<\dim\alpha$, for $0\leqslant 2j-1\leqslant t$, and $\phi'_0=\beta_1$ and $\phi'_t=\beta_2$.

We define Θ by the juxtaposition $\Theta=\Theta_1*\cdots*\Theta_k$ defined as follows. The index k is the largest integer with $2k-1\leqslant t$, and the edge-walks Θ_j, for $1\leqslant j\leqslant k$, are given by the next conditions:

1. $\Theta_j=(c(\phi'_{2j-2}))$ if $\phi'_{2j-2}=\phi'_{2j}$;
2. $\Theta_j=(c(\phi'_{2j-2}),c(\phi'_{2j-1}),c(\phi'_{2j}))$ if $\phi'_{2j-2}\neq\phi'_{2j}$ and ϕ'_{2j-1} is (O,S)-regular;
3. Θ_j is any of the edge-walks given by the induction hypothesis applied to ϕ'_{2j-1} and its faces ϕ'_{2j-2},ϕ'_{2j} whenever $\phi'_{2j-2}\neq\phi'_{2j}$ and ϕ'_{2j-1} is an (O,S)-singular cell.

By construction one readily checks that Θ satisfies properties (1) and (2) in the lemma. Moreover, after deleting the reducible vertices (if any) in Θ we can assume that Θ is also an irreducible edge-walk. $\square$

Remark 35. Observe that, given $\Delta \in \mathrm{pre}\mathscr{D}(\Gamma)$, any edge-loop $\Omega \in \mathrm{pre}^2\mathscr{D}(\Delta)$ is, by construction, equivalent to Δ, and hence to Γ by Lemma 33. Moreover, it is irreducible and (O,S)-regular.

Finally, we define the family $\mathscr{D}(\Gamma)$ of digital representatives of Γ as follows

Definition 36. Let Γ be an arbitrary edge-loop in $\mathscr{A}_{O\cup S}\setminus\mathscr{A}_S$ based at $c(\sigma)$, with $\sigma \in O$. We define the set $\mathscr{D}(\Gamma)$ of *digital representatives* of Γ by

$$\mathscr{D}(\Gamma) = \bigcup_{\Delta\in \mathrm{pre}\mathscr{D}(\Gamma)} \left(\bigcup_{\Omega\in \mathrm{pre}^2\mathscr{D}(\Delta)} \mathscr{D}(\Omega) \right).$$

Remark 37. (1) Let ∇ be and edge-loop in $\mathscr{A}_{O\cup S}\setminus\mathscr{A}_S$ obtained by removing from Γ any of its reducible vertices. Then $\bar{\nabla} = \bar{\Gamma}$ and hence $\mathscr{D}(\Gamma) = \mathscr{D}(\nabla)$. In particular, $\mathscr{D}(\Gamma) = \mathscr{D}(\bar{\Gamma})$, where $\bar{\Gamma}$ is the irreducible edge-loop obtained from Γ by removing all its reducible vertices.

(2) If Γ is an (O,S)-regular edge-loop in $\mathscr{A}_{O\cup S}\setminus\mathscr{A}_S$ then the irreducible edge-loop $\bar{\Gamma}$ is also (O,S)-regular. Thus, $\mathrm{pre}^2\mathscr{D}(\Gamma) = \{\bar{\Gamma}\}$, and all the digital representatives in $\mathscr{D}(\Gamma)$ are d-homotopic by Proposition 32.

(3) If ϕ_r is a S-loop in O, the family $\mathscr{D}(c(\phi_r))$ of digital representatives of the (O,S)-regular edge-loop $c(\phi_r) = (c(\phi_r(i/2)))_{i=0}^{2r}$ consists of a single element ψ_s, where $s \leqslant r$, by Remark 31. Moreover, ψ_s and ϕ_r are d-homotopic by Remark 27(c).

We are now ready to prove

Theorem 38. *Let (K,f) be an arbitrary digital space. For any two disjoint digital objects $O,S \subseteq \mathrm{cell}_n(K)$ the homomorphism*

$$h : \pi_1^d(O/S,\sigma) \to \pi_1(\mathscr{A}_{O\cup S}\setminus\mathscr{A}_S, c(\sigma))$$

is onto.

