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Infinite Convergent String-rewriting Systems and Cross-sections for Finitely Presented Monoids[†]

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A finitely presented monoid has a decidable word problem if and only if it can be presented by some left-recursive convergent string-rewriting system if and only if it has a recursive cross-section. However, regular cross-sections or even context-free cross-sections do not suffice. This is shown by presenting examples of finitely presented monoids with decidable word problems that do not admit regular cross-sections, and that, hence, cannot be presented by left-regular convergent string-rewriting systems. Also examples of finitely presented monoids with decidable word problems are presented that do not even admit context-free cross-sections. On the other hand, it is shown that each finitely presented monoid with a decidable word problem has a finite presentation that admits a cross-section which is a Church–Rosser language. Finally we address the notion of left-regular convergent string-rewriting systems that are tractable.

1. Introduction

The class of finite convergent rewriting systems has received a lot of attention in the literature because these systems yield an elegant, syntactically very simple algorithm for solving word problems. See, e.g. Dershowitz and Jouannaud (1990) and Avenhaus and Madlener (1990) for surveys on term-rewriting systems in general, and Book and Otto (1993) and Jantzen (1988) for the special case of string-rewriting systems. If R is a finite convergent string-rewriting system on some alphabet Σ, then the set of irreducible strings $\mathrm{IRR}(R)$, which is the complement of the ideal generated by the finite set $\mathrm{dom}(R)$ of left-hand sides of rules of R, is a regular language, which forms a complete set of unique representatives for the monoid M_R presented by $(\Sigma; R)$. In the following we will call complete sets of unique representatives *cross-sections*. Given a string $w \in \Sigma^*$, its representative $w_0 \in \mathrm{IRR}(R)$ can be determined simply by reduction. Now two strings $u, v \in \Sigma^*$ are equivalent with respect to the Thue congruence $\leftrightarrow_R^*$ induced by R if and only if their respective representatives u_0 and v_0 are identical. This is the basic observation underlying the so-called *normal form algorithm* for the word problem for R.

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Unfortunately, the finite convergent string-rewriting systems do not suffice to present all those finitely presented monoids that have solvable word problems. Indeed, as shown by Squier (1987), a monoid satisfies the homological finiteness condition FP_3 if it can be presented through a finite convergent string-rewriting system. In addition, Squier presents a sequence S_k ($k \geq 1$) of finitely presented monoids such that, for each $k \geq 2$, the monoid S_k does not satisfy the condition FP_3. Thus, none of these monoids can be presented by a finite convergent string-rewriting system. However, each of these monoids can be presented by an infinite convergent string-rewriting system with a regular set of left-hand sides. We will call systems of this form *left-regular*. Actually, the left-regular convergent systems given by Squier are sufficiently simple so that they still can be used as a basis for the normal form algorithm for solving the word problem. Obviously, a left-regular convergent system yields a regular cross-section for the monoid it presents.

However, even the left-regular convergent string-rewriting systems are not sufficient to present all those finitely presented monoids that have solvable word problems. This fact has been established by Kobayashi (1995) by presenting an example of a finitely presented monoid with a decidable word problem that has intermediate growth. This monoid cannot be presented by any left-regular convergent string-rewriting system, but it admits a left-context-free convergent system. However, this does not answer the question of whether this monoid has a regular cross-section. In fact, we may ask more generally whether there exist at all finitely presented monoids with decidable word problems that do not have regular cross-sections. Furthermore, the above result suggests the following question: which classes of infinite convergent string-rewriting systems suffice to present all finitely presented monoids with decidable word problems?

Recall that each finitely generated monoid can be presented through an infinite convergent string-rewriting system (Avenhaus, 1986). This includes in particular the finitely generated monoids with undecidable word problems. However, for a finitely presented monoid M the following three statements are equivalent:

(1) M has a decidable word problem.
(2) M has a presentation through a left-recursive convergent string-rewriting system.
(3) M has a recursive cross-section.

Is it possible to improve upon this characterization? In Section 3 we will consider classes of languages that satisfy certain closure properties and that we call *complementary* classes. If $\mathbb{C}$ is such a class, then a finitely presented monoid M has a presentation through a left-$\mathbb{C}$ convergent string-rewriting system if and only if there exists an admissible, well-founded partial ordering such that the set of minimal strings with respect to this ordering is a cross-section for M belonging to $\mathbb{C}$. For which complementary classes do we obtain all finitely presented monoids with decidable word problems in this way? Examples of complementary classes are the recursive sets, the context-sensitive languages, and the regular sets. Thus, the question above entails the question of whether every finitely presented monoid with a decidable word problem can be presented by a left-context-sensitive convergent string-rewriting system. In fact, we can also ask this question for the class of context-free languages, even though this class is not complementary.

In Section 4 we concentrate on regular cross-sections. Based on the pumping lemma for regular sets we develop a criterion that can be used to prove that a finitely generated monoid does not have a regular cross-section (Proposition 4.4). Then we present a sequence of finitely presented monoids M_t ($t \in \mathbb{N}$) extending the example of Kobayashi

(1995). It turns out that each of these monoids can be presented by a left-context-free convergent string-rewriting system, but that none of them has a regular cross-section. A second sequence N_t ($t \in \mathbb{N}_+$) of examples is presented that have the same properties for each $t \geq 3$, but which in addition have polynomial growth. This answers a question raised by Kobayashi (1995). The growth of monoids is considered in Section 5.

In Section 6 we consider the existence of context-free cross-sections. From Ogden's lemma we obtain a technical result that can be used for proving that a finitely generated monoid does not have a context-free cross-section (Proposition 6.3). Using this result we then show that none of the monoids $M_t (t \in \mathbb{N})$ or $N_t (t \geq 5)$ has a context-free cross-section, even though they all are presented by left-context-free convergent string-rewriting systems. The monoid N_3 is of particular interest. As shown in Section 4 it does not have a regular cross-section, but it has a context-free cross-section. Thus, context-free cross-sections are in fact a more powerful tool than regular cross-sections.

