The Terwilliger algebra of the Johnson schemes

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Received 13 October 2004; received in revised form 18 August 2006; accepted 2 September 2006
Available online 15 November 2006

Abstract

One of the classical families of association schemes is known as the Johnson schemes \( J(n, d) \). In this paper we compute the Terwilliger algebra associated to them when \( 3d \leq n \). We give its decomposition into simple ideals and the decomposition of its standard module into irreducible submodules.

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Keywords: Association schemes; Johnson schemes; Terwilliger algebra; Irreducible \( T \)-modules

1. Introduction

The theory of association schemes provided a way of approaching several combinatorial objects from an algebraic point of view.

A new tool for the study of association was first introduced in [10] as the subconstituent algebra. It is now called the Terwilliger algebra and we are going to denote it by \( T \). It is a non-commutative, finite dimensional, semisimple \( \mathbb{C} \) algebra. It has been studied for \( P \)- and \( Q \)-polynomial association schemes in [5], group association schemes in [1] and [3], strongly regular graphs in [13]. In [7] there is a detailed study of the irreducible modules of the \( T \)-algebra of the hypercube \( H(d, 2) \). In [9] the \( T \)-algebra of a Hamming scheme \( H(d, q) \) is described as the symmetric \( d \)-tensors on the \( T \)-algebra of \( H(1, q) \).

In this paper we focus on the Johnson schemes \( J(n, d) \) whose parameters satisfy \( 3d \leq n \). In these cases the isomorphism class of the \( T \)-algebra is independent of the parameter \( n \). In Theorem 6.8 we give the structure of the corresponding \( T \)-algebra and in Theorem 7.1, the decomposition of its standard module into irreducible \( T \)-modules.

2. Association schemes

Given \( X \) a finite set and \( R_0, \ldots, R_d \) non-empty subsets of \( X \times X \), we say that \( \Gamma = (X, \{R_i\}_{0 \leq i \leq d}) \) is a symmetric association scheme with diameter \( d \) if the following conditions hold:

(i) \( R_0 = \{(x, x) : x \in X\} \).
(ii) \( X \times X = R_0 \cup R_1 \cup \cdots \cup R_d \) and \( R_i \cap R_j = \emptyset \) if \( i \neq j \).
(iii) \( R_i^\Gamma = R_i \) for \( 0 \leq i \leq d \) where \( R_i^\Gamma = \{(y, x) : (x, y) \in R_i\} \).

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doi:10.1016/j.disc.2006.09.012
(iv) For all \(0 \leq h, i, j \leq d\) and for all \(x, y \in X\) such that \((x, y) \in R_h\),
\[
|\{z \in X : (x, z) \in R_i; \ (y, z) \in R_j\}|
\]
is a constant denoted by \(p^h_{ij}\) which is independent of \((x, y) \in R_h\).

The elements of \(X\) are called the vertices of \(\Gamma\) and \(R_i\) the relations of \(\Gamma\). From now on, when we refer to an association scheme, we mean a symmetric association scheme.

### 2.1. Bose–Mesner algebra

Let \(\Gamma = (X, \{R_i\}_{0 \leq i \leq d})\) denote an association scheme. Let \(\text{Mat}_X(\mathbb{C})\) denote the \(\mathbb{C}\)-algebra of matrices with complex entries, where the rows and columns are indexed by elements in \(X\). For \(0 \leq i \leq d\) let \(A_i\) denote the matrix in \(\text{Mat}_X(\mathbb{C})\) that has entries
\[
(A_i)_{xy} = \begin{cases} 
1 & \text{if } (x, y) \in R_i, \\
0 & \text{if } (x, y) \notin R_i.
\end{cases}
\]
We call \(A_i\) the \(i\)th adjacency matrix of \(\Gamma\). We abbreviate \(A = A_1\) and call this the adjacency matrix of \(\Gamma\). By (i)–(iv) above the adjacency matrices satisfy: (i') \(A_0 = I\) where \(I\) is the identity matrix in \(\text{Mat}_X(\mathbb{C})\); (ii') \(A_0 + \cdots + A_d = J\) where \(J\) is the all 1's matrix in \(\text{Mat}_X(\mathbb{C})\); (iii') \(A_i^2 = A_i\); (iv') \(A_iA_j = \sum_{h=0}^d p^h_{ij}A_h\) \((0 \leq i, j \leq d)\). It follows from (i')–(iv') that \(A_0, \ldots, A_d\) form a basis for a subalgebra \(M\) of \(\text{Mat}_X(\mathbb{C})\). We call \(M\) the Bose–Mesner algebra of \(\Gamma\).

Let \(C^X\) denote the vector space over \(\mathbb{C}\) consisting of all column vectors whose rows are indexed by \(X\) and whose entries are in \(\mathbb{C}\). We observe that for \(0 \leq i \leq d\) the subspace \(E_iV\) is a common eigenspace for \(A\). For \(0 \leq i \leq d\) let \(\lambda_i\) denote the eigenvalue of \(A\) associated with \(E_i\). We index the primitive idempotents so that \(\lambda_0 > \lambda_1 > \cdots > \lambda_d\).

### 2.2. Dual Bose–Mesner algebra and Terwilliger algebra (see [10])

Let \(\Gamma = (X, \{R_i\}_{0 \leq i \leq d})\) denote an association scheme and fix \(x \in X\). For \(0 \leq i \leq d\) let \(E_i^\ast\) denote the diagonal matrix in \(\text{Mat}_X(\mathbb{C})\) that has entries
\[
(E_i^\ast)_{xy} = \begin{cases} 
1 & \text{if } (x, y) \in R_i, \\
0 & \text{if } (x, y) \notin R_i.
\end{cases}
\]
We call \(E_i^\ast\) the \(i\)th dual idempotent of \(\Gamma\) with respect to \(x\).

For notational convenience we set \(E_i^\ast = 0\) for \(i > d\) or \(i < 0\). The \(\{E_i^\ast\}_{i=0}^d\) satisfy: (ix) \(E_0^\ast + E_1^\ast + \cdots + E_d^\ast = I\); (x) \(E_i^\ast E_j^\ast = E_i^\ast\); (xi) \(E_i^\ast E_j^\ast = \delta_{ij}E_i^\ast\).

Notice that \(\{E_i^\ast\}_{i=0}^d\) form a basis for commutative subalgebra of \(\text{Mat}_X(\mathbb{C})\); we denote this subalgebra by \(M^\ast = M^\ast(x)\). We call \(M^\ast\) the Dual Bose–Mesner algebra of \(\Gamma\).

The subconstituent algebra or Terwilliger algebra of \(\Gamma\) with respect to \(x\) is the subalgebra of \(\text{Mat}_X(\mathbb{C})\) generated by \(M\) and \(M^\ast\).

### 3. Johnson schemes

#### 3.1. Definition

Let \(S\) be a set of cardinality \(n\) and let \(d\) be a positive integer such that \(2d \leq n\). Let \(X\) be the set whose elements are the \(d\)-subsets of \(S\). Note that \(|X| = \binom{n}{d}\). Given \(x, y \in X\), we define \(\delta(x, y) := d - |x \cap y|\). Then for \(0 \leq i \leq d\) we
define in $X \times X$ the following relation:

$$R_i = \{(x, y) \in X \times X : \partial(x, y) = i\}.$$ 

It is known (see [2]) that the configuration $(X, \{R_i\}_{0 \leq i \leq d})$ gives a symmetric association scheme with diameter $d$. It is called Johnson scheme and it is denoted by $J(n, d)$.

It satisfy the $P$-polynomial property (see [2]): for $0 \leq i \leq d$ and for each $A_i$ there exists a polynomial $p_i$ of degree $i$ such that $A_i = p_i(A)$. It implies that the Bose–Mesner algebra $M$ of a $J(n, d)$ is generated by $A$, so in order to compute the Terwilliger algebra $T$ of such a scheme, we will study in detail the adjacency matrix of the scheme.

