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# A three-term theta function identity and its applications

Zhi-Guo Liu[1]

*Department of Mathematics, East China Normal University, Shanghai 200062, China*

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Dedicated to Bruce Berndt on his 65th birthday

**Abstract**

In this paper, we establish a three-term theta function identity using the complex variable theory of elliptic functions. This simple identity in form turns out to be quite useful and it is a common origin of many important theta function identities. From which the quintuple product identity and one general theta function identity related to the modular equations of the fifth order and many other interesting theta function identities are derived. We also give a new proof of the addition theorem for the Weierstrass elliptic function $\wp$. An identity involving the products of four theta functions is given and from which one theta function identity by McCullough and Shen is derived. The quintuple product identity is used to derive some Eisenstein series identities found in Ramanujan's lost notebook and our approach is different from that of Berndt and Yee. The proofs are self contained and elementary.


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*E-mail addresses:* liuzg18@hotmail.com, zgliu@math.ecnu.edu.cn.

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## 1. Introduction

We suppose throughout this paper that $q = \exp(2\pi i\tau)$, where $\tau$ has positive imaginary part. We will use the familiar notation

$$(z;q)_\infty = \prod_{n=0}^{\infty}(1-zq^n), \tag{1.1}$$

and sometimes write

$$(a,b,c,\ldots;q)_\infty = (a;q)_\infty(b;q)_\infty(c;q)_\infty\ldots. \tag{1.2}$$

To carry out our study, we need some basic facts about the Jacobi theta function $\theta_1(z|\tau)$ which is defined as

$$\begin{aligned}\theta_1(z|\tau) &= -iq^{1/8}\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n+1)/2}e^{(2n+1)iz}\\ &= 2q^{1/8}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}\sin(2n+1)z\end{aligned} \tag{1.3}$$

(see, for example, [25, p. 463]). From this we readily find that

$$\theta_1(z+\pi|\tau) = -\theta_1(z|\tau) \quad \text{and} \quad \theta_1(z+\pi\tau|\tau) = -q^{-1/2}e^{-2iz}\theta_1(z|\tau). \tag{1.4}$$

Using the well-known Jacobi triple product identity

$$(q,z,q/z;q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(n-1)/2}z^n \tag{1.5}$$

(see [2, pp. 21–22; 5, p. 35; 12,13]), we can deduce the infinite product representation for $\theta_1(z|\tau)$, namely,

$$\begin{aligned}\theta_1(z|\tau) &= 2q^{1/8}(\sin z)(q,qe^{2iz},qe^{-2iz};q)_\infty\\ &= iq^{1/8}e^{-iz}(q,e^{2iz},qe^{-2iz};q)_\infty\end{aligned} \tag{1.6}$$

(see, for example, [25, p. 469]). In this paper, we use $\theta_1'(z|\tau)$ to denote the partial derivative of $\theta_1(z|\tau)$ with respect to $z$. Differentiating (1.6) with respect to $z$ and then putting $z=0$ we have

$$\theta_1'(0|\tau) = 2q^{1/8}(q;q)_\infty^3. \tag{1.7}$$

In [15], we use the complex variable theory of elliptic functions to establish a general theta function identity. We then derive some remarkable theta function identities related to the modular equations of degree 5; in particular, we give new proofs of the two fundamental identities satisfied by the Rogers–Ramanujan continued fraction. In [17], we set up a general theta function identity with four parameters in the same way and this identity plays a central role to the cubic theta function identities. In this paper, we establish the following three-term theta function identity in the same spirit. This simple theta function identity in form turns out to be a fundamental theta function identity. From which, with a little calculus, we can derive the quintuple product identity and many other interesting theta function identities.

**Theorem 1.** *Suppose $f(z)$ is an entire function satisfying the functional equations*

$$f(z+\pi) = f(z) \quad \textit{and} \quad f(z+\pi\tau) = q^{-2}e^{-8iz}f(z). \tag{1.8}$$

*Then there is a constant $C$ independent of $z$ such that*

$$f(z) - f(-z) = C\theta_1(2z|\tau). \tag{1.9}$$

The rest of the paper are organized as follows. In Section 2, we prove Theorem 1 with the classical theory of elliptic functions. In Section 3, we first prove the following theorem using Theorem 1.

**Theorem 2.** *Suppose $h(z)$ is an entire function satisfying the functional equations*

$$h(z+\pi) = -h(z) \quad \textit{and} \quad h(z+\pi\tau) = -q^{-3/2}e^{-6iz}h(z). \tag{1.10}$$

*Then we have*

$$h(z) + h(-z) = \frac{\theta_1(2z|\tau)}{\theta_1(z|\tau)}h(0). \tag{1.11}$$

Then we derive the quintuple product identity and some other interesting theta function identities from this theorem. In Section 4, we study some Eisenstein series identities in Ramanujan's lost notebook and our approach is different from that of Berndt and Yee [8]. In Section 5, we establish the following general theta function identity using Theorem 1.

**Theorem 3.** *If $f_1(z)$ and $f_2(z)$ are two different entire functions satisfying the functional equations*

$$f(z+\pi) = -f(z) \qquad \textit{and} \qquad f(z+\pi\tau) = -q^{-5/2}e^{-10iz}f(z), \tag{1.12}$$

*Then there is a constant C independent of z such that*

$$C\theta_1(z|\tau)\theta_1(2z|\tau) = f_2(0)(f_1(z) + f_1(-z)) - f_1(0)(f_2(z) + f_2(-z)). \tag{1.13}$$

This is the main result in [15], but the proof of this paper is much simpler than that of [15]. In Section 6, we provide a new proof of the addition theorem for Weierstrass $\wp$ function. In Section 7, we prove the following identity using Theorem 1.

**Theorem 4.** *We have*

$$\begin{aligned}&\theta_1(z+x_1|\tau)\theta_1(z+x_2|\tau)\theta_1(z+x_3|\tau)\theta_1(z-x_1-x_2-x_3|\tau)\\&\quad -\theta_1(z-x_1|\tau)\theta_1(z-x_2|\tau)\theta_1(z-x_3|\tau)\theta_1(z+x_1+x_2+x_3|\tau)\\&\quad = -\theta_1(x_1+x_2|\tau)\theta_1(x_1+x_3|\tau)\theta_1(x_2+x_3|\tau)\theta_1(2z|\tau). \end{aligned}\tag{1.14}$$

From this we deduce the following identity by McCullough and Shen, which has been used to study the Sezgö kernel of an annulus [19].