Proof. Given any edge-loop Γ in $\mathscr{A}_{O\cup S}\setminus\mathscr{A}_S$ based at $c(\sigma)$, we consider any edge-loop $\Omega \in \mathrm{pre}^2\mathscr{D}(\Gamma)$ which is equivalent to Γ by Remark 35. Then the result follows from Proposition 32. $\square$

Remark 39. To show that the homomorphism h is injective it is required, as a necessary condition, that $\phi^1 \simeq_d \phi^2$ rel. σ for any pair $\phi^1,\phi^2 \in \mathscr{D}(\Gamma)$ of digital representatives of an arbitrary edge-loop Γ in $\mathscr{A}_{O\cup S}\setminus\mathscr{A}_S$ (see Proposition 32). The construction of the family $\mathscr{D}(\Gamma)$ suggests that this fact may not be true in general. The main problem is that, from the available data, we cannot derive a d-homotopy between $\phi^1 \in \mathscr{D}(\Omega^1)$ and $\phi^2 \in \mathscr{D}(\Omega^2)$ whenever $\Omega^1 \neq \Omega^2$ in $\mathrm{pre}^2\mathscr{D}(\Gamma)$. However, we conjecture that this d-homotopy will be found if, for each (O,S)-singular cell α, the set $\cup\{\mathring{\omega}; \omega \in \alpha(S; O\cup S)\}$ is required to be simply connected instead of just connected as we require in Definition 3(5b). The results in next Section also suggest that, for the set of axioms considered in this paper, the search of a digital space (R^n, f) for which h is not an isomorphism should be carried out for $n \geqslant 4$.

3.2. A case of isomorphism

For important cases, the family of digital representatives $\mathscr{D}(\Gamma)$ in Definition 36 is dramatically simplified. Recall that, in general, the family $\mathscr{D}(\Gamma)$ is obtained by a three-steps procedure that involves the definition of two auxiliary families of edge-loops $\mathrm{pre}\mathscr{D}(\Gamma)$ and $\mathrm{pre}^2\mathscr{D}(\Gamma)$. In this Section we will give a large class of digital spaces (K, f) for which the families $\mathrm{pre}\mathscr{D}(\Gamma)$ and $\mathrm{pre}^2\mathscr{D}(\Gamma)$ are reduced to singletons; so that, the difficulties pointed out in Remark 39 vanish. This will allow us to show that the epimorphism h in Theorem 38 is an isomorphism for a large class of digital spaces, which includes those most used in image processing. Namely, we will prove below

Theorem 40. *Let (K, f) be any digital space which is strongly local except possibly in 1-cells; that is, for any digital object $O \subseteq \mathrm{cell}_n(K)$ and any cell $\alpha \in K$ with $\dim\alpha \neq 1$, $f(O,\alpha) = f(\mathrm{st}_n(\alpha; O), \alpha)$. Then the homomorphism $h : \pi_1^d(O/S, \sigma) \to \pi_1(\mathscr{A}_{O\cup S} \setminus \mathscr{A}_S, c(\sigma))$ is an isomorphism for any pair of disjoint objects $O, S \subseteq \mathrm{cell}_n(K)$.*

Corollary 41. *Let (K, f) be a strongly local digital space. For disjoint digital objects $O, S \subseteq \mathrm{cell}_n(K)$ the homomorphism h is an isomorphism.*

Recall that a digital space (K, f) is said to be strongly local if $f(O,\alpha) = f(\mathrm{st}_n(\alpha; O), \alpha)$ for all $\alpha \in K$ and $O \subseteq \mathrm{cell}_n(K)$; see Definition 3. For each pair $(p, q) \neq (6, 6)$, with $p, q \in \{6, 18, 26\}$, it can be found a strongly local lighting function $f_{p,q}$ on the device model R^3 such that the digital space $(R^3, f_{p,q})$ represents the (p, q)-connectivity on the grid $\mathbb{Z}^3$; and, moreover, all the (p, q)-surfaces, as defined by Kong and Roscoe in [10], are digital surfaces in $(R^3, f_{p,q})$; see Theorem 13 in [1]. Notice also that, for an arbitrary device model K, the digital spaces $(K, f_{\max})$ and (K, g) given in Example 6(1) are strongly local. Hence, for these relevant examples, Corollary 41 holds.