The isomorphism class of $T(x)$ is independent of $x$ since the group of automorphism of the scheme acts transitively. We will denote $T := T(x)$.

### 3.2. Adjacency matrix of a $J(n, d)$ scheme

In this section we fix parameters $n$, $d$ such that $2d \leq n$ and we analyze the structure of the adjacency matrix associated to a Johnson scheme with these parameters. From now on, we will consider that $V = [n] := \{1, \ldots, n\}$ so $X$ consist of the $d$-subset of the set $[n]$ and every time we write “$x \in J(n, d)$” we will mean “$x$ is a $d$-subset of $[n]”.$

We fix $x_0 = [d] \in J(n, d)$ and for $0 \leq i \leq d$ we consider $\Omega_i := \{\sigma_d \in J(n, d) : \partial(x_0, \sigma_d) = i\}$. We call $\{\Omega_i\}_{i=0}^d$ the orbits of the scheme $J(n, d)$ with respect to the vertex $x_0$. We have the partition $X = \bigsqcup_{i=0}^d \Omega_i$.

Let us consider $A$ as a block-matrix with respect to this partition.

Take $\sigma \in \Omega_i$, that is $\partial(\sigma, [d]) = i$. We may consider $\sigma = z_{d-i} \beta_i := z_{d-i} \bigcup_i \beta_i$ where $z_{d-i}$ is a $(d-i)$-subset of $[d]$ and $\beta_i$ is an $i$-subset of $\{d+1, d+2, \ldots, n\}$. That is $z_{d-i} \in J(d, d-i)$ and $\beta_i$ can be considered as an element of $J(n-d, i)$, identifying the set $\{d+1, d+2, \ldots, n\}$ with $[n-d]$.

In this way, given two elements $\sigma \in \Omega_i$, $\sigma' \in \Omega_j$, $\sigma = z_{d-i} \beta_i$, $\sigma' = z_{d-j}' \beta'_j$ we have $\partial(\sigma, \sigma') = d-i - |z_{d-i} \cap z_{d-j}'| - |\beta_i \cap \beta'_j|$.

In the following lemma we describe the diagonal blocks of $A$ with respect to this partition.

**Lemma 3.1.** Let $I(\frac{v}{k})$ be the identity matrix of size $(\frac{v}{k})$ and $A(\frac{i}{k})$ the adjacency matrix of a $J(v, k)$. Then, for $0 \leq i \leq d$ we have

$$A|_{\Omega_i \times \Omega_i} = I(d-1) \otimes A(n-d) + A(d-1) \otimes I(n-d),$$

where “$\otimes$” denotes the Kronecker product of matrices.

**Proof.** Using the fact that an arbitrary element of the block $\Omega_i \times \Omega_i$ is indexed by elements $\sigma, \sigma'$, where $\sigma = z_{d-i} \beta_i$, $\sigma' = z_{d-j} \beta'_i$, with $z_{d-i}, z_{d-j} \in J(d, d-i)$ $\beta_i, \beta'_i \in J(n-d, i)$ we have $\partial(\sigma, \sigma') = d-i - |z_{d-i} \cap z_{d-j}'| + i - |\beta_i \cap \beta'_i|$.

Since $(A \otimes B)_{\sigma, \sigma'} = A|_{z_{d-i}, z_{d-j}} B_{\beta_i, \beta'_i}$ we can compare both sides of (1) and get the result. \[\square\]

**Remark 3.2.** We have proved that $A|_{\Omega_i \times \Omega_i} \in \mathcal{B}(d-1) \otimes \mathcal{B}(n-d)$ where $\mathcal{B}(\frac{i}{k})$ denotes the Bose–Mesner algebra of a $J(v, k)$ and “$\otimes$” the tensor product of algebras.

In order to describe the off-diagonal blocks of $A$ we need to introduce a generalization of adjacency matrices.

**Definition 3.3 (Generalized adjacency matrices).** Let $H^r_{v, i, j}$ be the matrix indexed by the elements of $J(v, i) \times J(v, j)$ and defined by

$$H^r_{v, i, j}(x, y) = \begin{cases} 1 & \text{if } |x \cap y| = r, \\ 0 & \text{otherwise.} \end{cases}$$

In order to simplify the notation we set $H^r_{i, j} := H^r_{v, i, j}$ omitting $v$ when it is clear from the context.
Remark 3.4.

- If \( i = j \Rightarrow H_{i,j} = A_{i-r} \) the \((i-r)\)th adjacency matrix indexed by the elements of \( J(v,i) \times J(v,i) \).
- If \( i < j \) we denote

\[
H_{i,j}(\alpha_i, \alpha_j) := H^i_{i,j}(\alpha_i, \alpha_j) = \begin{cases} 1, & \alpha_i \subseteq \alpha_j, \\ 0, & \text{otherwise}. \end{cases}
\]

These matrices are defined in [8] as “incidence matrices”.
- Given \( i, j \), \( H_{i,j} \neq 0 \) for \( \max(0, i+j-v) \leq r \leq \min(i, j) \).

Lemma 3.5. For \( i < j \) we have

\[
A|_{\Omega_i \times \Omega_j} = \begin{cases} H_{d-i,d-j} \otimes H_{i,j} & \text{if } j - i = 1, \\ 0 & \text{if } j - i > 1. \end{cases}
\]

Proof. Given \( \sigma \in \Omega_i, \sigma' \in \Omega_j \),

\[
\sigma = \alpha_{d-i} \beta_i, \quad \sigma' = \alpha_{d-j} \beta_j' \quad \text{with } \alpha_{d-i} \in J(d, d - i), \alpha_{d-j} \in J(d, d - j), \beta_i \in J(n - d, i), \beta_j' \in J(n - d, j)
\]

the assertion follows from the fact that \( \tilde{\sigma}(\sigma, \sigma') = 1 \iff |\sigma \cap \sigma'| = d - 1 \) and that \( \tilde{\sigma}(\sigma, \sigma') = d - |\alpha_{d-i} \cap \alpha_{d-j}| - |\beta_i \cap \beta_j'| \). \( \square \)

4. T-Algebra

In this section we fix \( n, d \) such that \( 3d \leq n \) and we consider a Johnson scheme \( J(n,d) = (X, \{ R_i \}_{0 \leq i \leq d}) \). In subsection 4.1 we will associate to \( J(n, d) \) a subspace \( \mathcal{M} \subseteq \text{Mat}_X(\mathbb{C}) \), and in subsection 4.2 we will prove that it is in fact an algebra.

4.1. Definition of the subspace \( \mathcal{M} \subseteq \text{Mat}_X(\mathbb{C}) \)

Recall that \( \mathcal{B}_{\binom{n}{d}} \) is the Bose–Mesner algebra corresponding to a \( J(v, k) \) and when \( v < 2k \) we take \( \mathcal{B}_{\binom{n}{d}} \cong \mathcal{B}_{\binom{v}{k}} = \text{span}_\mathbb{C}\{E_0, E_1, \ldots, E_{v-k}\} \).

Let \( J(n,d) = (X, \{ R_i \}_{0 \leq i \leq d}) \) be a Johnson scheme whose parameters satisfy \( 3d \leq n \).