**Theorem 5.** *We have*

$$\begin{aligned}&\frac{\theta_1'}{\theta_1}(x_1|\tau)+\frac{\theta_1'}{\theta_1}(x_2|\tau)+\frac{\theta_1'}{\theta_1}(x_3|\tau)-\frac{\theta_1'}{\theta_1}(x_1+x_2+x_3|\tau)\\&\quad =\theta_1'(0|\tau)\frac{\theta_1(x_1+x_2|\tau)\theta_1(x_1+x_3|\tau)\theta_1(x_2+x_3|\tau)}{\theta_1(x_1|\tau)\theta_1(x_2|\tau)\theta_1(x_3|\tau)\theta_1(x_1+x_2+x_3|\tau)}.\end{aligned}\tag{1.15}$$

## 2. The proof of Theorem 1

To prove the theorem, we require the following Lemma 1. Lemma 1 is a fundamental theorem of elliptic functions and can be found in [1, p. 6] or [10, p. 22]. Recently, in [15–18], we have used Lemma 1 to set up some important theta function identities.

**Lemma 1.** *The sum of the residues of an elliptic functions at its poles in any period parallelogram is zero.*

Now we begin to prove Theorem 1 by using Lemma 1.

**Proof.** Suppose that $f(u)$ is the given function satisfying the functional equations (1.8). Then we consider the function

$$g(u) = \frac{f(u)}{\theta_1(u-y|\tau)\theta_1(u+y|\tau)\theta_1(u-z|\tau)\theta_1(u+z|\tau)}. \tag{2.1}$$

Here we temporarily assume that $0 < y, z < \pi$ be two distinct parameters different from the zero points of $f(u)$. Using the functional equations in (1.4), we can verify that $g(u+\pi) = g(u)$ and $g(u+\pi\tau) = g(u)$. Hence $g(u)$ is an elliptic function with periods $\pi$ and $\pi\tau$. It is obvious that $y, \pi - y, z, \pi - z$ are its only poles and all its poles are simple poles.

In this paper we use $\operatorname{res}(g; \alpha)$ to denote the residue of $g$ at $\alpha$. Then Lemma 1 gives

$$\operatorname{res}(g; y) + \operatorname{res}(g; \pi - y) + \operatorname{res}(g; z) + \operatorname{res}(g; \pi - z) = 0. \tag{2.2}$$

Now we begin to compute the residues. By L'Hopital's rule, we have

$$\begin{aligned} \operatorname{res}(g; y) &= \lim_{u \to y} (u - y) g(u) \\ &= \lim_{u \to y} \frac{f(u)}{\theta_1(u+y|\tau)\theta_1(u-z|\tau)\theta_1(u+z|\tau)} \times \lim_{u \to y} \frac{u-y}{\theta_1(u-y|\tau)} \\ &= \frac{f(y)}{\theta_1'(0|\tau)\theta_1(2y|\tau)\theta_1(y-z|\tau)\theta_1(y+z|\tau)}. \end{aligned} \tag{2.3}$$

Replacing $y$ with $\pi - y$ and then using the relations $f(\pi - y) = f(-y)$, $\theta_1(2\pi - 2y|\tau) = -\theta_1(2y|\tau)$, $\theta_1(y - \pi - z|\tau) = -\theta_1(y+z|\tau)$, $\theta_1(y - \pi + z|\tau) = -\theta_1(y - z|\tau)$ in the resulting equation, we find that

$$\operatorname{res}(g; \pi - y) = -\frac{f(-y)}{\theta_1'(0|\tau)\theta_1(2y|\tau)\theta_1(y-z|\tau)\theta_1(y+z|\tau)}. \tag{2.4}$$

We note that $g(u)$ is symmetric in $y$ and $z$ and so we interchange $y$ and $z$ in (2.3) and (2.4), respectively, we arrive at

$$\operatorname{res}(g; z) = -\frac{f(z)}{\theta_1'(0|\tau)\theta_1(2z|\tau)\theta_1(y-z|\tau)\theta_1(y+z|\tau)}, \tag{2.5}$$

$$\operatorname{res}(g; \pi - z) = \frac{f(-z)}{\theta_1'(0|\tau)\theta_1(2y|\tau)\theta_1(y-z|\tau)\theta_1(y+z|\tau)}. \tag{2.6}$$

We substitute (2.3)–(2.6) into (2.2) and then cancel out the factor $\theta_1'(0|\tau)\theta_1(y - z|\tau)\theta_1(y+z|\tau)$ to obtain

$$\frac{f(z) - f(-z)}{\theta_1(2z|\tau)} = \frac{f(y) - f(-y)}{\theta_1(2y|\tau)}. \tag{2.7}$$

This identity indicate that $(f(z) - f(-z))/\theta_1(2z|\tau)$ is independent of $z$ and so it must be a constant, say $C$. Thus we obtain (1.6). By analytic continuation, we know that (1.6) holds for any complex $z$ and so this completes the proof of Theorem 1. $\square$

## 3. The proof of Theorem 2 and the quintuple product identity

In this section we first prove Theorem 2 and then discuss its applications.

**Proof of Theorem 2.** The function $h(z)$ satisfies the functional equations in (1.10) and $\theta_1(z|\tau)$ satisfies the functional equations in (1.4). Thus we know that $h(z)\theta_1(z|\tau)$ satisfies the functional equations in (1.8). So we can take $f(z) = h(z)\theta_1(z|\tau)$ in (1.9) to obtain

$$\theta_1(z|\tau)(h(z) + h(-z)) = C\theta_1(2z|\tau). \tag{3.1}$$

Dividing both sides by $\theta_1(z|\tau)$ we have

$$h(z) + h(-z) = C\frac{\theta_1(2z|\tau)}{\theta_1(z|\tau)}. \tag{3.2}$$

We set $z = 0$ in this equation to find that $C = h(0)$. Substituting $C = h(0)$ back to (3.2) we arrive at (1.11) and this completes the proof of Theorem 2. □

Next we prove the following identity by employing Theorem 2.