Moreover, as an immediate consequence of Remark 4(3), we have also

Corollary 42. *The homomorphism h is an isomorphism for digital spaces (K, f) with $\dim K \leqslant 2$.*

For non strongly local three-dimensional digital spaces we have the following

Lemma 43. *Let (R^3, f) be any digital space with R^3 the standard cubical decomposition of the Euclidean space $\mathbb{R}^3$. Moreover, assume $|\mathscr{A}_{R^3}| = \mathbb{R}^3$. Then the two following statements are equivalent.*

(i) *For each $O \subseteq \mathrm{cell}_3(R^3)$ and $\alpha \in R^3$ with $\dim\alpha = 2$, $f(O,\alpha) = 1$ if and only if $\alpha \in \mathrm{supp}(O)$.*
(ii) *(R^3, f) is strongly local except possibly for 1-cells.*

Proof. (i) implies (ii). It is clear that $\alpha \in \text{supp}(O)$ if and only if $\alpha \in \text{supp}(\text{st}_3(\alpha; O))$. Therefore, for $\dim \alpha = 2$ and $\alpha \in \text{supp}(O)$ we have $f(O, \alpha) = f(\text{st}_3(\alpha; O), \alpha) = 1$ by (i). Otherwise, in case $\alpha \notin \text{supp}(O)$, then $f(O, \alpha) = 0$ and $f(\text{st}_3(\alpha; O), \alpha) = 0$ by axiom (2) in Definition 3. For cells $\alpha \in R^3$ with $\dim \alpha \in \{0, 3\}$ the result follows from Remark 4(3).

(ii) implies (i). For any object $O \subseteq \text{cell}_3(R^3)$ and any 2-dimensional cell $\alpha \in R^3$ one easily checks that $\alpha \in \text{supp}(O)$ if and only if $\text{st}_3(\alpha; O) = \text{st}_3(\alpha; R^3)$. Hence $f(O, \alpha) = 0$ whenever $\alpha \notin \text{supp}(O)$ by axiom (2) in Definition 3 while $f(O, \alpha) = f(\text{st}_3(\alpha; R^3), \alpha) = f(\text{cell}_3(R^3), \alpha) = 1$ if $\alpha \in \text{supp}(O)$. Here we use that $|\mathscr{A}_{R^3}| = \mathbb{R}^3$. □

Then, we easily derive from Theorem 40 and Lemma 43 the following:

Theorem 44. *The homomorphism h is an isomorphism for the non strongly local digital space* (R^3, f^{BM}) *given in* [3].

We point out that the digital surfaces in (R^3, f^{BM}) coincide with the strong 26-surfaces defined by Malgouyres and Bertrand [12].

The rest of this section is devoted to the proof of Theorem 40. We start with the following result, whose proof is immediate from definitions.

Lemma 45. *Let* (K, f) *be a digital space which is strongly local at the cell* $\alpha \in K$. *Then this cell is* (O, S)*-regular for any pair of disjoint digital objects* $O, S \subseteq \text{cell}_n(K)$ *for which* $c(\alpha) \in \mathscr{A}_{O \cup S} \setminus \mathscr{A}_S$.

Lemma 46. *Let* (K, f) *be any digital space which is strongly local except possibly in* 1*-cells, and let* O, S *be two disjoint digital objects in* (K, f). *For any edge-loop* Γ *in* $\mathscr{A}_{O \cup S} \setminus \mathscr{A}_S$ *based at a vertex* $c(\sigma)$, *with* $\sigma \in O$, *the set* $\text{pre}^2\mathscr{D}(\Gamma) = \{\Omega_\Gamma\}$ *is a singleton. In particular, all the digital representatives of* Γ *are d-homotopic by Proposition* 32.