As the context-free cross-sections do not suffice to capture all finitely presented monoids with decidable word problems, the question arises of which class of languages would be sufficiently rich. It is fairly easily seen that the class of context-sensitive languages has this property. In fact, in Section 7 we will see that each finitely presented monoid M with a decidable word problem has a finite presentation that contains a cross-section for M which is a Church–Rosser language. This class of languages was defined by McNaughton *et al.* (1988) using length-reducing and confluent string-rewriting systems. It forms a proper subclass of the class of growing context-sensitive languages that is incomparable to the class of context-free languages (Buntrock and Otto, 1998). Unfortunately, as this class of languages is not closed under morphisms, it appears that the property of admitting a cross-section that is a Church–Rosser language actually depends on the chosen finite monoid-presentation.

So far, we have classified infinite string-rewriting systems solely on the form of their sets of left-hand sides. However, in order to use an infinite system to perform reductions effectively, we must also be able to determine the right-hand side of a rule from its left-hand side. If this task can be solved in polynomial time, then we call an infinite string-rewriting system *tractable*. In Section 8 we consider some classes of left-regular string-rewriting systems that are tractable. These include the *f-regular* systems of Ó'Dúnlaing (1983) and the *gsm-regular* systems of Benninghofen *et al.* (1987). The classes of tractable, left-regular, convergent string-rewriting systems considered form a strict hierarchy with respect to their ability to present finitely generated monoids with decidable word problems, but it remains open whether a corresponding result also holds when only finitely presented monoids are considered.

2. Definitions and Notation

Here we restate in short the basic definitions on string-rewriting systems and monoid-presentations that we will need throughout the paper. Our main reference for these topics is the monograph by Book and Otto (1993).

Let Σ be a finite alphabet. Then Σ^* denotes the set of all strings over Σ including the empty string λ, and $\Sigma^+ := \Sigma^* \smallsetminus \{\lambda\}$. As usual the length of a string w will be denoted as $|w|$. Further, if $w = xyz$ is a string over Σ, then x is called a *prefix*, y a *substring*, and z a *suffix* of w.

A *string-rewriting system* on Σ is a subset of $\Sigma^* \times \Sigma^*$, the elements of which are called (*rewrite*) *rules*. For a string-rewriting system R on Σ, $\mathrm{dom}(R) = \{\ell \in \Sigma^* \mid \exists r \in \Sigma^* :$

$(\ell, r) \in R\}$ and $\mathrm{range}(R) = \{r \in \Sigma^* \mid \exists \ell \in \Sigma^* : (\ell, r) \in R\}$. The system R is called *length-reducing* if $|\ell| > |r|$ holds for each rule $(\ell, r) \in R$. It is called *weight-reducing* if there exists a *weight function* $f : \Sigma \to \mathbb{N}_+$, where $\mathbb{N}_+ := \{n \in \mathbb{N} \mid n > 0\}$, such that $f(\ell) > f(r)$ holds for each rule $(\ell, r) \in R$. Here f is extended to a morphism from Σ^* to $\mathbb{N}$.

On Σ^*, R induces a *reduction relation* $\to_R^*$, which is the reflexive, transitive closure of the *single-step reduction relation* $\to_R := \{(u\ell v, urv) \mid u, v \in \Sigma^*, \text{ and } (\ell, r) \in R\}$. By $\to_R^+$ we denote the transitive closure of $\to_R$. The reflexive, symmetric, and transitive closure $\leftrightarrow_R^*$ of $\to_R$ is the *Thue congruence* generated by R. For $u, v \in \Sigma^*$, if $u \to_R^* v$, then u is an *ancestor* of v, and v is a *descendant* of u modulo R. By $\Delta_R^*(u)$ we denote the set $\Delta_R^*(u) = \{v \in \Sigma^* \mid u \to_R^* v\}$ of all descendants of u.

If there is no string v satisfying $u \to_R v$, then u is called *irreducible* modulo R, otherwise, u is *reducible* modulo R. By $\mathrm{IRR}(R)$ we denote the set of all irreducible strings, and $\mathrm{RED}(R) := \Sigma^* \smallsetminus \mathrm{IRR}(R)$ is the set of all reducible strings.

For $w \in \Sigma^*$, $[w]_R = \{u \in \Sigma^* \mid u \leftrightarrow_R^* w\}$ is the *congruence class* of w modulo R. By M_R we denote the factor monoid $\Sigma^*/\leftrightarrow_R^*$. If M is a monoid that is isomorphic to M_R, then the ordered pair $(\Sigma; R)$ is called a *monoid-presentation* for M with *generators* Σ and *defining relations* R. A monoid M is called *finitely generated* if it has a monoid-presentation $(\Sigma; R)$ with a finite set Σ of generators. It is called *finitely presented* if it has a finite monoid-presentation $(\Sigma; R)$, that is, both the set Σ of generators and the set R of defining relations are finite.

A string-rewriting system R is called *noetherian*, if there is no infinite sequence of reduction steps of the form $w_0 \to_R w_1 \to_R w_2 \to_R \cdots$. It is called *confluent*, if, for all $u, v, w \in \Sigma^*$, $u \to_R^* w$ and $u \to_R^* v$ imply that $w \to_R^* x$ and $v \to_R^* x$ hold for some $x \in \Sigma^*$. If R is both noetherian and confluent, then it is called *convergent*.

If R is a convergent string-rewriting system on Σ, then each congruence class $[w]_R$ contains a unique irreducible string $w_0 \in \mathrm{IRR}(R)$. Thus, in this situation $\mathrm{IRR}(R)$ is a *cross-section* for the monoid M_R, that is, it is a complete set of unique representatives for M_R. Obviously, the set $\mathrm{IRR}(R)$ is *substring-closed*, *s-closed* for short, that is, each substring of a string from $\mathrm{IRR}(R)$ does itself belong to $\mathrm{IRR}(R)$.

As $\mathrm{IRR}(R)$ is a cross-section for M_R, the word problem for R can be solved by the so-called *normal form algorithm*:

INPUT: $u, v \in \Sigma^*$.

begin reduce u to its irreducible descendant u_0;
reduce v to its irreducible descendant v_0;
if $u_0 = v_0$ **then** OUTPUT ("$u \leftrightarrow_R^* v$")
else OUTPUT ("$u \not\leftrightarrow_R^* v$").
end.