**Theorem 6.** *We have*

$$(q;q)_\infty \frac{\theta_1(2z|\tau)}{\theta_1(z|\tau)} = 2\sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2+n)/2} \cos(6n+1)z. \tag{3.3}$$

**Proof.** We can verify that $e^{2iz}\theta_1(3z + \pi\tau|3\tau)$ satisfies (1.10) by using (1.4), so we can take $h(z) = e^{2iz}\theta_1(3z + \pi\tau|3\tau)$ in (1.11). Using the infinite product representation for $\theta_1(z|\tau)$ and a direct computation, we find $h(0) = \theta_1(\pi\tau|3\tau) = iq^{-1/8}(q;q)_\infty$. Hence we have

$$iq^{-1/8}(q;q)_\infty \frac{\theta_1(2z|\tau)}{\theta_1(z|\tau)} = e^{2iz}\theta_1(3z + \pi\tau|3\tau) - e^{-2iz}\theta_1(3z - \pi\tau|3\tau). \tag{3.4}$$

Using the series expansion for $\theta_1(z|\tau)$ in (1.3) we infer that

$$e^{2iz}\theta_1(3z + \pi\tau|3\tau) = iq^{-1/8}\sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2+n)/2} e^{-(6n+1)zi}. \tag{3.5}$$

Replacing $z$ by $-z$ we have

$$e^{-2iz}\theta_1(3z-\pi\tau|3\tau) = -iq^{-1/8}\sum_{n=-\infty}^{\infty}(-1)^n q^{(3n^2+n)/2}e^{(6n+1)zi}. \tag{3.6}$$

Using the Euler identity $2\cos z = e^{iz}+e^{-iz}$ we readily find

$$\begin{aligned} &e^{2iz}\theta_1(3z+\pi\tau|3\tau) - e^{-2iz}\theta_1(3z-\pi\tau|3\tau)\\ &\quad = 2iq^{-1/8}\sum_{n=-\infty}^{\infty}(-1)^n q^{(3n^2+n)/2}\cos(6n+1)z. \end{aligned} \tag{3.7}$$

We substitute (3.7) into the right-hand side of (3.4) and cancel out the factor $iq^{-1/8}$ to get (3.3). Thus we complete the proof of Theorem 6. □

Using the infinite product representation for $\theta_1(z|\tau)$ in the left-hand side of (3.3) and then making the substitution of $e^{2iz}$ to $-z$, we derive the quintuple product identity.

**Theorem 7.** *We have*

$$(q,z,q/z;q)_\infty(z^2q,q/z^2;q^2)_\infty = \sum_{n=-\infty}^{\infty} q^{n(3n+1)/2}(z^{-3n}-z^{3n+1}). \tag{3.8}$$

For an interesting account of the quintuple product identity, one may consult [5, p. 83].

Using (1.4), it is easy to verify that $\theta_1(z+x|\tau)\theta_1(z+y|\tau)\theta_1(z-x-y|\tau)$ satisfies the conditions of Theorem 2. We take $h(z)=\theta_1(z+x|\tau)\theta_1(z+y|\tau)\theta_1(z-x-y|\tau)$ in Theorem 2 to obtain

**Theorem 8.** *We have*

$$\begin{aligned} &\theta_1(z+x|\tau)\theta_1(z+y|\tau)\theta_1(z-x-y|\tau)\\ &\quad -\theta_1(z-x|\tau)\theta_1(z-y|\tau)\theta_1(z+x+y|\tau)\\ &\qquad = -\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x+y|\tau)\frac{\theta_1(2z|\tau)}{\theta_1(z|\tau)}. \end{aligned} \tag{3.9}$$

We can check that $f(z)=\theta_1^3(z+\frac{\pi}{3}|\tau)$ satisfies all the conditions of Theorem 2; and it is easy to see that $\theta_1(\frac{\pi}{3}|\tau)=\sqrt{3}q^{1/8}(q^3;q^3)_\infty$ using the infinite product representation for $\theta_1(z|\tau)$. Thus we get the following identity.

**Theorem 9.** *We have*

$$\theta_1^3\left(z+\frac{\pi}{3}|\tau\right)-\theta_1^3\left(z-\frac{\pi}{3}|\tau\right)=3\sqrt{3}q^{3/8}(q^3;q^3)_\infty^3\frac{\theta_1(2z|\tau)}{\theta_1(z|\tau)}. \tag{3.10}$$

## 4. Some Eisenstein series identities in Ramanujan's lost notebook

For brevity we use $\frac{\theta_1'}{\theta_1}(z|\tau)$ to denote the logarithmic derivative of $\theta_1(z|\tau)$ with respect to $z$. We begin with the following lemma

**Lemma 2.** *Let $\theta_1(z|\tau)$ be the Jacobi theta function defined in* (1.3). *Then we have*

$$\frac{\theta_1'}{\theta_1}(z|\tau)=\frac{1}{z}+\sum_{k=1}^{\infty}(-1)^k\frac{2^{2k}B_{2k}}{(2k)!}E_{2k}(\tau)z^{2k-1}. \tag{4.1}$$

Here $B_k$ are the Bernoulli numbers defined as the coefficients in the power series

$$\frac{x}{e^x-1}=\sum_{k=0}^{\infty}B_k\frac{x^k}{k!},\quad |x|<2\pi \tag{4.2}$$

and $E_{2k}(\tau)$ are the normalized Eisenstein series defined by

$$E_{2k}(\tau)=1-\frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\frac{n^{2k-1}q^n}{1-q^n}. \tag{4.3}$$

It is easy to show that $B_{2k+1}=0$ for $k\geqslant 1$, and the first few values of $B_k$ are

$$\begin{aligned}B_0&=1,\quad B_1=-\frac{1}{2},\quad B_2=\frac{1}{6},\quad B_4=-\frac{1}{30},\\ B_6&=\frac{1}{42},\quad B_8=-\frac{1}{30},\quad B_{10}=\frac{5}{66},\quad B_{12}=-\frac{691}{2730};\end{aligned} \tag{4.4}$$

and the first few $E_{2k}(\tau)$ are

$$E_2(\tau)=1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \tag{4.5}$$

$$E_4(\tau)=1+240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}, \tag{4.6}$$

$$E_6(\tau) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}. \tag{4.7}$$

**Proof of Lemma 2.** The Laurent series expansion for $\cot z$ about $z = 0$ is

$$\begin{aligned}\cot z &= \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} - \cdots \\ &= \frac{1}{z} + \sum_{k=1}^{\infty}(-1)^k\frac{2^{2k}B_{2k}}{(2k)!}z^{2k-1}\end{aligned} \tag{4.8}$$

and the Taylor expansion for $\sin z$ is

$$\sin z = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{z^{2k-1}}{(2k-1)!}. \tag{4.9}$$

The trigonometric series expansion for the logarithmic derivative of $\theta_1(z|\tau)$ [25, p. 489] is

$$\frac{\theta_1'}{\theta_1}(z|\tau) = \cot z + 4\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin 2nz. \tag{4.10}$$

Substituting (4.8) and (4.9) into (4.10) and inverting the order of summation, we arrive at (4.1). Thus we complete the proof of Lemma 2. □

Now we state the main result of this section.