For \( 0 \leq i \leq d \), let \( \{ \Omega_i \}_{i=0}^{d} \) be the orbits of \( J(n,d) \) defined in 3.2. Any matrix \( Y \) indexed by elements in \( \Omega_i \times \Omega_j \) can be embedded into \( \text{Mat}_X(\mathbb{C}) \) by

\[
(L(Y))_{\Omega_i \times \Omega_m} = \begin{cases} Y & \text{if } l = i, m = j, \\ 0 & \text{otherwise}. \end{cases}
\]

For \( 0 \leq i \leq d \), we take \( H_{d-i,d-j} \) indexed by \( J(d, d - i) \times J(d, d - j) \) and \( H_{i,j} \) indexed by \( J(n - d, i) \times J(n - d, j) \) the generalized adjacency matrices defined in 3.3. We consider the Kronecker product of matrices denoted by “\( \otimes \)” and for \( 0 \leq i, j \leq d \) we define:

\[
M_{i,j} = \mathcal{B}_{\binom{d}{d-i}} H_{d-i,d-j} \otimes \mathcal{B}_{\binom{n-d}{i}} H_{i,j}, \quad i \leq j,
\]

\[
M_{i,j} = M_{j,i}^*, \quad i > j,
\]

\[
\mathcal{M}_{i,j} = L(M_{i,j})
\]

where \( \mathcal{B}_{\binom{n-d}{i}} H_{i,j} = \text{span}_\mathbb{C}\{H_{i,j}, A_{\binom{n-d}{i}} H_{i,j}, \ldots, A_{\binom{n-d}{i}} H_{i,j} \} \). Abusing the notation we will identify \( M_{i,j} \) with its image \( \mathcal{M}_{i,j} \).
Remark 4.1.

\[
\dim \mathcal{M}_{i,j} = \min(d - i, i)j,
\]

\[
\mathcal{M}_{i,j} = \left\{ \begin{pmatrix} E_m^{(d-i)} \otimes E_n^{(n-d)} \\ m, n = 0 \end{pmatrix} \right\},
\]

\[
\mathcal{M}_{i,j} = \left\{ \begin{pmatrix} E_m^{(d-i)} H_{d-i,d-j} \otimes E_n^{(n-d)} H_{i,j} \\ m, n = 0 \end{pmatrix} \right\},
\]

where the supraindex of the projectors, indicate the Bose–Mesner algebra which they belong to.

Definition 4.2. Let \( J(n, d) = (X, \{ R_i \}) \) be a Johnson scheme whose parameters satisfy \( 3d \leq n \). For \( 0 \leq i, j \leq d \), let \( \mathcal{M}_{i,j} \subseteq \text{Mat}_X(\mathbb{C}) \) be as above.

We define \( \mathcal{M} = \bigoplus_{i,j = 0}^{d} \mathcal{M}_{i,j} \).

4.2. \( \mathcal{M} \) is a subalgebra of \( \text{Mat}_X(\mathbb{C}) \)

The following lemmas will be useful to prove that \( \mathcal{M} \) is an algebra.

Lemma 4.3. Let \( A_m^{(v, k)} \) be the \( m \)-th adjacency matrix of \( J(v, k) \) and \( H_{k,k+l} \) indexed by \( J(v, k) \times J(v, k + l) \) be the generalized adjacency matrices given in Definition 3.3. Then

\[
A_m^{(v, k)} H_{k,k+l} = \sum_{j=0}^{\min(l,m)} \binom{m+l-j}{m} \binom{k-m+j}{k-m} H_{k,k+l}^{k-m+j},
\]

\[
H_{k,k+l} A_m^{(v, k)} = \sum_{j=\max(0,l-m)}^{l} \binom{m+j}{j} \binom{v-k-m-j}{l-j} H_{k,k+l}^{k+l-m-j}.
\]

Proof. We first prove (2).

Given \( x_k \in J(v, k) \), \( x_{k+l} \in J(v, k + l) \) such that \( |x_k \cap x_{k+l}| = k - m + j \), the element \( (x_k, x_{k+l}) \) of the LHS of (2) is given by

\[
(A_m^{(v, k)} H_{k,k+l})(x_k, x_{k+l}) = \sum_{\beta_k \in J(v, k)} A_m^{(v, k)} (x_k, \beta_k) H_{k,k+l}(\beta_k, x_{k+l})
\]

\[
= \#\{\beta_k \in J(v, k) : |x_k \cap \beta_k| = k - m, \beta_k \subseteq x_{k+l}\}
\]

\[
= \sum_{j=0}^{\min(l,m)} \binom{m+l-j}{m} \binom{k-m+j}{k-m} H_{k,k+l}^{k-m+j}(x_k, x_{k+l}).
\]

The equality (3) can be proved similarly. \( \square \)

Lemma 4.4. Let \( \mathcal{B}_{(v, k)} \) be the Bose–Mesner algebra corresponding to a \( J(v, k) \) and \( H_{k,k+l} \) as in the previous lemma. Define \( R := \{ r \in H_{k,k+l}^{(v, k)} : r \neq 0 \} \). Then we have that

\[
\left\{ H_{k,k+l}, A_m^{(v, k)} H_{k,k+l}, \ldots, A_m^{(v, k)} H_{k,k+l}, A_{R^{-1}}^{(v, k)} H_{k,k+l} \right\}
\]

is a basis of \( \mathcal{B}_{(v, k)} H_{k,k+l} \),

\[
\left\{ H_{k,k+l}, H_{k,k+l} A_m^{(v, k+i)}, \ldots, H_{k,k+l} A_m^{(v, k+i)}, A_{R^{-1}}^{(v, k+i)} \right\}
\]

is a basis of \( H_{k,k+l} \mathcal{B}_{(v, k+i)} \).
and each of these basis can be expressed as linear combinations of $H_{k,k+l}^r \in R$, therefore $B(v) H_{k,k+l} = H_{k,k+l} B(v) = \langle \{H_{k,k+l}^r\}_{r \in R} \rangle$.

**Proof.** Let us consider the first set. By (2) each one of the elements of the set of generators

$$\left\{ H_{k,k+l}, A^{(i)} \right\}_{k,k+l}, \ldots, A^{(\bar{m})} \right\}_{k,k+l}$$

of $B(v) H_{k,k+l}$ is a linear combination of $\{H_{k,k+l}^r\}_{r \in R}$. Furthermore, the $|R| \times |R|$ linear system given by $\{A^{(i)} \}_{k,k+l}$ for $i = 0, \ldots, |R| - 1$ expressed as a linear combination of $\{H_{k,k+l}^r\}_{r \in R}$ is triangular with positive diagonal $(\begin{pmatrix} i \cdot 0 \cdot (i+1) \cdot \ldots \cdot (|R| - 1) \end{pmatrix})$, therefore it is invertible. Since $\{H_{k,k+l}^r\}_{r \in R}$ are linearly independent, this implies that $\{A^{(i)} \}_{k,k+l}$ for $i = 0, \ldots, |R| - 1$ as well as the set $\{H_{k,k+l}^r\}_{r \in R}$ are a basis of $B(v) H_{k,k+l}$. Similarly, using (3) we can prove the second statement. \(\square\)

**Lemma 4.5.** Let $J(n,d) = (X, \{R_i\}_{0 \leq i \leq d})$ be a Johnson scheme whose parameters satisfy $3d \leq n$. For $0 \leq i, j, l \leq d$, let $M_{i,j}$ and $M_{j,l}$ be the subspaces of $\text{Max}(X)$ defined in 4.1. Consider $H_{d-i,d-j} \otimes H_{i,j} \in M_{i,j}$ and $H_{d-i,d-l} \otimes H_{j,l} \in M_{j,l}$. Then

$$(H_{d-i,d-j} \otimes H_{i,j}) (H_{d-j,d-l} \otimes H_{j,l}) \in B(d) H_{d-i,d-l} \otimes B(n-d) H_{i,l} = M_{i,l}.$$  

**Proof.** In order to prove the lemma we are going to work with the second coordinate of “$\otimes$”. Observe that it is enough to compute $H_{i,j} H_{j,l}$, in three different cases.