This algorithm terminates, because R is noetherian, and it yields the correct answer, because R is also confluent.

If R is a finite system, then the degree of complexity of this algorithm is closely related to the lengths of the reduction sequences it generates. Thus, the reduction strategy used influences the degree of complexity of the normal form algorithm. A particular strategy that is easily implemented is that of using left-most reduction steps only. Here a reduction step $w = x_1 \ell_1 y_1 \to_R x_1 r_1 y_1 = z$, $x_1, y_1 \in \Sigma^*$, $(\ell_1, r_1) \in R$, is called *left-most*, if, for all

$x_2, y_2 \in \Sigma^*$ and $\ell_2 \in \mathrm{dom}(R)$, $w = x_2\ell_2y_2$ implies that $x_1\ell_1$ is a proper prefix of $x_2\ell_2$, or $x_1\ell_1 = x_2\ell_2$ and x_1 is a prefix of x_2. A left-most reduction step is denoted as $w_L\!\!\to_R z$.

Finally, a string-rewriting system R is called *normalized* if $r \in \mathrm{IRR}(R)$ and $\ell \in \mathrm{IRR}(R \smallsetminus \{(\ell, r)\})$ hold for each rule $(\ell, r) \in R$, and two string-rewriting systems R and S on the same alphabet Σ are called *equivalent* if they both define the same Thue congruence on Σ^*.

3. Some Observations on Infinite Convergent String-rewriting Systems

A partial ordering $>$ on Σ^* is called

admissible, if $u > v$ implies $xuy > xvy$ for all strings $u, v, x, y \in \Sigma^*$;
well-founded, if there is no infinite descending sequence of strings of the form $w_0 > w_1 > w_2 \cdots$;
a *well-ordering*, if it is a well-founded linear ordering.

Let S be a string-rewriting system on Σ. It is called *compatible* with the partial ordering $>$, if $\ell > r$ holds for each rule (ℓ, r) of S. If the partial ordering $>$ is admissible, then this implies that $u > v$ whenever $u \to_S v$ holds. In particular, if $>$ is an admissible partial ordering that is well-founded, then the system S is noetherian.

The partial ordering $>$ is called *min-complete* for S, if each congruence class $[w]_S$ contains a smallest element with respect to $>$, that is, for each $w \in \Sigma^*$, there exists an element $w_{\min} \in [w]_S$ such that $u > w_{\min}$ holds for each u in $[w]_S \smallsetminus \{w_{\min}\}$. Obviously, each well-ordering is min-complete for S. By $\mathrm{MIN}(S, >)$ we denote the set of minimal elements:

$$\mathrm{MIN}(S, >) := \{w \mid w \text{ is the smallest element in } [w]_S \text{ with respect to } >\}.$$

If $>$ is an admissible partial ordering that is min-complete for S, then $\mathrm{MIN}(S, >)$ is an s-closed cross-section for S. In fact, if $uvw \in \mathrm{MIN}(S, >)$ and $v > v_1 \in [v]_S$, then $uvw > uv_1w$, because $>$ is admissible, and $uvw \leftrightarrow_S^* uv_1w$, contradicting the minimality of uvw.

Proposition 3.1. (Avenhaus, 1986) *Let M be a finitely generated monoid given by a presentation $(\Sigma; S)$. If $>$ is an admissible well-founded partial ordering on Σ^* that is min-complete for S, then*

$$R := \{w \to w_{\min} \mid w \in \Sigma^* \smallsetminus \mathrm{MIN}(S, >)\}$$

is a convergent string-rewriting system on Σ that is equivalent to S and that satisfies $\mathrm{IRR}(R) = \mathrm{MIN}(S, >)$. *Conversely, if R is a convergent string-rewriting system on Σ that is equivalent to S, then $> := \to_R^+$ is an admissible, well-founded partial ordering on Σ^* that is min-complete for S and that satisfies* $\mathrm{MIN}(S, >) = \mathrm{IRR}(R)$.

Proof. Straightforward.□

As there exist admissible well-orderings on Σ^*, we have the following immediate consequence.

Corollary 3.2. *Each finitely generated monoid can be presented by some (generally infinite) convergent string-rewriting system.*

As a finitely generated monoid can have an undecidable word problem, the convergent system provided by Corollary 3.2 will in general be non-recursive. We are, however, interested in those convergent systems for which the process of reduction is an effective one, thus leading to a solution for the word problem. Therefore, we need to restrict the class of infinite string-rewriting systems that we consider appropriately.

Let $\mathbb{C}$ be a class of formal languages. A string-rewriting system R on Σ is called *left-*$\mathbb{C}$, if $\mathrm{dom}(R)$ belongs to the class $\mathbb{C}$. In particular, we will call R *left-regular* (*left-context-free*, *left-context-sensitive*, *left-recursive*), if $\mathrm{dom}(R)$ is a regular (context-free, context-sensitive, recursive) language over Σ.

A class $\mathbb{C}$ of formal languages is called *complementary*, if it satisfies the following two conditions:

(1) $\mathbb{C}$ is closed under complement, that is, if $L \subseteq \Sigma^*$ belongs to $\mathbb{C}$, then so does $\Sigma^* \smallsetminus L$;
(2) if $L \subseteq \Sigma^*$ belongs to $\mathbb{C}$, then the ideal $\Sigma^* \cdot L \cdot \Sigma^*$ generated by L also belongs to $\mathbb{C}$.

Examples of complementary classes are the class REG of regular languages, the class CSL of context-sensitive languages, the class REC of recursive languages, and the class $\mathrm{E}_n\mathrm{L}(n \geq 1)$ of those languages whose characteristic functions belong to the n-th level E_n of the Grzegorczyk hierarchy (Grzegorczyk, 1953, see also Tourlakis (1984) and Weihrauch (1974)). On the other hand, the class CFL of context-free languages and the class DCFL of deterministic context-free languages are not complementary.

For complementary classes $\mathbb{C}$ we obtain the following correspondence between string-rewriting systems and partial orderings.