**Theorem 10.** *Let $T_{2k}(\tau)$ be defined as*

$$T_{2k}(\tau) = \sum_{n=-\infty}^{\infty}(-1)^n(6n+1)^{2k}q^{n(3n+1)/2}, \tag{4.11}$$

*and let $B_k$ be the Bernoulli numbers and $E_{2k}(\tau)$ the normalized Eisenstein series. Then we have*

$$T_{2m}(\tau) = \sum_{k=1}^{m}\frac{(2m-1)!2^{2k}(2^{2k}-1)B_{2k}}{(2k)!(2m-2k)!}E_{2k}(\tau)T_{2m-2k}(\tau). \tag{4.12}$$

**Proof.** The Taylor expansion for $\cos z$ is

$$\cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}. \tag{4.13}$$

Using this expansion we readily find that

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2+n)/2} \cos(6n+1)z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} T_{2k}(\tau). \tag{4.14}$$

Therefore (3.3) can be rewritten as

$$(q;q)_\infty \frac{\theta_1(2z|\tau)}{\theta_1(z|\tau)} = 2\sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} T_{2k}(\tau). \tag{4.15}$$

We differentiate this equation with respect to $z$ to get

$$\begin{aligned}(q;q)_\infty \frac{\theta_1(2z|\tau)}{\theta_1(z|\tau)} &\left\{ 2\frac{\theta_1'}{\theta_1}(2z|\tau) - \frac{\theta_1'}{\theta_1}(z|\tau) \right\} \\ &= 2\sum_{k=1}^{\infty} (-1)^k \frac{z^{2k-1}}{(2k-1)!} T_{2k}(\tau).\end{aligned} \tag{4.16}$$

Using Lemma 2 we can find that

$$2\frac{\theta_1'}{\theta_1}(2z|\tau) - \frac{\theta_1'}{\theta_1}(z|\tau) = \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!} E_{2k}(\tau) z^{2k-1}. \tag{4.17}$$

Combining (4.15)–(4.17) we immediately have

$$\begin{aligned}&\left\{ \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} T_{2k}(\tau) \right\} \\ &\quad \times \left\{ \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!} E_{2k}(\tau) z^{2k-1} \right\} \\ &= \sum_{k=1}^{\infty} (-1)^k \frac{z^{2k-1}}{(2k-1)!} T_{2k}(\tau).\end{aligned} \tag{4.18}$$

Equating the coefficient of $z^{2m-1}$ in this equation we arrive at (4.12). This completes the proof of Theorem 10. □

From (4.12) we readily find that

$$T_2(\tau) = E_2(\tau)T_0(\tau), \tag{4.19}$$

$$T_4(\tau) = 3E_2(\tau)T_2(\tau) - 2E_4(\tau)T_0(\tau), \tag{4.20}$$

$$T_6(\tau) = 5E_2(\tau)T_4(\tau) - 20E_4(\tau)T_2(\tau) + 16E_6(\tau)T_0(\tau), \tag{4.21}$$

$$T_8(\tau) = 7E_2(\tau)T_6(\tau) - 70E_4(\tau)T_4(\tau) + 330E_6(\tau)T_2(\tau) - 272E_8(\tau)T_0(\tau), \tag{4.22}$$

$$T_{10}(\tau) = 9E_2(\tau)T_8(\tau) - 168E_4(\tau)T_{16}(\tau) + 2016E_6(\tau)T_4(\tau) - 9792E_8(\tau)T_2(\tau) + 7936E_{10}(\tau)T_0(\tau), \tag{4.23}$$

$$T_{12}(\tau) = 11E_2(\tau)T_{10}(\tau) - 330E_4(\tau)T_8(\tau) + 7392E_6(\tau)T_6(\tau) - 89760E_8(\tau)T_4(\tau) + 436480E_{10}(\tau)T_2(\tau) - 353792E_{12}(\tau)T_0(\tau). \tag{4.24}$$

We note that $T_0(\tau) = (q;q)_\infty$ by letting $z = 0$ in (4.15). Next we will denote $E_2(\tau), E_4(\tau)$, and $E_6(\tau)$ by $P(\tau) := P, Q(\tau) := Q$, and $R(\tau) := R$ respectively. Now we begin to represent $T_{2k}(\tau)$ in terms of $P, Q$, and $R$ for $k = 1, 2, 3, 4, 5, 6$ using the above equations. The identity in (4.9) can be written as

$$\frac{T_2(\tau)}{(q;q)_\infty} = P. \tag{4.25}$$

Substituting this into (4.20) we find that

$$\frac{T_4(\tau)}{(q;q)_\infty} = 3P^2 - 2Q. \tag{4.26}$$

Substituting (4.25) and (4.26) into (4.21) we obtain

$$\frac{T_6(\tau)}{(q;q)_\infty} = 15P^3 - 30PQ + 16R. \tag{4.27}$$

In the same way we find that

$$\frac{T_8(\tau)}{(q;q)_\infty} = 105P^4 - 420P^2Q + 448PR - 132Q^2, \tag{4.28}$$

$$\frac{T_{10}(\tau)}{(q;q)_\infty} = 945P^5 - 6300P^3Q + 10080P^2R - 5940PQ^2 + 1216QR, \tag{4.29}$$

$$\frac{T_{12}(\tau)}{(q;q)_\infty} = 10395P^6 - 103950P^4Q + 221760P^3R - 196020P^2Q^2 + 80256PQR - 2712Q^3 - 9728R^2. \tag{4.30}$$

These identities can be found on page 188 of Ramanujan's lost notebook. Berndt and Yee [8] have proved these identities using Ramanujan's famous differential equations. Our approach is different from that of [8].

## 5. The proof of Theorem 3

In this section, we first prove Theorem 3 using Theorem 1 and then discuss its applications. Some results of this section have appeared in [15], but the proofs given here are more compact and attractive.

**Proof of Theorem 3.** We assume that $f_1(z)$ and $f_2(z)$ satisfy the functional equations (1.12). Then $\{f_2(0)f_1(z) - f_1(0)f_2(z)\}/\theta_1(z|\tau)$ satisfies the functional equations in (1.8). And so we can take $f(z) = \{f_2(0)f_1(z) - f_1(0)f_2(z)\}/\theta_1(z|\tau)$ in Theorem 1 and (1.9) in Theorem 1 becomes (1.13) in Theorem 3. This completes the proof of Theorem 3. □

We can derive many remarkable theta function identities by employing identity (1.13). We take $f_1(z) = \frac{\theta_1(2z|\tau)}{4\theta_1(z|\tau)}\theta_1(z-x|\tau)\theta_1(z+x|\tau)$ and $f_2(z) = \frac{\theta_1(2z|\tau)}{4\theta_1(z|\tau)}\theta_1(z-y|\tau)\theta_1(z+y|\tau)$ in Theorem 3 and then cancel out the factor $\theta_1(2z|\tau)$ in the resulting equation to obtain

$$C\theta_1^2(z|\tau) = \theta_1^2(x|\tau)\theta_1(z-y|\tau)\theta_1(z+y|\tau) - \theta_1^2(y|\tau)\theta_1(z-x|\tau)\theta_1(z+x|\tau). \tag{5.1}$$