Let $\min_{i,l} := \min(i, l)$, $\max_{i,l} := \max(i, l)$

**Case 1:** For $0 \leq i \leq j \leq l \leq d$, in [8] it is proved that

$$H_{i,j} H_{j,l} = \begin{pmatrix} l - i \\ l - j \end{pmatrix} H_{i,l}. \quad (4)$$

**Case 2:** For $0 \leq \max_{i,l} \leq j \leq d$.

$$H_{i,j} H_{j,l} = \sum_{m=0}^{\max_{i,l}} \binom{v - \max_{i,l} - m}{j - \max_{i,l} - m} H_{i,l}^\min_{i,l,m} \quad (5)$$

**Proof.** Given $(\beta_i, \beta_j) \in J(v, i) \times J(v, l)$ we compute

$$(H_{i,j} H_{j,l}) (\beta_i, \beta_j) = \sum_{\beta_j \in J(v, j)} H_{i,j}(\beta_i, \beta_j) H_{j,l}(\beta_j, \beta_l)$$

$$= \# \{ \beta_j : (\beta_i \cup \beta_j) \subseteq \beta_j \}$$

$$= \begin{pmatrix} v - l - m \\ j - l - m \end{pmatrix} \text{ if } |\beta_i \cap \beta_l| = i - m \text{ for } i \leq l$$

$$= \sum_{m=0}^{i-l} \binom{v - l - m}{j - l - m} H_{i,l}^{i-m} (\beta_i, \beta_l).$$

For $l < i$ we have $H_{i,j} H_{j,l} = \sum_{m=0}^{j-i} \binom{v - i - m}{j - i - m} H_{i,l}^{i-m}$. This proves Case 2. \(\square\)

Similarly we have

**Case 3:** For $0 \leq j \leq \min_{i,l} \leq d$

$$H_{i,j} H_{j,l} = \sum_{m=0}^{\min_{i,l} - j} \binom{\min_{i,l} - m}{j} H_{i,l}^{\min_{i,l} - m}. \quad (6)$$
Proposition 4.6. Under the same hypothesis of Lemma 4.5, we have

\[ \mathcal{M}_{i,j} = \mathcal{B}(d_{i,j}) H_{d-i,d-j} \mathcal{B}(d_{i,j}) \otimes \mathcal{B}(n-d_i) H_{i,j} \mathcal{B}(n-d_j), \]

\[ \mathcal{M}_{i,j} \mathcal{M}_{j,i} \subseteq \mathcal{M}_{i,1} \] and therefore \( \mathcal{M} \) is an algebra.

Proof. By Lemmas 3.1 and 3.5 we have

\[ \mathcal{M}_{i,j} \mathcal{M}_{j,i} = \mathcal{B}(d_{i,j}) H_{d-i,d-j} \mathcal{B}(d_{i,j}) \otimes \mathcal{B}(n-d_i) H_{i,j} \mathcal{B}(n-d_j), \]

now using also Lemma 4.5

\[ \mathcal{M}_{i,j} \mathcal{M}_{j,i} \subseteq \mathcal{M}_{i,1}. \]

5. \( T = \mathcal{M} \)

In this section we will prove that the \( T \)-algebra of a Johnson scheme \( J(n, d) \) with \( 3d \leq n \) is the algebra \( \mathcal{M} \) defined in 4.1. First, we will show:

Proposition 5.1. Given \( J(n, d) = (X, \{ R_i \}_{0 \leq i \leq d}) \) a Johnson scheme whose parameters satisfy \( 3d \leq n \), let \( T \) be the Terwilliger algebra and \( \mathcal{M} \) be the corresponding algebra defined in 4.1. Then \( T \subseteq \mathcal{M} \).

Proof. By Lemmas 3.1 and 3.5 we have

\[ A_{|\Omega_1 \times \Omega_1} = I^{(n-d)} \otimes A^{(n-d)} + A^{(d)} \otimes I^{(n-d)} \]

\[ \subseteq \mathcal{B}(d_{i,j}) \otimes \mathcal{B}(n-d_i), \]

\[ A_{|\Omega_1 \times \Omega_{i+1}} = H_{d-i,d-i-1} \otimes H_{i,i+1} \]

\[ \subseteq \mathcal{B}(d_{i,j}) H_{d-i,d-i-1} \otimes \mathcal{B}(n-d_{i+1}) H_{i,j+1}, \]

\[ A_{|\Omega_1 \times \Omega_{i+1}} = 0, \quad l \geq 2. \]

Then \( A \subseteq \mathcal{M} \). Also we have that \( E_n^m = L(H_{m,m}^m) \in \mathcal{M} \) therefore \( T \subseteq \mathcal{M} \). \( \square \)

Now we will prove \( \mathcal{M} \subseteq T \), that is \( \mathcal{M}_{i,j} \subseteq T \) for \( 0 \leq i, j \leq d \).

Since we have \( \mathcal{M}_{i,j} = \mathcal{M}_{i,j} (H_{d-i,d-j} \otimes H_{i,j}) \) it will be enough to show that \( \mathcal{M}_{i,j} \subseteq T \) and the following:

Proposition 5.2. For \( 0 \leq i, j \leq d \) we have \( H_{d-i,d-j} \otimes H_{i,j} \in T_{|\Omega_1 \times \Omega_1} \).

Proof. The case \( i = j \) is trivial since \( H_{i,i} = I \). If \( i < j \) we write \( j = i + l, l \in \mathbb{N} \) and we use induction on \( l \).

For \( l = 1 \) the assertion holds for every \( i \) since by Lemma 3.5, we have

\[ A_{|\Omega_1 \times \Omega_{i+1}} = H_{d-i,d-i-1} \otimes H_{i,i+1} \in T_{|\Omega_1 \times \Omega_{i+1}}. \]
Assume that
\[ H_{d-i,d-i-l} \otimes H_{i,l+1} \in T_{[\Omega_i] \times [\Omega_{i+l}]}, \] (8)
then we need to prove that \( H_{d-i,d-i-l-1} \otimes H_{i,l+1+1} \in T_{[\Omega_i] \times [\Omega_{i+l+1}]}. \)

From (7) we have \( H_{d-i,d-i-l-1} \otimes H_{i,l+1+1} \in T_{[\Omega_i] \times [\Omega_{i+l+1}]} \). Multiplying this expression on the left by (8) and using Case 1 of the proof of Lemma 4.5, we have that the following product belongs to \( T \):

\[ \begin{align*}
(H_{d-i,d-i-l-1} &\otimes H_{i,l+1})(H_{d-i,d-i-l-1} \otimes H_{i,l+1+1}) \\
&= H_{d-i,d-i-l} H_{d-i,d-i-l-1} \otimes H_{i,l+1} H_{i+l,l+1+1} \\
&= (l + 1) \binom{l + 1}{l} H_{d-i,d-i-l-1} \otimes H_{i,l+1+1} \\
&= (l + 1)^2 H_{d-i,d-i-l-1} \otimes H_{i,l+l+1}.
\end{align*} \]

The case \( i > j \) is similar, so the proposition is proved. \( \square \)

5.1. \( H_{i,i} \subseteq T \)

This is a critical case. We need to show that
\[ L(E_q^{(d-i)} \otimes E_p^{(n-d)}) \in T \quad \forall \quad q, p. \] (9)