Proposition 3.3. *Let M be a finitely generated monoid given by a presentation $(\Sigma; S)$, and let $\mathbb{C}$ be a complementary class of languages. Then the following two statements are equivalent:*

(a) there exists a left-$\mathbb{C}$ convergent system R on Σ that is equivalent to S;
(b) there exists an admissible, well-founded partial ordering $>$ on Σ^* that is min-complete for S such that $\mathrm{MIN}(S, >)$ belongs to $\mathbb{C}$.

Proof. (a) $\Rightarrow$ (b) If R is a left-$\mathbb{C}$ convergent string-rewriting system equivalent to S, then $> := \to_R^+$ is an admissible, well-founded, and min-complete partial ordering such that $\mathrm{MIN}(S, >) = \mathrm{IRR}(R)$ by Proposition 3.1. As $\mathrm{dom}(R) \in \mathbb{C}$ and $\mathbb{C}$ is complementary, it follows that $\mathrm{IRR}(R) = \Sigma^* \smallsetminus \Sigma^* \cdot \mathrm{dom}(R) \cdot \Sigma^* \in \mathbb{C}$.

(b) $\Rightarrow$ (a) If $>$ is an admissible, well-founded, and min-complete partial ordering on Σ^* such that $\mathrm{MIN}(S, >) \in \mathbb{C}$, then by Proposition 3.1 $R := \{w \to w_{\min} \mid w \notin \mathrm{MIN}(S, >)\}$ is a convergent system equivalent to S such that $\mathrm{IRR}(R) = \mathrm{MIN}(S, >)$. As $\mathrm{MIN}(S, >)$ is in $\mathbb{C}$, $\mathrm{dom}(R) = \Sigma^* \smallsetminus \mathrm{MIN}(S, >)$ is also in $\mathbb{C}$.□

Corollary 3.4. *Let M be a finitely generated monoid given by a presentation $(\Sigma; S)$, and let $\mathbb{C}$ be a complementary class of languages. If (a) (or (b)) of Proposition 3.3 holds for M, then M has an s-closed cross-section that belongs to $\mathbb{C}$.*

Proof. If (a) or (b) of Proposition 3.3 holds for M, then actually both these statements hold for M, and $\mathrm{IRR}(R) = \mathrm{MIN}(S, >)$ is an s-closed cross-section belonging to $\mathbb{C}$. □

Thus, if M admits a left-regular convergent string-rewriting system, then it has a regular cross-section. Observe that such a system can still have an undecidable word problem.

Example 3.5. Let $\Sigma = \{a, b, c\}$, let A be a non-recursive subset of $\mathbb{N}$, and let $R = \{ab^n c \to a \mid n \in A\} \cup \{ab^n c \to c \mid n \in \mathbb{N} \setminus A\}$. Then, $\mathrm{dom}(R) = \{ab^n c \mid n \in \mathbb{N}\} = a \cdot b^* \cdot c$, and hence, R is left-regular. Obviously, R is noetherian, and as R has no non-trivial critical pairs at all, it is also confluent. However, $ab^n c \leftrightarrow_R^* a$ if and only if $n \in A$. Thus, R has an undecidable word problem.

Notice, however, that the monoid M presented by $(\Sigma; R)$ of Example 3.5 is not finitely presented. The situation improves drastically when we restrict our attention to those monoids that are finitely presented.

Proposition 3.6. *Let M be a monoid that is given through a finite presentation $(\Sigma; S)$. Then the following statements are equivalent:*

(a) M has a decidable word problem;
(b) M has a recursive cross-section $C \subseteq \Sigma^*$;
(c) there exists a left-recursive convergent string-rewriting system R on Σ that is equivalent to S;
(d) there exists an admissible, well-founded partial ordering $>$ on Σ^* that is min-complete for S such that $\mathrm{MIN}(S, >)$ is a recursive set.

Proof. (a) ⇒ (d) Let $>_{\ell\ell}$ be the length-lexicographical ordering on Σ^* induced by some linear ordering on Σ. Then $>_{\ell\ell}$ is an admissible well-ordering on Σ^*. The set $\mathrm{MIN}(S, >_{\ell\ell})$ is a cross-section for M, and for all $u \in \Sigma^*$, u is in $\mathrm{MIN}(S, >_{\ell\ell})$ if and only if $u \not\leftrightarrow_S^* v$ for any $v \in \Sigma^*$ satisfying $u >_{\ell\ell} v$. But, for each $u \in \Sigma^*$, $\{v \in \Sigma^* \mid u >_{\ell\ell} v\}$ is a finite set that can effectively be determined from u. Thus, as M has a decidable word problem, $\mathrm{MIN}(S, >_{\ell\ell})$ is a recursive cross-section for M.

(d) ⇒ (c) Proposition 3.3.

(c) ⇒ (b) Corollary 3.4.

(b) ⇒ (a) Let C be a recursive cross-section for M. As S is finite, the congruence class $[w]_S$ is recursively enumerable for each $w \in \Sigma^*$. In fact, there is a uniform process that, given a string $w \in \Sigma^*$ as input, will enumerate the congruence class $[w]_S$. In this way we eventually find the unique representative $w_0 \in C$ of $[w]_S$. Now, $u \leftrightarrow_S^* v$ if and only if $u_0 = v_0$, where $u_0(v_0)$ denotes the unique representative of $[u]_S([v]_S)$. Thus, the word problem for M is decidable. □

The implication (c) ⇒ (a) above improves upon Corollary 2.7.2 of Sattler-Klein (1991), where it is stated only for infinite convergent string-rewriting systems that are generated from finite systems by the Knuth–Bendix completion procedure.

Proposition 3.6 shows that each finitely presented monoid with a decidable word problem can be presented by some left-recursive convergent system. As shown by Squier (1987) not every monoid with this property can be presented by a finite convergent system. In fact, even the left-regular convergent string-rewriting systems do not suffice to present all these monoids (Kobayashi, 1995).