Taking $z = x$ we find that $C = \theta_1(x-y|\tau)\theta_1(x+y|\tau)$. We substitute this back to (5.1) to obtain

**Theorem 11.** *We have*

$$\theta_1^2(z|\tau)\theta_1(x-y|\tau)\theta_1(x+y|\tau) = \theta_1^2(x|\tau)\theta_1(z-y|\tau)\theta_1(z+y|\tau) - \theta_1^2(y|\tau)\theta_1(z-x|\tau)\theta_1(z+x|\tau). \tag{5.2}$$

This identity plays a fundamental role in the paper [16]. Differentiating the above equation with respect to $z$, twice, using the method of logarithmic differentiation and then setting $z = 0$, we obtain

**Theorem 12.** *Let* $\theta_1(z|\tau)$ *be defined as in* (1.3). *Then we have*

$$\left(\frac{\theta_1'}{\theta_1}\right)'(x|\tau) - \left(\frac{\theta_1'}{\theta_1}\right)'(y|\tau) = \theta_1'(0|\tau)^2 \frac{\theta_1(x-y|\tau)\theta_1(x+y|\tau)}{\theta_1(x|\tau)^2\theta_1(y|\tau)^2}. \tag{5.3}$$

This is a fundamental identity in the theory of elliptic functions (see, for example, [25, p. 325, Eq. (1.7)]).

Taking $f_1(z) = e^{2iz}\theta_1(5z + \pi\tau|5\tau)$ and $f_2(z) = e^{4iz}\theta_1(5z + 2\pi\tau|5\tau)$ in (1.13), we have

$$\begin{aligned} &C\theta_1(z|\tau)\theta_1(2z|\tau) \\ &\quad = \theta_1(2\pi\tau|5\tau)\{e^{2iz}\theta_1(5z+\pi\tau|5\tau) - e^{-2iz}\theta_1(5z-\pi\tau|5\tau)\} \\ &\qquad - \theta_1(\pi\tau|5\tau)\{e^{4iz}\theta_1(5z+2\pi\tau|5\tau) - e^{-4iz}\theta_1(5z-2\pi\tau|5\tau)\}. \end{aligned} \tag{5.4}$$

Setting $z = \frac{\pi}{5}$ we have

$$C\theta_1\left(\frac{\pi}{5}|\tau\right)\theta_1\left(\frac{2\pi}{5}|\tau\right) = 2i\left(\cos\frac{4\pi}{5} - \cos\frac{2\pi}{5}\right)\theta_1(\pi\tau|5\tau)\theta_1(2\pi\tau|5\tau). \tag{5.5}$$

Using the infinite product representation for $\theta_1(z|\tau)$, we find that

$$\theta_1(\pi\tau|5\tau) = iq^{1/8}(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty, \tag{5.6}$$

$$\theta_1(2\pi\tau|5\tau) = iq^{-3/8}(q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty. \tag{5.7}$$

It follows that

$$\theta_1(\pi\tau|5\tau)\theta_1(2\pi\tau|5\tau) = -q^{-1/4}(q;q)_\infty(q^5;q^5)_\infty. \tag{5.8}$$

Similarly we have

$$\theta_1\left(\frac{\pi}{5}|\tau\right)\theta_1\left(\frac{2\pi}{5}|\tau\right) = \sqrt{5}q^{\frac{1}{4}}(q;q)_\infty(q^5;q^5)_\infty. \tag{5.9}$$

Substituting these two equations into (5.5) and then using the fact

$$\cos\frac{4\pi}{5} - \cos\frac{2\pi}{5} = \frac{\sqrt{5}}{2} \tag{5.10}$$

we conclude that

$$C = iq^{-1/2}. \tag{5.11}$$

**Theorem 13.** *We have*

$$\begin{aligned} &iq^{-1/2}\theta_1(z|\tau)\theta_1(2z|\tau) \\ &\quad = \theta_1(2\pi\tau|5\tau)\{e^{2iz}\theta_1(5z+\pi\tau|5\tau) - e^{-2iz}\theta_1(5z-\pi\tau|5\tau)\} \\ &\qquad - \theta_1(\pi\tau|5\tau)\{e^{4iz}\theta_1(5z+2\pi\tau|5\tau) - e^{-4iz}\theta_1(5z-2\pi\tau|5\tau)\}. \end{aligned} \tag{5.12}$$

Hirschhorn [12] first proved an equivalent form of the identity in (5.12) using only the Jacobi triple product identity and Farkas and Kra [11] rediscovered it using the theory of theta functions with rational characteristics.

Replacing $\tau$ by $5\tau$, setting $z = \pi\tau$ in the resulting equation and then using (5.8) gives

$$q^{-1}\frac{(q^{10}, q^{15}; q^{25})_\infty}{(q^5, q^{20}; q^{25})_\infty} - q\frac{(q^5, q^{20}; q^{25})_\infty}{(q^{10}, q^{15}; q^{25})_\infty} = \frac{(q;q)_\infty}{q(q^{25};q^{25})_\infty}. \tag{5.13}$$

Applying the imaginary transformation to (5.12) we obtain

**Theorem 14.** *We have*

$$\begin{aligned} &5\theta_1(z|5\tau)\theta_1(2z|5\tau) \\ &\quad = \theta_1\left(\frac{2\pi}{5}|\tau\right)\left\{\theta_1\left(z+\frac{\pi}{5}|\tau\right) - \theta_1\left(z-\frac{\pi}{5}|\tau\right)\right\} \\ &\qquad - \theta_1\left(\frac{\pi}{5}|\tau\right)\left\{\theta_1\left(z+\frac{2\pi}{5}|\tau\right) - \theta_1\left(z-\frac{2\pi}{5}|\tau\right)\right\}. \end{aligned} \tag{5.14}$$

Differentiating both sides of the above twice with respect to $z$ and then setting $z = 0$, we find that

$$\frac{\theta_1''}{\theta_1}\left(\frac{\pi}{5}\middle|\tau\right) - \frac{\theta_1''}{\theta_1}\left(\frac{2\pi}{5}\middle|\tau\right) = \frac{10\theta_1'(0|5\tau)^2}{\theta_1(\frac{\pi}{5}|\tau)\theta_1(\frac{2\pi}{5}|\tau)}. \tag{5.15}$$

Applying (1.7) and (5.9) to the left-hand side we have

$$\frac{\theta_1''}{\theta_1}\left(\frac{\pi}{5}\middle|\tau\right) - \frac{\theta_1''}{\theta_1}\left(\frac{2\pi}{5}\middle|\tau\right) = 8\sqrt{5}\frac{q(q^5;q^5)_\infty^5}{(q;q)_\infty}. \tag{5.16}$$

Using the infinite product representation and the method of logarithmic differentiation and the partial differential equation

$$\frac{\partial^2 \theta_1}{\partial z^2} = -8q \frac{\partial \theta_1}{\partial q} \tag{5.17}$$

we can readily find

$$\frac{\theta_1''}{\theta_1}(z|q) = -1 + 16\sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2} \cos 2nz + 8\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} \tag{5.18}$$

(see, for example, [23]).