For this we try the following approach: recall the identities,
\[ A_{[\Omega_i] \times [\Omega_i]} = I_{(d-i)}^{(d-i)} \otimes A_{(n-d)}^{(n-d)} + A_{(d-i)}^{(d-i)} \otimes I_{(n-d)}^{(n-d)}, \]
\[ A_{(k)}^{(l)} = \sum_{r=0}^{k} p_1^{(r)}(E_r) E_r^{(l)}, \]
where \( p_1^{(r)}(m) = k(v - k) - m(v + 1 - m) \) is the eigenvalue of \( A_{(k)}^{(l)} \) in the eigenspace \( V_m \) (see [2]). Denote \( p_1^{(a)}(r) \) by \( \lambda_r \) and \( p_1^{(d-i)}(s) \) by \( \mu_s \). Then,
\[ \begin{align*}
A_{[\Omega_i] \times [\Omega_i]} &= I_{(d-i)}^{(d-i)} \otimes A_{(n-d)}^{(n-d)} + A_{(d-i)}^{(d-i)} \otimes I_{(n-d)}^{(n-d)} \\
&= \min(d-i,i) \sum_{p'=0}^{d-i} E_p^{(d-i)} \otimes \sum_{p=0}^{d-i} \lambda_p E_p^{(n-d)} + \sum_{q=0}^{d-i} \mu_q E_q^{(d-i)} \otimes \sum_{q'=0}^{d-i} E_{q'}^{(n-d)} \\
&= \min(d-i,i) \sum_{p'=0}^{d-i} \sum_{p=0}^{d-i} \lambda_p E_p^{(d-i)} \otimes E_p^{(n-d)} + \sum_{q=0}^{d-i} \sum_{q'=0}^{d-i} \mu_q E_q^{(d-i)} \otimes E_{q'}^{(n-d)} \\
&= \sum_{p,q=0}^{d-i} (\lambda_p + \mu_q) E_q^{(d-i)} \otimes E_p^{(n-d)}
\end{align*} \]
so the system of equations given by the powers of \( A_{[\Omega_i] \times [\Omega_i]} \) as linear combinations of \( E_q^{(d-i)} \otimes E_p^{(n-d)} \) gives a Vandermonde matrix. Therefore if \( \lambda_l + \mu_m \neq \lambda_p + \mu_q \) whenever \( (l, m) \neq (p, q) \) then \( E_q^{(d-i)} \otimes E_p^{(n-d)} \in T \) and in these cases \( A_{[\Omega_i] \times [\Omega_i]} = \left\{ E_q^{(d-i)} \otimes E_p^{(n-d)} \right\} \) so the equation in (9) holds. This is not always the case, thus we make the following:
**Definition 5.3.** Given two non-zero idempotents
\[ E_q \left( \frac{d}{d-i} \right) \otimes E_p \left( \frac{n-d}{i} \right), E_s \left( \frac{d}{d-i} \right) \otimes E_r \left( \frac{n-d}{i} \right) \in B(\frac{d}{d-i}) \otimes B(\frac{n-d}{i}) \]
we say that they are in resonance if
\[ p(n-d + 1 - p) + q(d + 1 - q) = r(n - d + 1 - r) + s(d + 1 - s). \]

The strategy to break the resonant cases in \( M_{i,j} \) will be to find \( j \) such that exactly one of the idempotents \( E_q \left( \frac{d}{d-i} \right) \otimes E_p \left( \frac{n-d}{i} \right) \) or \( E_s \left( \frac{d}{d-i} \right) \otimes E_r \left( \frac{n-d}{i} \right) \) is non-zero in \( M_{j,j} \) and then “map” it (respectively, “pull it back”) to \( M_{i,j} \) if \( j < i \) (respectively, \( j > i \)). For this we need the following:

**Definition 5.4.** Given \( J(n, d) = (X, \{ R_i \}_{0 \leq i \leq d}) \) a Johnson scheme whose parameters satisfy \( 3d \leq n \), \( M \) the corresponding algebra defined in 4.1. and \( H_{i,j} \) the matrix defined in 3.3 we set
\[
\mathcal{L}_i : M_{i,j} \rightarrow M_{i+1,j+1}, \quad 0 \leq i < d,
\]
\[
E_q \left( \frac{d}{d-i} \right) \otimes E_p \left( \frac{n-d}{i} \right) \rightarrow H_{d-i-1,d-i-1} E_q \left( \frac{d}{d-i} \right) H_{d-i,d-i-1} \otimes H_{i+1,i} E_p \left( \frac{n-d}{i} \right) H_{i,i+1},
\]
\[
\mathcal{P}_i : M_{i,j} \rightarrow M_{i-1,j-1}, \quad 0 < i \leq d,
\]
\[
E_q \left( \frac{d}{d-i} \right) \otimes E_p \left( \frac{n-d}{i} \right) \rightarrow H_{d-i-1,d-i} E_q \left( \frac{d}{d-i} \right) H_{d-i,d-i+1} \otimes H_{i+1,i} E_p \left( \frac{n-d}{i} \right) H_{i,i+1}.
\]

**Remark 5.5.** If \( E_q \left( \frac{d}{d-i} \right) \otimes E_p \left( \frac{n-d}{i} \right) \in T \) then
\[
\mathcal{L}_i \left( E_q \left( \frac{d}{d-i} \right) \otimes E_p \left( \frac{n-d}{i} \right) \right) \in T \quad \text{since} \quad H_{d-i,d-i-1} \otimes H_{i,i+1} \in T.
\]

Similarly
\[
\mathcal{P}_i \left( E_q \left( \frac{d}{d-i} \right) \otimes E_p \left( \frac{n-d}{i} \right) \right) \in T.
\]

We have the following result:

**Lemma 5.6.**
\[
\mathcal{L}_i \left( E_s \left( \frac{d}{d-i} \right) \otimes E_r \left( \frac{n-d}{i} \right) \right) = p_{d,d-i,s} n_{d-i,i,r} E_s \left( \frac{d}{d-i-1} \right) \otimes E_r \left( \frac{n-d}{i+1} \right),
\]
\[
\mathcal{P}_i \left( E_s \left( \frac{d}{d-i} \right) \otimes E_r \left( \frac{n-d}{i} \right) \right) = l_{d,d-i,s} p_{n-d,i,r} E_s \left( \frac{d}{d-i+1} \right) \otimes E_r \left( \frac{n-d}{i-1} \right),
\]
where \( l_{v,k,r} = v - k + p_1^{(v)}(r) \) and \( p_{v,k,r} = k + p_1^{(v)}(r) \).

**Proof.** We will prove only the first equality, the computations are analogous for the second one. Observe that \( \mathcal{L}_i \) can be decomposed as \( \mathcal{L}_i^{(1)} \otimes \mathcal{L}_i^{(2)} \) where
\[
\mathcal{L}_i^{(1)} \left( E_q \left( \frac{d}{d-i} \right) \right) := H_{d-i-1,d-i} E_q \left( \frac{d}{d-i} \right) H_{d-i,d-i-1}, \quad \mathcal{L}_i^{(2)} \left( E_p \left( \frac{n-d}{i} \right) \right) := H_{i+1,i} E_p \left( \frac{n-d}{i} \right) H_{i,i+1}.
\]

Since the actions of \( \mathcal{L}_i^{(1)} \) and \( \mathcal{L}_i^{(2)} \) are similar we will consider only the latter. From Lemmas 4.4 and 4.5, we have that \( \mathcal{L}_i^{(2)} \left( E_p \left( \frac{n-d}{i} \right) \right) \in B(\frac{n-d}{i+1}) \). Now we will show that for \( 0 \leq r \leq i \), \( \mathcal{L}_i^{(2)} \left( E_p \left( \frac{n-d}{i} \right) \right) \) is a multiple of a primitive idempotent in \( B(\frac{n-d}{i+1}) \).
By definition we have that

$$
\mathcal{L}_i^{(2)} \left( E_r \binom{n-d}{i} \right) \mathcal{L}_i^{(2)} \left( E_r \binom{n-d}{i} \right) = H_{i+1,i} E_r \binom{n-d}{i} H_{i,i+1} E_r \binom{n-d}{i} H_{i,i+1}.
$$