What about the classes of left-context-free, or left-context-sensitive, convergent systems? Does one of these classes suffice to present all finitely presented monoids with decidable word problems? Observe that each finitely presented monoid $M = (\Sigma; S)$ with a decidable word problem has a presentation $(\Sigma; R)$ such that R is equivalent to S, and the set $R_\# := \{u\#v \mid (u, v) \in R\}$ is a context-sensitive language (Madlener and Otto, 1988). Here # is an extra letter just used for encoding purposes.

Another obvious question raised by Corollary 3.4 is the following: which s-closed cross-sections can actually occur as the sets of irreducible strings with respect to some convergent string-rewriting system? The following example shows that not all s-closed cross-sections can be obtained in this way.

Example 3.7. Let $\Sigma = \{a, b, c, d, e\}$ and let $S = \{ac \to bc, db \to da, dbc \to e, dac \to e\}$. Then S is not noetherian, as $dbc \to_S dac \to_S dbc$. On the other hand, S is strongly confluent, because

$$\begin{array}{ccc} dac & \to & e \\ \downarrow & \nearrow & \\ dbc & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} dbc & \to & e \\ \downarrow & \nearrow & \\ dac & & \end{array}$$

and these are the only critical pairs. Consider $C := \mathrm{IRR}(S) = \Sigma^* \smallsetminus \Sigma^* \cdot \{ac, db\} \cdot \Sigma^*$. Then C is an s-closed subset of Σ^*, and it can be shown that C is actually a cross-section for the monoid M presented by $(\Sigma; S)$.

Assume now that there exists a convergent string-rewriting system R on Σ equivalent to S such that $C = \mathrm{IRR}(R)$. Then by Proposition 3.1, $> := \to_R^+$ is an admissible, well-founded partial ordering that is min-complete for S, and $C = \mathrm{MIN}(S, >)$. Hence, as $[ac]_S = \{ac, bc\}$ and $[da]_S = \{da, db\}$, we have $ac > bc$ and $db > da$. As $>$ is admissible, this yields $dac > dbc > dac$, a contradiction. Thus, C cannot be the set of irreducible strings for any convergent system equivalent to S.

However, the following weaker result holds. Here a string-rewriting system is called *left-most terminating* if each left-most reduction sequence is finite.

Proposition 3.8. *Let M be a finitely generated monoid given by a presentation $(\Sigma; S)$, and let $\mathbb{C}$ be a complementary class of languages. For a language C over Σ, the following statements are equivalent:*

(a) there exists a left-$\mathbb{C}$, confluent, and left-most terminating string-rewriting system R on Σ equivalent to S such that $\mathrm{IRR}(R) = C$;
(b) C belongs to $\mathbb{C}$, and C is an s-closed cross-section for M.

Proof. (a) ⇒ (b) If R is a left-$\mathbb{C}$, confluent, and left-most terminating string-rewriting system on Σ that is equivalent to S, then $\mathrm{IRR}(R)$ is in $\mathbb{C}$, and it is an s-closed cross-section for M.

(b) ⇒ (a) Consider the system

$$R := \{w \to w_0 \mid w \in \Sigma^* \smallsetminus C, w_0 \in C \text{ is the representative of } [w]_S\}.$$

Then $\mathrm{dom}(R) = \Sigma^* \smallsetminus C$, and hence, R is a left-$\mathbb{C}$ system. As C is s-closed, $\Sigma^* \cdot \mathrm{dom}(R) \cdot \Sigma^* = \mathrm{dom}(R)$, and hence, $\mathrm{IRR}(R) = \Sigma^* \smallsetminus \mathrm{dom}(R) = C$. It is easily seen that R is confluent and equivalent to S. Finally, if $u_L\!\rightarrow_R v$ is a left-most reduction step, then $u = wu_1$ and $v = w_0u_1$, where w is the shortest prefix of u that does not belong to C. Let $\alpha(x) := |x| - \ell_x$ for $x \in \Sigma^*$, where ℓ_x is the length of the shortest prefix of x that is not in C. Then $u_L\!\rightarrow_R v$ implies that either $v \in C$ or $\alpha(u) > \alpha(v)$. Therefore, R is left-most terminating. □

Observe that the system S of Example 3.7 is actually left-most terminating. Moreover, the system S is equivalent to the system $S_0 := \{bc \to ac, db \to da, dbc \to e, dac \to e\}$, which is finite and convergent. This observation raises the following question: if a finitely presented monoid M has an s-closed cross-section that belongs to a complementary class $\mathbb{C}$, does then M have necessarily a presentation through some left-$\mathbb{C}$ convergent string-rewriting system? Observe that Proposition 3.6 answers this question in the affirmative for the class REC of recursive languages.

An s-closed set $C \subseteq \Sigma^*$ is called *finitely s-closed*, if $C = \Sigma^* \smallsetminus \Sigma^* \cdot T \cdot \Sigma^*$ for some finite set T. Related to the question above we also pose the following question: can each finitely presented monoid that has a finitely s-closed cross-section be presented through a finite convergent string-rewriting system?

4. Regular Cross-sections

If a finitely generated monoid M is presented by a left-regular convergent string-rewriting system R on some finite alphabet Σ, then $\mathrm{IRR}(R) \subseteq \Sigma^*$ is a regular cross-section for M. In this section we develop a criterion that will allow us to prove that certain monoids do not have regular cross-sections. For that we will need the following variant of the pumping lemma for regular languages.

Lemma 4.1. (Berstel, 1979) *Let $C \subseteq \Sigma^*$ be a regular language. Then there exists an integer $k(C) \geq 1$ such that, for each string $w \in C$ and each factorization $w = uyv$ with $|y| \geq k(C)$, y admits a factorization $y = y_1zy_2$ such that $1 \leq |z| \leq k(C)$ and $uy_1z^ny_2v \in C$ for all $n \in \mathbb{N}$.*

For example, for $k(C)$ we can simply take the number of states of the minimal deterministic finite-state acceptor for C.

Let M be a monoid. An element $m \in M$ is said to have *infinite order* if $m^k \neq_M m^\ell$ holds for all $k, \ell \in \mathbb{N}, k \neq \ell$. If there exist integers $n \geq 0$ and $k \geq 1$ such that $m^{n+k} =_M m^n$, then m has *finite order* in M.