Replacing $z$ by $\frac{\pi}{5}$ and $\frac{2\pi}{5}$ in the above, respectively, and then subtracting the two resulting equations and finally using the following elementary trigonometric function identities:

$$\csc^2 \frac{\pi}{5} - \csc^2 \frac{2\pi}{5} = \frac{4}{\sqrt{5}} \quad \text{and} \quad \cos \frac{2n\pi}{5} - \cos \frac{4n\pi}{5} = \frac{\sqrt{5}}{2} \left(\frac{n}{5}\right) \tag{5.19}$$

we can find that

$$\frac{\theta_1''}{\theta_1}\left(\frac{\pi}{5}\middle|\tau\right) - \frac{\theta_1''}{\theta_1}\left(\frac{2\pi}{5}\middle|\tau\right) = 8\sqrt{5}\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2}, \tag{5.20}$$

where $(\frac{n}{5})$ denote the Legendre symbol. Comparing (5.16) and (5.20) we conclude that

$$\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2} = \frac{q(q^5;q^5)_\infty^5}{(q;q)_\infty}. \tag{5.21}$$

Taking $x = \frac{\pi}{5}$ and $y = \frac{2\pi}{5}$ in (5.3) and then using (5.9) in the resulting equation we obtain

$$\left(\frac{\theta_1'}{\theta_1}\right)'\left(\frac{\pi}{5}\middle|\tau\right) - \left(\frac{\theta_1'}{\theta_1}\right)'\left(\frac{2\pi}{5}\middle|\tau\right) = -\frac{4}{\sqrt{5}}\frac{(q;q)_\infty^5}{(q^5;q^5)_\infty}. \tag{5.22}$$

From (4.10) and (5.19) we find that

$$\left(\frac{\theta_1'}{\theta_1}\right)'\left(\frac{\pi}{5}\middle|\tau\right) - \left(\frac{\theta_1'}{\theta_1}\right)'\left(\frac{2\pi}{5}\middle|\tau\right) = -\frac{4}{\sqrt{5}}\left\{1 - 5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^n}\right\}. \tag{5.23}$$

Comparing the above two equations we conclude that

$$1-5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^n}=\frac{(q;q)_\infty^5}{(q^5;q^5)_\infty}. \tag{5.24}$$

The Ramanujan identities (5.21) and (5.24) were recorded by Ramanujan in his lost notebook [21, pp. 139–140]. There are several proofs of these two identities in the literature. The first proof was given by Bailey by using his ${}_6\Psi_6$ summation formula [3,4]. Raghavan [20] proved them using the theory of modular forms. Chan [9] utilized the Hecke correspondence between Dirichlet series and Fourier expansions of modular forms to show that they are equivalent.

Taking $f_1(z)=\theta_1^5(z+\frac{\pi}{5}|\tau)$ and $f_2(z)=\theta_1^5(z+\frac{2\pi}{5}|\tau)$ in Theorem 3, we find that

$$\begin{aligned} &C\theta_1(z|\tau)\theta_1(2z|\tau)\\ &\quad=\theta_1^5\left(\frac{2\pi}{5}|\tau\right)\left\{\theta_1^5\left(z+\frac{\pi}{5}|\tau\right)-\theta_1^5\left(z-\frac{\pi}{5}|\tau\right)\right\}\\ &\qquad-\theta_1^5\left(\frac{\pi}{5}|\tau\right)\left\{\theta_1^5\left(z+\frac{2\pi}{5}|\tau\right)-\theta_1^5\left(z-\frac{2\pi}{5}|\tau\right)\right\}. \end{aligned} \tag{5.25}$$

Differentiating both sides of the above equation twice with respect to $z$ and then setting $z=0$, we find that

$$\begin{aligned} \frac{2C\theta_1'(0|\tau)^2}{5\theta_1^5(\frac{\pi}{5}|\tau)\theta_1^5(\frac{2\pi}{5}|\tau)} &= \frac{8C(q;q)_\infty}{125\sqrt{5}q(q^5;q^5)_\infty^5}\\ &=\frac{\theta_1''}{\theta_1}\left(\frac{\pi}{5}\middle|\tau\right)-\frac{\theta_1''}{\theta_1}\left(\frac{2\pi}{5}\middle|\tau\right)+4\left(\frac{\theta_1'}{\theta_1}\right)^2\left(\frac{\pi}{5}\middle|\tau\right)\\ &\quad-4\left(\frac{\theta_1'}{\theta_1}\right)^2\left(\frac{2\pi}{5}\middle|\tau\right). \end{aligned} \tag{5.26}$$

Using the simple differential identity

$$\left(\theta_1'/\theta_1\right)^2=\theta_1''/\theta_1-\left(\theta_1'/\theta_1\right)' \tag{5.27}$$

we can write (5.26) as

$$\begin{aligned} \frac{2C\theta_1'(0|\tau)^2}{5\theta_1^5(\frac{\pi}{5}|\tau)\theta_1^5(\frac{2\pi}{5}|\tau)} &= \frac{8C(q;q)_\infty}{125\sqrt{5}q(q^5;q^5)_\infty^5}\\ &=5\left\{\frac{\theta_1''}{\theta_1}\left(\frac{\pi}{5}\middle|\tau\right)-\frac{\theta_1''}{\theta_1}\left(\frac{2\pi}{5}\middle|\tau\right)\right\} \end{aligned}$$

$$-4\left\{\left(\frac{\theta_1'}{\theta_1}\right)'\left(\frac{\pi}{5}\middle|\tau\right)-\left(\frac{\theta_1'}{\theta_1}\right)'\left(\frac{2\pi}{5}\middle|\tau\right)\right\}. \tag{5.28}$$

Using (5.16) and (5.22) in the above equation we find that

$$C=250q(q;q)_\infty^4(q^5;q^5)_\infty^4+3125q^2\frac{(q^5;q^5)_\infty^{10}}{(q;q)_\infty^2}. \tag{5.29}$$