Proof. Notice that any cell $\alpha \in K$ with $c(\alpha) \in \mathscr{A}_{O \cup S} \setminus \mathscr{A}_S$ and $\dim \alpha \neq 1$ is (O, S)-regular by Lemma 45. So, the construction of the family $\mathscr{D}(\Gamma)$ is determined by the vertices $c(\alpha)$ with $\dim \alpha = 1$. More explicitly, if $\bar{\Gamma} = (c(\bar{\gamma}_i))_{i=0}^{2r}$ is the irreducible edge-loop in Lemma 26 then the family $\text{pre}\mathscr{D}(\Gamma) = \{\Delta_\Gamma\}$ consists of a unique (irreducible) edge-loop $\Delta_\Gamma = (c(\delta_i))_{i=0}^{2r}$ obtained by setting $\delta_{2i} = \bar{\gamma}_{2i}$ and replacing each vertex $c(\bar{\gamma}_{2i-1})$, with $\bar{\gamma}_{2i-1}$ an (O, S)-singular 1-cell, by $c(\delta_{2i-1})$ where $\delta_{2i-1} < \bar{\gamma}_{2i-1}$ is the unique vertex of $\bar{\gamma}_{2i-1}$ with $c(\delta_{2i-1}) \in \mathscr{A}_{O \cup S} \setminus \mathscr{A}_S$ or, equivalently, which is an (O, S)-regular cell. Here we use axiom (5) in Definition 3. Moreover, $\text{pre}^2\mathscr{D}(\Gamma) = \text{pre}^2\mathscr{D}(\Delta_\Gamma) = \{\Omega_\Gamma\}$ is also a singleton since, for any vertex $c(\delta_{2i})$ in Δ_Γ, with $0 < i < r$, such that $\delta_{2i} = \bar{\gamma}_{2i}$ is an (O, S)-singular 1-cell, axiom (5) in Definition 3 yields $\delta_{2i-1} = \delta_{2i+1}$. Hence Ω_Γ is determined by replacing the edge-walk $(c(\delta_{2i-1}), c(\delta_{2i}), c(\delta_{2i+1}))$ by the constant edge-walk $(c(\delta_{2i-1}))$. □

Lemma 47. *Let* (K, f) *be a digital space which is strongly local except possibly in* 1*-cells, and let* O, S *be two disjoint digital objects in* (K, f). *Then any edge-loop*

$\Gamma = (c(\gamma_i))_{i=0}^{k}$ in $\mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$ based at a $c(\sigma)$, with $\sigma \in O$, is equivalent to an (O,S)-regular edge-loop $\Gamma^* = (c(\gamma_i^*))_{i=0}^{k}$ called the regularization *of Γ. Moreover, if Σ is another edge-loop obtained from Γ by removing a vertex $c(\gamma_{i_0})$ via an equivalence transformation of type (a) or (b), then the regularization of Σ, Σ^*, can be derived from Γ^* after an equivalence transformation of the same type.*