For the special case of groups, the following result has been established by Gilman (1987).

Lemma 4.2. *Let M be a finitely generated monoid given by a presentation $(\Sigma; S)$, and let $C \subseteq \Sigma^*$ be a regular cross-section for M. Then the monoid M is infinite if and only if it contains an element of infinite order.*

Proof. Certainly, if M contains an element of infinite order, then M is infinite. Conversely, assume that M is infinite. Then also the regular set C is infinite, and so there is a string $w \in C$ such that $|w| > k(C)$. Hence, by the pumping lemma for regular languages, w can be factored as $w = xyz$ such that $y \neq \lambda$ and $xy^nz \in C$ for all $n \in \mathbb{N}$. As C is a cross-section for M, this means that $[xy^nz]_S \neq_M [xy^mz]_S$ for all $n \neq m$. Hence, also $[y^n]_S \neq_M [y^m]_S$ for all $n \neq m$, that is, y presents an element of infinite order of M. □

Thus, the infinite Burnside monoids $B(m, n)$ and the infinite Burnside groups $G(m)$ have no regular cross-sections (Adian, 1979). Observe that these monoids and groups are only finitely generated, but not finitely presented. In order to obtain finitely presented monoids without regular cross-sections, we need the following more involved application of the pumping lemma.

Let M be a monoid, and let $\alpha \in M$. An element $\beta \in M$ is called a *factor* of α if $\alpha =_M \gamma\beta\delta$ for some $\gamma, \delta \in M$. We say that the factor β is *torsional* in α, if there are positive integers $m, n, m \neq n$, such that $\gamma\beta^m\delta =_M \gamma\beta^n\delta$.

LEMMA 4.3. *Let M be a finitely generated monoid given by a presentation $(\Sigma; S)$, let $C \subseteq \Sigma^*$ be a regular cross-section for M, and let k denote the corresponding constant $k(C)$ of Lemma 4.1. Then, for each string $x \in C$ and each substring y of x satisfying $|y| \geq k$, there exists a substring z of y such that $[z]_S$ is not a torsional factor of $[x]_S$ in M.*

PROOF. Let $x \in C$, and let $x = uyv$ for some $y \in \Sigma^*$ satisfying $|y| \geq k$. By the pumping lemma y can be written as $y = y_1zy_2$ such that $z \neq \lambda$ and $uy_1z^ny_2v \in C$ for all $n \in \mathbb{N}$. As $z \neq \lambda$, $uy_1z^my_2v \neq uy_1z^ny_2v$ for all $n \neq m$. Thus, as C is a cross-section for M, $[uy_1z^my_2v]_S \neq_M [uy_1z^ny_2v]_S$ for all $n \neq m$. Hence, $[z]_S$ is not a torsional factor of $[x]_S$. □

Based on this lemma we can now formulate the following technical result which states a condition that guarantees that a monoid does not have a regular cross-section.

PROPOSITION 4.4. *Let M be a finitely generated monoid given by a presentation $(\Sigma; S)$. If M contains a subset A that satisfies the following condition $(*)$, then there is no regular cross-section $C \subseteq \Sigma^*$ for M:*

$(*)$ *For each $n \in \mathbb{N}$, there is an element $\alpha_n \in A$ such that each $x \in \Sigma^*$ satisfying $[x]_S = \alpha_n$ contains a substring y of length $|y| \geq n$ such that, for each substring z of y, $[z]_S$ is a torsional factor of $[x]_S$ in M.*

PROOF. Assume that $C \subseteq \Sigma^*$ is a regular cross-section for M, and let $n := k(C)$. From $(*)$ we see that there is an element $\alpha_n \in A$ satisfying the following condition: the representative $x \in C$ of α_n contains a substring y of length $|y| \geq n$ such that $[z]_S$ is a torsional factor of $[x]_S = \alpha_n$ for each substring z of y. However, by Lemma 4.3 y must contain a substring z such that $[z]_S$ is not a torsional factor of $[x]_S$ in M. This contradiction shows that M has no regular cross-section $C \subseteq \Sigma^*$. □

Let $(\Sigma_1; S_1)$ and $(\Sigma_2; S_2)$ be two finitely generated presentations of the same monoid M. Then there exists a morphism $f : \Sigma_1^* \to \Sigma_2^*$ such that, for every string $w \in \Sigma_1^*, [w]_{S_1} =_M [f(w)]_{S_2}$, that is, w and $f(w)$ both represent the same element of M. Thus, if $C \subseteq \Sigma_1^*$ is a cross-section for M, then so is the set $f(C) \subseteq \Sigma_2^*$, and with C also the set $f(C)$ is regular. Hence, if a finitely generated monoid M satisfies the hypothesis of Proposition 4.4, then no finitely generated presentation of M contains a regular cross-section for M. In addition, we have the following consequence.

Corollary 4.5. *If a finitely generated monoid satisfies the hypothesis of Proposition 4.4, then it cannot be presented by any left-regular convergent string-rewriting system.*

We now present some examples of finitely presented monoids that do satisfy the hypothesis of Proposition 4.4.

Example 4.6. Let $\Sigma = \{a, b, c, d\}$, let $t \in \mathbb{N}$, and let $S(t)$ be the following string-rewriting system on Σ:

$$S(t) := \{ba \to ab, bc \to aca, a^t cc \to d\} \cup \{de \to d, ed \to d \mid e \in \Sigma\}.$$

Using the syllable ordering $>_{\mathrm{syl}}$ induced by $b > c > a > d$ with status $\tau(b) =$ 'right' (Sattler-Klein, 1991), we see that $S(t)$ is noetherian. However, it is not confluent, and indeed, completion yields the following left-context-free convergent system:

$$R(t) = S(t) \cup \{a^{n+t} ca^n c \to d \mid n \geq 1\}.$$

It follows that the word problem for the monoid M_t presented by $(\Sigma; S(t))$ is decidable in polynomial time.