By (5) we now that $H_{i,i+1} H_{i+1,i} = (n - d - i) I \binom{n-d}{i} + A \binom{n-d}{i} \in \mathcal{B} \binom{n-d}{i}$, then

$$
\mathcal{L}_i^{(2)} \left( E_r \binom{n-d}{i} \right) \mathcal{L}_i^{(2)} \left( E_r \binom{n-d}{i} \right) = H_{i+1,i} E_r \binom{n-d}{i} (n - d - i) I \binom{n-d}{i} + A \binom{n-d}{i} E_r \binom{n-d}{i} H_{i,i+1}
$$

$$
= \left( n - d - i + p_1 \binom{n-d}{i} (t) \right) H_{i+1,i} E_r \binom{n-d}{i} E_r \binom{n-d}{i} H_{i,i+1}.
$$

With this expressions we can conclude that

$$
\mathcal{L}_i^{(2)} \left( E_r \binom{n-d}{i} \right) \mathcal{L}_i^{(2)} \left( E_r \binom{n-d}{i} \right) = \begin{cases} 
I_{n-d,i,r} \mathcal{L}_i^{(2)} \left( E_r \binom{n-d}{i} \right) & \text{if } r = t, \\
0 & \text{if } r \neq t,
\end{cases}
$$

where $I_{n-d,i,r} = (n - d - i) + p_1 \binom{n-d}{i} (r)$.

Let us analyze $I_{n-d,i,r}$ in relation with the parameters $n - d$ and $i$. From the general theory we know that the eigenvalues of a $J(n - d, i)$ are $p_1 \binom{n-d}{i} (r) = i(n - d - i) - r(n - d + 1 - r)$, and that they are strictly decreasing with minimal eigenvalue $p_1 \binom{n-d}{i} (i) = -i$.

Then we have that $(n - d - i) + p_1 \binom{n-d}{i} (r) \geq n - d - 2i$ and therefore

$$
I_{n-d,i,r} = (n - d - i) + p_1 \binom{n-d}{i} (r) = 0 \iff r = i \text{ and } n - d = 2i.
$$

From this equations we can assure that if $n - d > 2i$, then every idempotent of $\mathcal{B} \binom{n-d}{i}$ can be lifted to one in $\mathcal{B} \binom{n-d}{i+1}$ (since $I_{n-d,i,r} > 0$).

Let us prove that it is exactly $E_r \binom{n-d}{i+1}$. From Lemma 4.3 we know that

$$
A \binom{n-d}{i} H_{i,i+1} = 2 H_{i,i+1}^{i-1} + i H_{i,i+1},
$$

$$
H_{i,i+1} A \binom{n-d}{i+1} = (n - d - i - 1) H_{i,i+1} + 2 H_{i,i+1}^{i-1}
$$

$$
= (n - d - i - 1) H_{i,i+1} + A \binom{n-d}{i} H_{i,i+1} - i H_{i,i+1}
$$

$$
= (n - d - 2i - 1) H_{i,i+1} + A \binom{n-d}{i} H_{i,i+1}.
$$
Therefore
\[
\mathcal{L}_i^{(2)}\left( E_r^{(n-d)} \right) A^{(n-d)}_{i+1} = \left( H_{i+1} E_r^{(n-d)} H_{i,i+1} \right) A^{(n-d)}_{i+1}
\]
\[
= H_{i+1} E_r^{(n-d)} \left( (n - d - 2i - 1)H_{i,i+1} + A^{(n-d)}_{i+1} \right)
\]
\[
= \left( n - d - 2i - 1 + p_i^{(n-d)}(r) \right) \left( H_{i+1} E_r^{(n-d)} H_{i,i+1} \right)
\]
\[
= p_i^{(n-d)}(r) \mathcal{L}_i^{(2)}\left( E_r^{(n-d)} \right)
\]
so we can conclude that
\[
\mathcal{L}_i^{(2)}\left( E_r^{(n-d)} \right) = I_{n-d,i,r} E_r^{(n-d)} \in \mathcal{B}^{(n-d)}_{i+1}.
\]

The following corollary will be used in the next chapter.

**Corollary 5.7.**
\[
E_r^{(d-i)} H_{d-i,d-j} \otimes E_r^{(n-d)} H_{i,j} = H_{d-i,d-j} E_r^{(d-i)} \otimes H_{i,j} E_r^{(n-d)}.
\]

**Proof.** When \( j = i \) there is nothing to prove since \( H_{i,i} \) and \( H_{d-i,d-i} \) are identities matrices. For \( j = i + 1 \)
\[
E_r^{(n-d)} H_{i,i+1} = p_{n-d,i+1,8} \mathcal{P}_{i+1}^{(n-d)} E_r^{(n-d)} H_{i,i+1}
\]
\[
= p_{n-d,i+1,8} H_{i,i+1} E_r^{(n-d)} H_{i,i+1}
\]
\[
= p_{n-d,i+1,8} H_{i,i+1} E_r^{(n-d)} ((i + 1)I + A)
\]
\[
= H_{i,i+1} E_r^{(n-d)}
\]
Using (4) we can iterate the equality and generalize this for \( j > i + 1 \). Similarly for \( j < i \). □

**Example 5.8.** If we consider \( J(12, 4) \), for \( A_{12} \otimes A_2 \) we have that
\[
\left\{ l_0^{(\frac{1}{2})}, l_1^{(\frac{1}{2})}, l_2^{(\frac{1}{2})} \right\} = \{12, 4, -2\} \quad \text{and} \quad \left\{ \mu_0^{(\frac{1}{2})}, \mu_1^{(\frac{1}{2})}, \mu_2^{(\frac{1}{2})} \right\} = \{4, 0, -2\}.
\]
Therefore the only resonant case is \( l_2^{(\frac{1}{2})} + \mu_0^{(\frac{1}{2})} = l_1^{(\frac{1}{2})} + \mu_2^{(\frac{1}{2})} \). We have \( E_0^{(\frac{1}{2})} \otimes E_2^{(\frac{1}{2})} + E_2^{(\frac{1}{2})} \otimes E_1^{(\frac{1}{2})} \in T_{12} \otimes A_2 \) so to conclude
\[
T_{12} \otimes A_2 = \mathcal{B}^{(\frac{1}{2})}_{\frac{1}{2}} \otimes \mathcal{B}^{(\frac{3}{2})}_{\frac{3}{2}} = M_{2,2}
\]
(10)
it is enough to show \( E_0^{(\frac{1}{2})} \otimes E_2^{(\frac{3}{2})} \in T_{12} \otimes A_2 \).
It is clear from (1) that \( M_{4,4} = T_{12} \otimes A_4 \). So the claim follows from
\[
E_0^{(\frac{1}{2})} \otimes E_2^{(\frac{3}{2})} \leftarrow E_0^{(\frac{1}{2})} \otimes E_2^{(\frac{3}{2})} \leftarrow E_0^{(\frac{3}{2})} \otimes E_2^{(\frac{3}{2})}
\]
since \( E_0^{(\frac{1}{2})} \otimes E_2^{(\frac{3}{2})} \in T \) and \( E_0^{(\frac{1}{2})} \otimes E_2^{(\frac{3}{2})} = \mathcal{P}_3^{(\mathcal{P}_3)}\left( E_0^{(\frac{1}{2})} \otimes E_2^{(\frac{3}{2})} \right) \in T \) by Remark 5.5.
Theorem 5.9. Let $T$ be the Terwilliger algebra of a $J(n, d)$ with $3d \leq n$ and $\mathcal{M}$ be the algebra of Definition 4.2. Then

$$T = \mathcal{M}, \text{ that is } T_{|\Omega_1 \times \Omega_i} = \text{span}_C \left\{ E_r^{(d-i)} \otimes E_s^{(n-d)} : r, s = 0 \right\}_{r, s = 0}^{\min(d-i, i)} = \mathcal{B} \left( d^{(d-i)} \right) \otimes \mathcal{B} \left( n^{(n-d)} \right).$$

Proof. It is enough to prove that

$$T_{|\Omega_1 \times \Omega_i} = \text{span}_C \left\{ E_r^{(d-i)} \otimes E_s^{(n-d)} : r, s = 0 \right\}_{r, s = 0}^{\min(d-i, i)} = \mathcal{B} \left( d^{(d-i)} \right) \otimes \mathcal{B} \left( n^{(n-d)} \right).$$

We will use induction on $i (i$ decreasing from $d$ to $\lfloor d/2 \rfloor$).