Proof. We construct the edge-loop Γ^* as follows. If γ_i is (O,S)-regular we set $\gamma_i^* = \gamma_i$. Otherwise, $\dim \gamma_i = 1$ by Lemma 45 and we take γ_i^* to be the unique vertex $\gamma_i^* < \gamma_i$ such that $c(\gamma_i^*) \in \mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$ or, equivalently, which is (O,S)-regular. Here we use Axiom 5 in Definition 3. In order to show that $\Gamma^* = (c(\gamma_i^*))_{i=0}^{k}$ is an edge-loop equivalent to Γ we consider the set $\mathscr{R}\mathrm{eg}(\Gamma)$ consisting of finite sequences $\Lambda = (c(\lambda_i))_{i=0}^{k}$ such that $\lambda_i = \gamma_i$ if γ_i is (O,S)-regular and $\lambda_i \in \{\gamma_i, \gamma_i^*\}$ otherwise. Notice that $\{\Gamma, \Gamma^*\} \subseteq \mathscr{R}\mathrm{eg}(\Gamma)$. Next we show inductively that each $\Lambda \in \mathscr{R}\mathrm{eg}(\Gamma)$ is an edge-loop in $\mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$ based at $c(\sigma)$ which is equivalent to Γ. For this, let $t(\Lambda)$ be the number of vertices $c(\lambda_i)$ with $\lambda_i \neq \gamma_i$. If $t(\Lambda) = 0$ then $\Lambda = \Gamma$ and the result is obvious. Assume the result holds for $t(\Lambda) \leqslant t-1$ and take $\Lambda \in \mathscr{R}\mathrm{eg}(\Gamma)$ with $t(\Lambda) = t$. Given any vertex $c(\lambda_i) \in \Lambda$ with $\lambda_i = \gamma_i^* < \gamma_i$ we consider the sequence $\tilde{\Lambda} \in \mathscr{R}\mathrm{eg}(\Gamma)$ with $\tilde{\lambda}_j = \lambda_j$ if $j \neq i$ and $\tilde{\lambda}_i = \gamma_i$. Notice that $0 < i < n$ since $\gamma_0 = \gamma_k = \sigma = \lambda_0 = \lambda_k \in O$ is an (O,S)-regular cell by Remark 29(a). By the induction hypothesis $\tilde{\Lambda}$ is an edge-loop in $\mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$ based at $c(\sigma)$ which is equivalent to Γ. Moreover, we have the following possible face relations between $\tilde{\lambda}_j = \lambda_j$ $(j = i-1, i+1)$ and $\tilde{\lambda}_i = \gamma_i$: (1) $\tilde{\lambda}_j < \tilde{\lambda}_i$, or (2) $\tilde{\lambda}_j \geqslant \tilde{\lambda}_i$. In the first case we have that $\lambda_j = \gamma_i^* = \lambda_i$, while in the second $\lambda_j \geqslant \gamma_i > \gamma_i^* = \lambda_i$. In any case, Λ is an edge-loop equivalent to $\tilde{\Lambda}$, and hence to Γ, via two transformations of type (b).

Let us now assume that $c(\gamma_{i_0})$ can be removed from Γ by an equivalence transformation. Then one of the following cases necessarily occurs:

1. $0 < i_0 < t$; $\gamma_{i_0-1} < \gamma_{i_0} > \gamma_{i_0+1}$,
2. $0 < i_0 < t$; $\gamma_{i_0-1} > \gamma_{i_0} < \gamma_{i_0+1}$,
3. $\gamma_{i_0-1} = \gamma_{i_0}$,
4. $\gamma_{i_0-1} < \gamma_{i_0} < \gamma_{i_0+1}$,
5. $\gamma_{i_0-1} > \gamma_{i_0} > \gamma_{i_0+1}$,
6. $\gamma_{i_0} = \gamma_{i_0+1}$.

Let $(1)^* \ldots (6)^*$ denote the corresponding statements for the vertices in Γ^*. The reader can easily check that $(i) \Rightarrow (i)^*$ if no (O,S)-singular cell is involved. In case γ_{i_0-1} is singular then both (2) and (5) yield $(3)^*$, and for the rest of statements we get $(i) \Rightarrow (i)^*$. If γ_{i_0} is singular then (1) yields $\gamma_{i_0-1}^* = \gamma_i^* = \gamma_{i_0+1}^*$, while $(4) \Rightarrow (3)^*$ and $(5) \Rightarrow (6)^*$, and $(i) \Rightarrow (i)^*$ for the other cases. Finally if γ_{i_0+1} is singular we derive $(6)^*$ from both (2) and (4), while for the remaining cases $(i) \Rightarrow (i)^*$. From this, and analyzing all the possible cases, it is not difficult to show that $c(\gamma_{i_0}^*)$ can also be removed from Γ^* by an equivalence trasformation. $\square$

Proof of Theorem 40. We already know that h is onto by Theorem 38. So, it will suffice to prove that any two S-loops, ϕ, ψ, in O define the same element in $\pi_1^d(O/S, \sigma)$ provided $h([\phi]) = [c(\phi)] = [c(\psi)] = h([\psi])$.