We claim that M_t has no regular cross-section. To prove this, consider the set $A := \{\alpha_n \mid n \geq 0\}$, where $\alpha_n := [a^n ca^{n+1} ca^{n+2} c]_{S(t)}$. The congruence class α_n consists exactly of those strings $w \in \Sigma^*$ that satisfy the following condition:

$$w = xcyca^i c \text{ for some } i \geq 0, x, y \in \{a, b\}^*, |x| = n, |x|_b + |y| = n+1, \text{ and } |y|_b + i = n+2.$$

For $n \geq 1$, let w be an arbitrary string in Σ^* satisfying $[w]_{S(t)} = \alpha_n$. Then $w = xcyca^i c$ according to the above observation. We consider the substring x of w, which satisfies $|x| = n$. Let $x = x_1 z x_2$ for a non-empty string z, and let $k \geq t+1$. As $x \in \{a, b\}^*$, we have

$$x_1 z^{k+1} x_2 cyca^i c \leftrightarrow^*_{S(t)} z^k x_1 z x_2 cyca^i c \leftrightarrow^*_{S(t)} z^k a^n ca^{n+1} ca^{n+2} c.$$

If $|z|_a \geq 1$, then $z \leftrightarrow^*_{S(t)} z_1 a$ for some $z_1 \in \{a, b\}^*$, and hence,

$$z^k a^n ca^{n+1} ca^{n+2} c \leftrightarrow^*_{S(t)} z_1^k a^{n+k} ca^{n+1} ca^{n+2} c = z_1^k a^{k-(t+1)} a^{n+1+t} ca^{n+1} ca^{n+2} c \to^*_{R(t)} d.$$

If $|z|_a = 0$, then $z = b^r$ for some $r \geq 1$. Hence,

$$z^k a^n ca^{n+1} ca^{n+2} c \leftrightarrow^*_{S(t)} a^n b^{r \cdot k} ca^{n+1} ca^{n+2} c \leftrightarrow^*_{S(t)} a^{n+r \cdot k} ca^{n+1+r \cdot k} ca^{n+2} c.$$

As $r \cdot k \geq k \geq t+1$, we see that $n + 1 + r \cdot k \geq n + 2 + t$, and hence,

$$a^{n+r \cdot k} ca^{n+1+r \cdot k} ca^{n+2} c \to^*_{R(t)} d.$$

Thus, in any case $[z]_{S(t)}$ is a torsional factor of $[w]_{S(t)}$. Hence, by Proposition 4.4 the monoid M_t has no regular cross-section.

For $t = 1$ the monoid M_t is the example considered by Kobayashi (1995). He shows that the monoid M_1 has intermediate growth, and that it cannot be presented by any left-regular convergent string-rewriting system. We will consider the growth of monoids shortly in the next section. Before we come to that, we present another example.

EXAMPLE 4.7. Let t be a positive integer, let $\Sigma_t := \{a_1, \ldots, a_t, b, c_1, \ldots, c_t, d\}$, let $X_t := \{a_i^2, a_i c_i, c_j a_{j+1}, a_j b, b a_j, c_j b, b c_j, b^2 \mid i = 1, \ldots, t,\ j = 1, \ldots, t-1\}$, let $S(t)$ be the following string-rewriting system on Σ_t,

$$\begin{array}{rcl} S(t) & := & \{x \to d \mid x \in \Sigma_t^2 \smallsetminus X_t\} \cup \{b a_i \to a_i b, b c_i \to a_i c_i a_{i+1} \mid i = 1, \ldots, t-1\} \\ & \cup & \{de \to d, ed \to d \mid e \in \Sigma_t\}, \end{array}$$

and let N_t be the monoid presented by $(\Sigma_t; S(t))$. Using an appropriate ordering it is easily seen that $S(t)$ is noetherian. It is, however, not confluent, and completion yields the following left-context-free convergent system:

$$R(t) := S(t) \cup \{a_i^n c_i a_{i+1}^n c_{i+1} \to d \mid i = 1, \ldots, t-1, n \geq 1\}.$$

It follows that the word problem for the monoid N_t is decidable in polynomial time.

We claim that N_t has no regular cross-section for $t \geq 3$. Consider the set $A := \{\alpha_n \mid n \geq 0\}$, where $\alpha_n := [a_1^n c_1 a_2^{n+1} c_2 a_3^{n+2} c_3]_{S(t)}$. If $w \in \alpha_n$, then $w = x c_1 y c_2 a_3^i c_3$ for some $i \geq 0$, $x \in \{a_1, b\}^*$, $y \in \{a_2, b\}^*$, $|x| = n, |x|_b + |y| = n+1$, and $|y|_b + i = n+2$. Now it follows as in Example 4.6 that the set A satisfies the hypothesis of Proposition 4.4. Thus, the monoid $N_t (t \geq 3)$ has no finitely generated presentation containing a regular cross-section for N_t.

Notice, however, that the system $S(1)$ is convergent, that is, N_1 has a finitely s-closed cross-section, and that the system $S(2)$ is equivalent to the following left-regular convergent string-rewriting system

$$\begin{array}{rcl} S_\infty(2) & := & \{x \to d \mid x \in \Sigma_2^2 \smallsetminus X_2\} \cup \{de \to d, ed \to d \mid e \in \Sigma_2\} \\ & \cup & \{a_1 b \to b a_1, a_1 c_1 a_2 \to b c_1\} \\ & \cup & \{c_1 b^n a_1 \to d, c_1 b^n c_1 \to d, b a_1^n c_1 b \to d \mid n \geq 0\}. \end{array}$$

Thus, N_2 has a regular, s-closed cross-section.

The monoids $M_t (t \geq 0)$ of Example 4.6 and the monoids $N_t (t \geq 3)$ of Example 4.7 cannot be presented by any left-regular convergent string-rewriting systems. Also, if $(\Sigma; S)$ is a finitely generated presentation for one of these monoids, and $>$ is an admissible, well-founded partial ordering on Σ^* that is min-complete for S, then the set $\mathrm{MIN}(S, >)$ is not a regular language (cf. Proposition 3.3).

Finally, from the proof of Proposition 4.4 we immediately obtain the following consequence.