For $i = d$ it is true since $A_{\Omega_1 \times \Omega_d} = I_{(d)} \otimes A_{(n-d)} \in T_{|\Omega_1 \times \Omega_d}$ and $(\lambda_p + \mu_0) \neq (\lambda_q + \mu_0)$ for $p \neq q$. Therefore

$$\mathcal{B} \left( d^{(d-i)} \right) \otimes \mathcal{B} \left( n^{(n-d)} \right) = \text{span}_C \left\{ E_r^{(d-i)} \otimes E_s^{(n-d)} : r, s = 0 \right\}_{r, s = 0}^{d} \subseteq T_{|\Omega_1 \times \Omega_d}.$$

Using the inductive hypothesis

$$\bigcup_{j=0}^{d-i-1} \left\{ E_j^{(d-i)} \otimes E_s^{(n-d)} : r, s = 0 \right\}_{r, s = 0}^{i+1} \subseteq T_{|\Omega_j \times \Omega_{i+1}},$$

we want to prove

$$\bigcup_{j=0}^{d-i} \left\{ E_j^{(d-i)} \otimes E_s^{(n-d)} : r, s = 0 \right\}_{r, s = 0}^{i} \subseteq T_{|\Omega_1 \times \Omega_i}.$$

Then for $0 \leq r \leq d - i - 1, 0 \leq s \leq i$ we have

$$E_r^{(d-i)} \otimes E_s^{(n-d)} = \mathcal{B}_{i+1} \left( E_r^{(d-i)} \otimes E_s^{(n-d)} \right) \in T_{|\Omega_1 \times \Omega_i}.$$

Since we know that

$$A_{\Omega_1 \times \Omega_i} = \sum_{r=0}^{d-i-1} \sum_{s=0}^{i} (\lambda_s + \mu_r) E_r^{(d-i)} \otimes E_s^{(n-d)} + \sum_{s=0}^{i} (\lambda_s + \mu_d - i) E_{d-i}^{(d-i)} \otimes E_s^{(n-d)}$$

with the RHS $\in T_{|\Omega_1 \times \Omega_d}$ and $(\lambda_p + \mu_d - i) \neq (\lambda_q + \mu_d - i)$ if $p \neq q$, then

$$\left\{ E_{d-i}^{(d-i)} \otimes E_s^{(n-d)} : r, s = 0 \right\}_{r, s = 0}^{i} \subseteq T_{|\Omega_1 \times \Omega_i}$$

so the result is true for $\lfloor d/2 \rfloor < i$. The proof for $i \leq \lfloor d/2 \rfloor$ is similar. \(\square\)

Remark 5.10. Notice that the same proof also holds for $n = 3d - 1$.

Corollary 5.11. Let $f, l, m$ be positive integers such that $3f \leq \min(l, m)$ and $T_f^{(l)}, T_f^{(m)}$ be the $T$-algebra of $J(l, f)$ and $J(m, f)$, respectively. Then

$$T_f^{(l)} \simeq T_f^{(m)}.$$
The first product is zero if

For 3

Remark 6.2.

\[ rs T \]

and

\[ \text{Proof.} \]

Definition 6.3.

For 0

\[ \text{Proof.} \]

Remark 6.4.

\[ \text{Definition 6.1.} \]

Given integers \( r, s \) with \( 0 \leq r \leq d/2 \), \( 0 \leq s \leq d \), we define

\[ e_{r,s} := \min \left\{ i : 0 \neq E_r^{(d-i)} \otimes E_s^{(n-d)} \in M_{i,l} \right\}, \]

\[ d_{r,s} := \# \left\{ i : 0 \neq E_r^{(d-i)} \otimes E_s^{(n-d)} \in M_{i,d} \right\} - 1. \]

Remark 6.2. For \( 3d \leq n \), \( e_{r,s} = \max(r, s) \), \( e_{r,s} + d_{r,s} = d - r \).

Definition 6.3. For \( 0 \leq i, j \leq d \) and \( 0 \leq r \leq \min(d - i, i) \), \( 0 \leq s \leq i \) we define

\[ rs T \]

and \( rs T \subseteq T \) as the subspace generated by these matrices:

\[ \text{Remark 6.4.} \]

\[ rs T \neq 0 \iff \max(r, s) \leq i, j \leq d - r. \]

Proposition 6.5.

\[ rs T \subseteq T \text{ is a bilateral ideal and } rs T \mathbf{p} T = 0 \iff (r, s) \neq (p, q). \]

Proof. For \( 0 \leq r, p \leq \min(d - i, i) \), \( 0 \leq s, q \leq i \) we need to show that

\[ \left( E_r^{(d-i)} H_{d-i,d-j} \otimes E_s^{(n-d)} H_{i,l} \right) \left( E_p^{(d-i)} H_{d-l,d-m} \otimes E_q^{(n-d)} H_{l,m} \right) \subseteq rs T, \]

\[ \left( E_p^{(d-i)} H_{d-l,d-m} \otimes E_q^{(n-d)} H_{l,m} \right) \left( E_r^{(d-i)} H_{d-i,d-j} \otimes E_s^{(n-d)} H_{i,l} \right) \subseteq rs T. \]

The first product is zero if \( j \neq l \). For simplicity, we consider only the second coordinate of the tensor product, then we check that

\[ E_s^{(n-d)} H_{i,j} E_q^{(n-d)} H_{j,l} = \delta_{s,q} \beta E_s^{(n-d)} H_{i,l} \]

for some scalar \( \beta \). This follows from the Corollary 5.7 and the fact that \( H_{i,j} H_{j,l} \in \mathcal{B}_{(n-d)} H_{i,l} \), then

\[ H_{i,j} H_{j,l} = \sum_p \alpha_p E_p^{(n-d)} H_{i,l} \]

for some coefficients \( \alpha_p \). \[ \square \]
Now we will give a normalized basis for \( rs T \). We have seen that \( rs \tilde{T}_{ij} \leftrightarrow rs \tilde{T}_{jl} \) is a multiple of \( rs \tilde{T}_{il} \).

**Definition 6.6.** Given the ideal \( rs T \), for \( i \leq j \) set

\[
\begin{align*}
  n^r_{ij} &:= \sum_{m=0}^{j-i} \binom{d-j-m}{d-i} p_m^{d-i}(r), \\
  n^s_{ij} &:= \sum_{m=0}^{j-i} \binom{n-d-i-m}{j-i-m} p_m^{n-d}(s),
\end{align*}
\]

\[
rs T_{ij} := \frac{1}{\sqrt{n^r_{ij} n^s_{ij}}} E_r^{d-i} H_{d-i,d-j} \otimes E_s^{n-d}(i) H_{i,j}, \quad rs T_{ji} := rs T^t_{ij}.
\]

**Lemma 6.7.** \( rs T_{ij} \leftrightarrow rs T_{jl} \) therefore \( rs T \approx \text{End}(C^{d_{rs}+1}) \) so \( rs T \) is simple.