Since $c(\phi)$ and $c(\psi)$ are equivalent edge-loops, there exists a sequence $\alpha_0, \alpha_1, \dots, \alpha_k$ of edge-loops in $\mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$ such that $\alpha_0 = c(\phi)$, $\alpha_k = c(\psi)$ and α_{i-1}, α_i are related by an equivalence transformation of type (a) or (b). Moreover, by Remark 31, $c(\phi)$ and $c(\psi)$ are (O,S)-regular, and the regularized edge-loops $c(\phi) = \alpha_0^*, \alpha_1^*, \dots, \alpha_k^* = c(\psi)$ define also a sequence of equivalent edge-loops by Lemma 47. Then, Remark 37(2) and Lemma 48 below yields that every S-loop in $\cup_{i=0}^{k} \mathscr{D}(\alpha_i^*)$ defines the same element in $\pi_1^d(O/S, \sigma)$. Hence ϕ and ψ are d-homotopic by Remark 37(3). □

Next lemma is an extension of Lemma 4.14 in [4] which corresponds to the special case $S = \emptyset$.

Lemma 48. *Let $O, S \subseteq \mathrm{cell}_n(K)$ be two disjoint digital objects in an arbitrary digital space (K, f), and let $\Gamma = (c(\gamma_i))_{i=0}^{t}$ be an (O,S)-regular edge-loop in $\mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$ based at $c(\sigma)$, with $\sigma \in O$. Assume that an edge-loop Σ is obtained by removing a vertex $c(\gamma_{i_0})$ from Γ after an equivalence transformation of type (a) or (b). Then, for each S-loop $\phi \in \mathscr{D}(\Gamma)$ there exist a digital representative $\psi \in \mathscr{D}(\Sigma)$ and a d-homotopy $\phi \simeq_d \psi$ rel. σ.*

Proof. The hypothesis leads to one of the following cases:

(1) $0 < i_0 < t$, the centroids $c(\gamma_{i_0-1}), c(\gamma_{i_0}), c(\gamma_{i_0+1})$ span a simplex in $\mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$ and $\gamma_{i_0-1} < \gamma_{i_0} > \gamma_{i_0+1}$.
(2) $0 < i_0 < t$, the centroids $c(\gamma_{i_0-1}), c(\gamma_{i_0}), c(\gamma_{i_0+1})$ span a simplex in $\mathscr{A}_{O\cup S} \setminus \mathscr{A}_S$ and $\gamma_{i_0-1} > \gamma_{i_0} < \gamma_{i_0+1}$.
(3) $c(\gamma_{i_0})$ is a reducible vertex in γ.
(4) $\gamma_{i_0} = \gamma_{i_0+1}$, and hence the vertex $c(\gamma_{i_0+1})$ is reducible.

In cases (3) and (4) the edge-loop Σ is obtained by dropping a reducible vertex from Γ, so $\mathscr{D}(\Gamma) = \mathscr{D}(\Sigma)$ by Remark 37(1) and the result follows from Remark 37(2). Therefore we concentrate our efforts in proving the lemma for the case (1) since case (2) is settled in a similar way.

We start by considering the number $n(\Gamma)$ of reducible vertices of Γ in the set

$$V_\Gamma = \{c(\gamma_j); 0 \leqslant j \leqslant i_0 - 2\} \cup \{c(\gamma_j); i_0 + 2 \leqslant j \leqslant t\}.$$

Since any reducible vertex in V_Γ is also a reducible vertex of Σ we can remove all of them from both Γ and Σ. This way we replace Γ and Σ by two new edge-loops Γ' and Σ' respectively such that $n(\Gamma') = 0$. Moreover, by Remark 37(1), $\mathscr{D}(\Gamma) = \mathscr{D}(\Gamma')$ and $\mathscr{D}(\Sigma) = \mathscr{D}(\Sigma')$. Hence, by Remark 37(2), there is no loss of generality in assuming $\Gamma = \Gamma'$ and $\Sigma = \Sigma'$.