Thus we have

**Theorem 15.** *Let C be defined as in* (5.28). *Then we have*

$$\begin{aligned}&C\theta_1(z|\tau)\theta_1(2z|\tau)\\&\quad=\theta_1^5\left(\frac{2\pi}{5}\middle|\tau\right)\left\{\theta_1^5\left(z+\frac{\pi}{5}\middle|\tau\right)-\theta_1^5\left(z-\frac{\pi}{5}\middle|\tau\right)\right\}\\&\qquad-\theta_1^5\left(\frac{\pi}{5}\middle|\tau\right)\left\{\theta_1^5\left(z+\frac{2\pi}{5}\middle|\tau\right)-\theta_1^5\left(z-\frac{2\pi}{5}\middle|\tau\right)\right\}.\end{aligned} \tag{5.30}$$

Applying the imaginary transformation to (5.30) we obtain

**Theorem 16.** *We have*

$$\begin{aligned}&C\theta_1(z|5\tau)\theta_1(2z|5\tau)\\&\quad=\theta_1^5(2\pi\tau|5\tau)\left\{e^{2iz}\theta_1^5(z+\pi\tau|5\tau)-e^{-2iz}\theta_1^5(z-\pi\tau|5\tau)\right\}\\&\qquad-\theta_1^5(\pi\tau|5\tau)\left\{e^{4iz}\theta_1^5(z+2\pi\tau|5\tau)-e^{-4iz}\theta_1^5(z-2\pi\tau|5\tau)\right\},\end{aligned} \tag{5.31}$$

*where*

$$q^{5/2}C=10q(q;q)_\infty^4(q^5;q^5)_\infty^4+\frac{(q;q)_\infty^{10}}{(q^5;q^5)_\infty^2}. \tag{5.32}$$

Taking $z=\pi\tau$ in (5.31) we find that

$$q^{-1}\frac{(q^2,q^3;q^5)_\infty^5}{(q,q^4;q^5)_\infty^5}-11-q\frac{(q,q^4;q^5)_\infty^5}{(q^2,q^3;q^5)_\infty^5}=\frac{(q;q)_\infty^6}{q(q^5;q^5)_\infty^6}. \tag{5.33}$$

The identities in (5.13) and (5.33) are well-known Ramanujan identities about the Rogers–Ramanujan continued fraction, which can be found in Ramanujan's second notebook [22, pp. 265–267] and were first proved by Watson [24]. They were used by Berndt et al. [6] in deriving general formulas for the explicit evaluation of the

Rogers–Ramanujan continued fraction. They were also used by Liu and Lewis [14] to provide simpler proofs of two Lambert series identities of Ramanujan.

## 6. The addition theorem for Weierstrass $\wp$ function

We first prove the following lemma using Theorem 8 and the technique of logarithmic differentiation.

**Lemma 3.** *We have*

$$\left\{\frac{\theta_1'}{\theta_1}(x|\tau)+\frac{\theta_1'}{\theta_1}(y|\tau)-\frac{\theta_1'}{\theta_1}(x+y|\tau)\right\}^2$$
$$=-1+24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-\left(\frac{\theta_1'}{\theta_1}\right)'(x|\tau)-\left(\frac{\theta_1'}{\theta_1}\right)'(y|\tau)-\left(\frac{\theta_1'}{\theta_1}\right)'(x+y|\tau). \quad (6.1)$$

**Proof.** We write identity (3.9) in Theorem 6 in the form

$$f_1(z)-f_2(z)=-\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x+y|\tau)f_3(z), \quad (6.2)$$

where

$$f_1(z)=\theta_1(z+x|\tau)\theta_1(z+y|\tau)\theta_1(z-x-y|\tau), \quad (6.3)$$

$$f_2(z)=\theta_1(z-x|\tau)\theta_1(z-y|\tau)\theta_1(z+x+y|\tau), \quad (6.4)$$

$$f_3(z)=\frac{\theta_1(2z|\tau)}{\theta_1(z|\tau)}. \quad (6.5)$$

Differentiating (6.2) with respect to $z$, twice, and then setting $z=0$, we have

$$f_1''(0)-f_2''(0)=-\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x+y|\tau)f_3''(0). \quad (6.6)$$

Now we begin to compute $f_1''(0)$, $f_2''(0)$, and $f_3''(0)$. Differentiating $f_1(z)$ with respect to $z$, twice, using the method of logarithmic differentiation, we readily find that

$$f_1''(z)=f_1(z)\left\{\phi(z)^2+\phi'(z)\right\}, \quad (6.7)$$

where

$$\phi(z) = \frac{\theta_1'}{\theta_1}(z+x|\tau) + \frac{\theta_1'}{\theta_1}(z+y|\tau) + \frac{\theta_1'}{\theta_1}(z-x-y|\tau). \tag{6.8}$$

It follows that

$$\begin{aligned} f_1''(0) = & -\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x+y|\tau) \\ & \times \left\{ \left( \frac{\theta_1'}{\theta_1}(x|\tau) + \frac{\theta_1'}{\theta_1}(y|\tau) - \frac{\theta_1'}{\theta_1}(x+y|\tau) \right)^2 \right. \\ & \left. + \left( \frac{\theta_1'}{\theta_1} \right)'(x|\tau) + \left( \frac{\theta_1'}{\theta_1} \right)'(y|\tau) + \left( \frac{\theta_1'}{\theta_1} \right)'(x+y|\tau) \right\}. \end{aligned} \tag{6.9}$$

By a direct computation, we find that

$$f_2''(0) = -f_1''(0). \tag{6.10}$$

We proceed as in (6.7) to obtain

$$f_3''(z) = \frac{\theta_1(2z|\tau)}{\theta_1(z|\tau)} \left\{ \left( 2\frac{\theta_1'}{\theta_1}(2z|\tau) - \frac{\theta_1'}{\theta_1}(z|\tau) \right)^2 + \left( 2\frac{\theta_1'}{\theta_1}(2z|\tau) - \frac{\theta_1'}{\theta_1}(z|\tau) \right)' \right\}. \tag{6.11}$$

From (4.1) we find that

$$2\frac{\theta_1'}{\theta_1}(2z|\tau) - \frac{\theta_1'}{\theta_1}(z|\tau) = -E_2(\tau)z + O(z^3). \tag{6.12}$$

Hence we have

$$f_3''(0) = -E_2(\tau). \tag{6.13}$$

We substitute (6.9), (6.10), and (6.13) into (6.6) and cancel out the factor $-\theta_1(x|\tau)\theta_1(y|\tau)\theta_1(x+y|\tau)$ to obtain (6.1). This completes the proof of Lemma 3. □

We will use $\wp(z|\tau)$ to represent the Weierstrass $\wp$ function of periods $\pi$ and $\pi\tau$. Then we have the following addition formula for $\wp(z|\tau)$ (see, for example, [10, p. 34, Eq. (4.1)]).