**Proof.** The scalars \( n^r_{ij}, n^s_{ij} \) were defined so that the following equations are satisfied

\[
\begin{align*}
  E_r^{d-i} H_{d-i,d-j} H_{d-j,d-i} E_r^{d-i} &= n^r_{ij} E_r^{d-i}, \\
  E_s^{n-d}(i) H_{i,j} H_{j,i} E_s^{n-d}(i) &= n^s_{ij} E_s^{n-d}(i).
\end{align*}
\]

Therefore with this normalization we have \( rs T_{ij} \leftrightarrow rs T_{jl} \).

For the general case \( rs T_{ij} \leftrightarrow rs T_{jl} \), we need to prove

\[
\frac{1}{\sqrt{n^r_{ij} n^s_{ij}}} E_s^{n-d}(i) H_{i,j} E_s^{n-d}(j) H_{j,i} = \frac{1}{\sqrt{n^r_{ij} n^s_{ij}}} E_s^{n-d}(j) H_{j,i} E_s^{n-d}(i) H_{i,j} \]

and the corresponding equation for the first component of the tensor product. The equality follows from the fact that both sides are multiple of each other (by Corollary 5.7 and (11)) and multiplying each one by its transpose they become \( E_r^{d-i} \). \( \square \)

We have the following decomposition:

**Theorem 6.8.** Let \( T \) be the Terwilliger algebra of the Johnson scheme \( J(n, d) \) with \( 3d \leq n \). Then

\[
T = \bigoplus_{r=0}^{d/2} \bigoplus_{s=0}^{d/2} rs T \approx \bigoplus_{r=0}^{d/2} \bigoplus_{s=0}^{d/2} \text{End}(C^{d_{rs}+1}),
\]

where \( rs T \subseteq T \) are simple ideals defined in 6.3.

**Proof.** From Theorem 5.9 we have the equalities

\[
T_{[i] \times [i]} = \mathcal{B}^{d}_{d-i} H_{d-i,d-j} \otimes \mathcal{B}^{n-d}_{n-d}(i) H_{i,j} \quad \text{for } i \leq j,
\]

\[
T_{[i] \times [j]} = (T_{[j] \times [i]})^t \quad \text{if } i > j
\]

therefore \( T = \bigoplus_{r,s=0}^{d/2} rs T \). From Proposition 6.5 it is a direct sum. \( \square \)

**7. Decomposing \( C^{d(n)} \) into irreducible \( T \)-submodules**

In this section we consider the action of the algebra \( T \) on \( C^{d(n)} \), \( T \times C^{d(n)} \rightarrow C^{d(n)} \) and give a decomposition of such space into irreducibles \( T \)-submodules.
7.1. Isotypic T-submodules and their projections

Suppose, in general, that we have $T$ the Terwilliger algebra of an arbitrary association scheme and a decomposition $T = \oplus T_i$ of the algebra as a finite direct sum of simple ideals $T_i$. To each simple ideal $T_i$ we can associate the $T$-submodule $T_i \subseteq T$. It is indeed a submodule since $TT_i \subseteq T_i$. We call such a $T$-submodule “isotypic”. In our case we have

$$\mathbb{C}^{(\overline{n})}_{d} = \bigoplus_{r,s=0}^{\lfloor d/2 \rfloor} rs T \mathbb{C}^{(\overline{n})}_{d}.$$ 

For $0 \leq r \leq \lfloor d/2 \rfloor$, $0 \leq s \leq d$, we define the projections $\pi_{r,s} : \mathbb{C}^{(\overline{n})}_{d} \rightarrow rs T \mathbb{C}^{(\overline{n})}_{d}$ where $\pi_{r,s}(Z) = \sum_{i=0}^{d} rs T_{ii} Z \forall Z \in \mathbb{C}^{(\overline{n})}_{d}$. They satisfy $\pi_{r,s}^2 = \pi_{r,s}$, $\sum_{r,s=0}^{\lfloor d/2 \rfloor} \pi_{r,s} = I \in T$. Therefore we have $\dim rs T \mathbb{C}^{(\overline{n})}_{d} = r(k(r,s)).$

Let $\tilde{e} = e_{r,s}$, $\tilde{d} = d_{r,s}$ be the parameters of Definition 6.1. We also have $rk(\pi_{r,s}) = \sum_{i=0}^{\tilde{d}+\tilde{e}} rk(rs T_{ii}) = (\tilde{d}+1)rk(rs T_{\tilde{e},\tilde{e}})$ since $rk(rs T_{\tilde{e},\tilde{e}})$ irreducible $T$-submodules of $rs T \mathbb{C}^{(\overline{n})}_{d}$ are irreducible for the ideal $rs T$, therefore they are of dimension $\tilde{d} + 1$. From the equality above we have $rs T \mathbb{C}^{(\overline{n})}_{d}$ decomposes as a sum of $rk(rs T_{\tilde{e},\tilde{e}})$ irreducible $T$-submodules (here $rk(M)$ denotes rank of the matrix $M$). Any decomposition of $rs T \mathbb{C}^{(\overline{n})}_{d}$ into irreducible $T$-submodules can be found from decomposing the projection $rs T_{\tilde{e},\tilde{e}} = \sum_{j=1}^{rk(rs T_{\tilde{e},\tilde{e}})} rs T_{\tilde{e},\tilde{e}}^{(j)}$ as a sum of rank one projectors. Each projector $rs T^{(j)}_{\tilde{e},\tilde{e}}$ is associated to the following irreducible $T$-submodule:

$$W_{rs T^{(j)}_{\tilde{e},\tilde{e}}} = tr_{rs T^{(j)}_{\tilde{e},\tilde{e}}}(\mathbb{C}^{(\overline{n})}_{d}) + tr_{rs T_{\tilde{e},\tilde{e}}^{(j)}}(\mathbb{C}^{(\overline{n})}_{d}) + \cdots + tr_{rs T_{\tilde{e},\tilde{e}}^{(j)}}(\mathbb{C}^{(\overline{n})}_{d}).$$

Therefore one can prove that $rs T \mathbb{C}^{(\overline{n})}_{d} = \bigoplus_{j=1}^{rk(rs T_{\tilde{e},\tilde{e}})} W_{rs T^{(j)}_{\tilde{e},\tilde{e}}}$ and we have:

**Theorem 7.1.** Let $T$ be the Terwilliger algebra of the Johnson scheme $J(n, d)$ with $3d \leq n$. Let $W_{rs T^{(j)}_{\tilde{e},\tilde{e}}}$ be the irreducible $T$-submodules defined above. Then we have the following decomposition:

$$\mathbb{C}^{(\overline{n})}_{d} = \bigoplus_{r,s=0}^{\lfloor d/2 \rfloor} \bigoplus_{j=1}^{rk(rs T_{\tilde{e},\tilde{e}})} W_{rs T^{(j)}_{\tilde{e},\tilde{e}}}. $$

**8. Concluding remarks**

In the proof of Theorem 5.9 we showed an algorithm to break every resonant case when $3d - 1 \leq n$. In the case $n = 2d$ we have that $E_{rs,i}^{(a-d)} \otimes E_{r,s}^{(a-d)}$ and $E_{rs,i}^{(a-d)} \otimes E_{r,s}^{(a-d)}$ are resonant in all $M_{i,j}$, also $e_{r,s} = e_{r,s}$ and $d_{r,s} = d_{r,s}$ then one can show that it is not possible to break the resonance. Therefore the $T$-algebra is strictly contained in $\mathcal{M}$.

A way to attack the cases not considered in this paper ($2d < n < 3d - 1$) would be to prove that it is possible to break the resonant cases when $e_{r,s} \neq e_{p,q}$ or $d_{r,s} \neq d_{p,q}$ and to reduce the algebra $\mathcal{M}$ when $e_{r,s} = e_{p,q}$ and $d_{r,s} = d_{p,q}$.  

**References**