Next we consider all possible face relations among the pairs of cells $(\gamma_{i_0-2}, \gamma_{i_0-1})$, $(\gamma_{i_0+1}, \gamma_{i_0+2})$ and $(\gamma_{i_0-1}, \gamma_{i_0+1})$. Notice that the two elements in each pair may be equal, and in case (2) it is also possible that $i_0 = 1$ or $i_0 = t - 1$. The proof requires in general the four steps below whatever be the face relations we consider. For illustrating the proof we give a detailed account of these steps for the case (1) and the face relations

$$\gamma_{i_0-2} > \gamma_{i_0-1} < \gamma_{i_0} > \gamma_{i_0+1} < \gamma_{i_0+2} \tag{I}$$

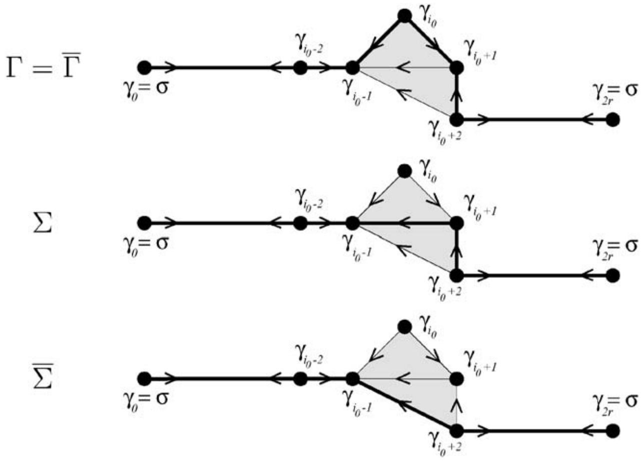

Fig. 4.

and

$$\gamma_{i_0-1} < \gamma_{i_0+1}. \tag{II}$$

Step A. Describe the irreducible edge-loops $\bar{\Gamma}$ and $\bar{\Sigma}$.

The face relations (I) and (II) yield that Γ has not reducible vertices, so that $\Gamma = \bar{\Gamma}$ is an edge-loop of even length $t = 2r$ by Remark 27(a) and, moreover, i_0 is an even number too. In addition, the irreducible edge-loop $\bar{\Sigma}$ associated to Σ is

$$\bar{\Sigma} = (c(\gamma_0), \dots, c(\gamma_{i_0-2}), c(\gamma_{i_0-1}), c(\gamma_{i_0+2}), \dots, c(\gamma_{2r}))$$

since $c(\gamma_{i_0+1})$ is reducible in Σ by the face relations (I) and (II); see Fig. 4. Therefore, any digital representative of Γ is an S-loop of length r, while digital representatives of Σ have length $r - 1$.

Notice that under a different set of face relations Γ and $\bar{\Gamma}$ may be distinct. In any case, the length of $\bar{\Gamma}$ is always greater than or equal to the length of $\bar{\Sigma}$, and the same happens for the digital representatives of Γ and Σ.

Step B. Given a digital representative $\phi \in \mathscr{D}(\Gamma)$ of Γ, derive a digital representative $\psi \in \mathscr{D}(\Sigma)$ of Σ.

Given $\phi = \phi_r \in \mathscr{D}(\Gamma)$, it is not difficult to check from Step A that the S-loop $\psi = \psi_{r-1}$, given by $\psi_{r-1}(j/2) = \phi_r(j/2)$, for $0 \leqslant j \leqslant i_0 - 1$, and $\psi_{r-1}(j/2) = \phi_r(j/2 + 1)$, for $i_0 \leqslant j \leqslant 2r - 2$, is a digital representative of the edge-loop Σ.

Step C. Obtain a new S-loop $\tilde{\psi}$ d-homotopic to ψ and such that $\tilde{\psi}$ and ϕ have the same length.

By Definition 17, the S-loops $\psi = \psi_{r-1}$ and $\psi_{r-1} * \psi_1^{\sigma}$ are d-homotopic, where ψ_1^{σ} is the constant S-loop of length 1 at $\sigma = \psi_{r-1}(0) = \psi_{r-1}(r-1)$. Then, Proposition 18 yields the following d-homotopy

$$\psi_{r-1} * \psi_1^{\sigma} \simeq_d \psi_{i_0/2} * \psi_1^{\tau} * \psi_{r-1-i_0/2} = \tilde{\psi}_r,$$

where $\psi_{i_0/2}$ and $\psi_{r-1-i_0/2}$ are the S-walks in O given by $\psi_{i_0/2}(j/2) = \psi_{r-1}(j/2)$, for $0 \leqslant j \leqslant i_0$ and $\psi_{r-1-i_0/2}(j/2) = \psi_{r-1}((j+i_0)/2)$, for $0 \leqslant j \leqslant 2r - i_0 - 2$, respectively, and moreover ψ_1^{τ} is the constant S-loop of length 1 at $\tau = \psi_{r-1}(i_0/2)$.