**Theorem 17.** *We have*

$$\wp(x|\tau)+\wp(y|\tau)+\wp(x+y|\tau)=\frac{1}{4}\left(\frac{\wp'(x|\tau)-\wp'(y|\tau)}{\wp(x|\tau)-\wp(y|\tau)}\right)^2. \tag{6.14}$$

**Proof.** It is well known that [25, p. 460]

$$\begin{aligned}\wp(z|\tau) &= \csc^2 -8\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\cos 2nz-\frac{1}{3}E_2(\tau)\\ &= -\frac{1}{3}E_2(\tau)-\left(\frac{\theta_1'}{\theta_1}\right)'(z|\tau)\end{aligned} \tag{6.15}$$

and so (5.3) can be rewritten as

$$\wp(x|\tau)-\wp(y|\tau)=-\theta_1'(0|\tau)^2\frac{\theta_1(x+y|\tau)\theta_1(x-y|\tau)}{\theta_1^2(x|\tau)\theta_1^2(y|\tau)}. \tag{6.16}$$

Writing $x$ as $x+z$ and $y$ as $y+z$ we obtain

$$\wp(x+z|\tau)-\wp(y+z|\tau)=-\theta_1'(0|\tau)^2\frac{\theta_1(x-y|\tau)\theta_1(x+y+2z|\tau)}{\theta_1^2(x+z|\tau)\theta_1^2(y+z|\tau)}. \tag{6.17}$$

Logarithmic differentiation about $z$ gives

$$\begin{aligned}&\frac{\wp'(x+z|\tau)-\wp'(y+z|\tau)}{\wp(x+z|\tau)-\wp(y+z|\tau)}\\ &\quad=-2\left(\frac{\theta_1'}{\theta_1}(x+z|\tau)+\frac{\theta_1'}{\theta_1}(y+z|\tau)-\frac{\theta_1'}{\theta_1}(x+y+2z|\tau)\right).\end{aligned} \tag{6.18}$$

Setting $z=0$ the equation reduces to

$$\frac{\wp'(x|\tau)-\wp'(y|\tau)}{\wp(x|\tau)-\wp(y|\tau)}=-2\left(\frac{\theta_1'}{\theta_1}(x|\tau)+\frac{\theta_1'}{\theta_1}(y|\tau)-\frac{\theta_1'}{\theta_1}(x+y|\tau)\right). \tag{6.19}$$

In light of (6.15), identity in (6.1) can be written as

$$\left\{\frac{\theta_1'}{\theta_1}(x|\tau)+\frac{\theta_1'}{\theta_1}(y|\tau)-\frac{\theta_1'}{\theta_1}(x+y|\tau)\right\}^2$$

$$= -E_2(\tau) - \left(\frac{\theta_1'}{\theta_1}\right)'(x|\tau) - \left(\frac{\theta_1'}{\theta_1}\right)'(y|\tau) - \left(\frac{\theta_1'}{\theta_1}\right)'(x+y|\tau)$$
$$= \wp(x|\tau) + \wp(y|\tau) + \wp(x+y|\tau). \tag{6.20}$$

Combining the above equations we arrive at (6.14). This completes the proof of the Theorem. □

## 7. The proofs of Theorems 4 and 5

We first prove Theorem 4 using Theorem 1.

**Proof of Theorem 4.** The function $\theta_1(u+x_1|\tau)\theta_1(u+x_2|\tau)\theta_1(u+x_3|\tau)\theta_1(u-x_1-x_2-x_3|\tau)$ satisfies the functional equations in (1.8) and so we can take

$$f(u) = \theta_1(u+x_1|\tau)\theta_1(u+x_2|\tau)\theta_1(u+x_3|\tau)\theta_1(u-x_1-x_2-x_3|\tau) \tag{7.1}$$

in (1.9) to obtain

$$\theta_1(z+x_1|\tau)\theta_1(z+x_2|\tau)\theta_1(z+x_3|\tau)\theta_1(z-x_1-x_2-x_3|\tau)$$
$$-\theta_1(z-x_1|\tau)\theta_1(z-x_2|\tau)\theta_1(z-x_3|\tau)\theta_1(z+x_1+x_2+x_3|\tau)$$
$$= C\theta_1(2z|\tau). \tag{7.2}$$

Setting $z = x_1$ and using the fact $\theta_1(0|\tau) = 0$ we find that

$$-\theta_1(2x_1|\tau)\theta_1(x_1+x_2|\tau)\theta_1(x_1+x_3|\tau)\theta_1(x_2+x_3|\tau) = C\theta_1(2x_1|\tau). \tag{7.3}$$

Hence we have

$$C = -\theta_1(x_1+x_2|\tau)\theta_1(x_1+x_3|\tau)\theta_1(x_2+x_3|\tau). \tag{7.4}$$

Combining this with (7.1) we arrive at (1.14). This completes the proof of Theorem 4. □

Now we come to prove Theorem 5 using Theorem 4 and the method of logarithmic differentiation. The identity in (1.14) can be written in the form

**Proof of Theorem 5.**

$$f(z) - f(-z) = C\theta_1(2z|\tau), \tag{7.5}$$

where $f(z)$ and $C$ are defined by (7.1) and (7.4), respectively. Differentiating the above equation with respect to $z$ and then setting $z = 0$ we conclude that

$$f'(0) = C\theta_1'(0|\tau). \tag{7.6}$$

Using the method of logarithmic differentiation we find that

$$\begin{aligned} f'(z) \;=\; f(z)\left\{\frac{\theta_1'}{\theta_1}(z+x_1|\tau)+\frac{\theta_1'}{\theta_1}(z+x_2|\tau)\right.\\ \left.+\frac{\theta_1'}{\theta_1}(z+x_3|\tau)+\frac{\theta_1'}{\theta_1}(z-x_1-x_2-x_3|\tau)\right\}. \end{aligned} \tag{7.7}$$

It follows that

$$f'(0) = f(0)\left\{\frac{\theta_1'}{\theta_1}(x_1|\tau)+\frac{\theta_1'}{\theta_1}(x_2|\tau)+\frac{\theta_1'}{\theta_1}(x_3|\tau)-\frac{\theta_1'}{\theta_1}(x_1+x_2+x_3|\tau)\right\}. \tag{7.8}$$

From (7.1) we immediately have

$$f(0) = -\theta_1(x_1|\tau)\theta_1(x_2|\tau)\theta_1(x_3|\tau)\theta_1(x_1+x_2+x_3|\tau). \tag{7.9}$$

Combining (7.4), (7.6), (7.8), and (7.9) we arrive at (1.14). We complete the proof of Theorem 5. □

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