In general, different constant S-loops may be required for other sets of face relations. Notice also that this step could be not necessary in case the original digital representatives ϕ and ψ have the same length.

Step D. Describe a d-homotopy between ϕ and $\tilde{\psi}$, and then the lemma follows.

From the face relations (I) and (II) it is not difficult to show that the d-map given by

$$H\left(\frac{j}{2}, \frac{k}{2}\right) = \begin{cases} \phi_r(j/2) & \text{if } k = 0 \text{ and } 0 \leqslant j \leqslant 2r, \\ \phi_r(j/2) & \text{if } k = 1 \text{ and } 0 \leqslant j \leqslant i_0 - 1 \text{ or } i_0 + 1 \leqslant j \leqslant 2r, \\ \gamma_{i_0+1} & \text{if } k = 1 \text{ and } j = i_0, \\ \tilde{\psi}_r(j/2) & \text{if } k = 2 \text{ and } 0 \leqslant j \leqslant 2r \end{cases}$$

is a d-homotopy between ϕ_r and the S-loop $\bar{\psi}_r \simeq_d \psi_{r-1}$. $\square$

Other sets of face relations lead to possibly different d-homotopies although all are of the same nature.

4. Further remarks

As it was mentioned in the Introduction, Khalismky [7] (see also [5]) has considered a different approach to discrete loops and homotopies leading to an alternative definition of digital fundamental group π_1^{Kh}. Our group π_1^d in Definition 13 is closely related to Khalimsky's group. More explicitly, recall that the group π_1^{Kh} is constructed within the class of locally finite T_0 topological spaces (here Khalimsky's spaces) where "intervals" and "rectangles" are suitably defined. By Remark 1, any device model K, and more generally a subset of its cells, yields a locally finite T_0 space in such a way that 1- and 2-dimensional windows in Definition 13 correspond to "intervals" and "rectangles", respectively. In this way, digital loops and homotopies defined in Section 2 can be regarded as loops and homotopies in Khalimsky's setting. In the former, however, one can check that they are d-maps satisfying condition (2) in Definition 11 but need not satisfy condition (1); that is, they preserve the face realitionship but the top dimensional cells in windows (open points in the corresponding Khalimsky's space) may be mapped to arbitrary cells of the device model (arbitrary points in the associated Khalimsky's space). This way Khalimsky considers a larger class of loops and homotopies than us in order to define his digital fundamental group. Nevertheless, these additional loops seem not be very appropriate from a digital point of view since they may entirely consist of lower dimensional cells. In contrast, the loops in Definition 15 are forced, by condition

11(1), to pass from a pixel to another one through a common face. This keeps close our group π_1^d to Kong's graph-theoretical approach to the digital fundamental group [8] in which no others than adjacent pixels form a discrete loop.

For two disjoint digital objects O and S in a digital space (K, f), we conjecture that the group π_1^{Kh} of the space associated to the light body $\mathrm{Lb}(K(O \cup S)/S)$ is always isomorphic to the ordinary fundamental group of the difference of continuous analogues $|\mathscr{A}_{O \cup S}| - |\mathscr{A}_S|$, even in case the function f only satisfies Axiom (1) in Definition 3. If this conjecture holds, for well-behaved digital spaces (i.e., satisfying all the five axioms in Definition 3), Theorem 23 in [4] would give an isomorphism between our group π_1^d and Khalimsky's group π_1^{Kh} for $S = \emptyset$. And, more generally, Theorem 40 above would extend this result when $S \neq \emptyset$ for a large class of digital spaces including those often used in image processing (see Section 3.2). Of course, these conjectures must be carefully analyzed as well as other relations among the different notions of digital fundamental groups in the literature. In particular, the digital fundamental groups introduced by Kong for digital objects and their complements should be compared with our groups $\pi_1^d(O, \sigma)$ and $\pi_1^d(O/(\mathrm{cell}_n(K) - O), \sigma)$. We expect to take over this task in a future paper.

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