COROLLARY 4.8. *Let M_1 be a finitely generated monoid that satisfies the hypothesis of Proposition 4.4, and let M_2 be a monoid that is given through a finitely generated presentation $(\Sigma_2; S_2)$ satisfying $[\lambda]_{S_2} = \{\lambda\}$. Then the free product $M_1 * M_2$ has no finitely generated presentation that contains a regular cross-section for $M_1 * M_2$.*

PROOF. Let $(\Sigma_1; S_1)$ be a finitely generated presentation for M_1. Then $M_1 * M_2$ is the monoid presented through the presentation $(\Sigma_1 \cup \Sigma_2; S_1 \cup S_2)$, where we assume without loss of generality that $\Sigma_1 \cap \Sigma_2 = \emptyset$. Let A be the subset of M_1 according to the hypothesis of Proposition 4.4. Since $[\lambda]_{S_2} = \{\lambda\}$, we have $[w]_{S_1 \cup S_2} = [w]_{S_1}$ for each string $w \in \Sigma_1^*$, that is, no string containing any occurrences of letters from Σ_2 defines an element of the monoid M_1. Thus, property $(*)$ still holds for A, even when this set is considered as a subset of $M_1 * M_2$. Thus, by Proposition 4.4 $M_1 * M_2$ has no regular cross-section $C \subseteq (\Sigma_1 \cup \Sigma_2)^*$. $\square$

Hence, the free products $M_t * \Gamma^*(t \geq 0)$ and $N_t * \Gamma^*(t \geq 3)$ of the monoids M_t (Example 4.6) and the monoids N_t (Example 4.7) with the free monoid Γ^* have no regular cross-sections, either. The same arguments apply to the operation of direct product, too. Thus, also the direct products $M_t \times \Gamma^*(t \geq 0)$ and $N_t \times \Gamma^*(t \geq 3)$ have no regular cross-sections.

5. Regular Cross-sections and the Growth of Monoids

Kobayashi (1995) shows that the monoid M_1 presented by the string-rewriting system $S(1)$ of Example 4.6 has intermediate growth. He then asks whether there exists a finitely presented monoid with polynomial growth that cannot be presented by any left-regular convergent string-rewriting system, either. Here we will answer this question in the affirmative.

Let M be a finitely generated monoid, and let $(\Sigma; S)$ be a finitely generated presentation of M. The *growth function* $g_M : \mathbb{N} \to \mathbb{N}$ of M with respect to Σ is the following function:

$$g_M(n) := |\{[w]_S \mid w \in \Sigma^* \text{and} |w| \leq n\}|.$$

For two functions $f, g : \mathbb{N} \to \mathbb{N}$, we write $f \preceq g$ if there exists a constant $c \in \mathbb{R}_+$ such that $f(n) \leq g(c \cdot n)$ holds for all sufficiently large $n \in \mathbb{N}$. The functions f and g are called *equivalent* if $f \preceq g$ and $g \preceq f$ both hold (see, e.g., Borho and Kraft (1976)). By $[f]$ we denote the equivalence class of f with respect to this relation.

The class $[g_M]$ of the growth function g_M does not depend on the actually chosen set of generators Σ, and hence, it is an invariant of the monoid M. It is called the *growth* of M.

If $[g_M(n)] = [n^d]$ for some $d \geq 0$, then M is said to have *polynomial growth* of *degree d*. If $[g_M(n)] = [2^n]$, then M is said to have *exponential growth*. If M has neither polynomial nor exponential growth, then it has *intermediate growth*.

For a subset $L \subseteq \Sigma^*$, the *growth function* g_L of L is defined by $g_L(n) := |\{w \in L \mid |w| \leq n\}|$. We will be interested in the growth functions g_C of cross-sections C of monoids M and their relationship to the growth functions of the monoids M themselves. The first observation is rather straightforward.

PROPOSITION 5.1. *Let M be a finitely generated monoid. If M has a cross-section C that grows exponentially, then M grows exponentially.*

However, a corresponding result does not hold in general for polynomial growth as shown by the following example.

EXAMPLE 5.2. Let $\Sigma = \{a, b\}$ and $S = \{ab \to ba^2\}$. Then S is a finite convergent string-rewriting system on Σ, and hence, $\mathrm{IRR}(S) = b^* \cdot a^*$ is a regular, s-closed cross-section for the monoid M presented by $(\Sigma; S)$. Since $g_{\mathrm{IRR}(S)}(n) = \sum_{i=0}^{n}(i+1) = \frac{(n+1)\cdot(n+2)}{2}$, this cross-section has polynomial growth of degree 2.

The system $R := \{ba^2 \to ab\}$ is equivalent to S, and it is also convergent. Thus, $\mathrm{IRR}(R) = a^* \cdot (b^+ \cdot a)^* \cdot b^*$ is another regular, s-closed cross-section for M. However, as $\mathrm{IRR}(R) \supseteq \{b, ba\}^*$, we see that this cross-section has exponential growth, and hence, so does the monoid M.

Let M be a finitely generated monoid, and let $(\Sigma; S)$ be a finitely generated presentation of M. By $\mathrm{MIN}(S)$ we denote the set $\mathrm{MIN}(S) := \mathrm{MIN}(S, >_{\ell\ell})$, where $>_{\ell\ell}$ is a length-lexicographical ordering on Σ^*. Then $\mathrm{MIN}(S)$ is an s-closed cross-section for M, and $g_{\mathrm{MIN}(S)}$ coincides with the growth function g_M (with respect to the set of generators Σ).

A cross-section $C \subseteq \Sigma^*$ of M is called *polynomially mild*, if there exists a polynomial p such that $|w| \leq p(|\min_S(w)|)$ holds for all $w \in C$. Here $\min_S(w)$ denotes the minimal element in $[w]_S$ with respect to the well-ordering $>_{\ell\ell}$. If the polynomial p is of degree 1, then we say that the cross-section C is *linearly mild*.

For polynomially mild cross-sections we obtain the following result, which can be seen as a partial converse of Proposition 5.1.

Proposition 5.3. *Let M be a finitely generated monoid. If M has a polynomially mild cross-section that grows polynomially, then M grows polynomially.*

Proof. Let $(\Sigma; S)$ be a finitely generated presentation for M, and let $C \subseteq \Sigma^*$ be a polynomially mild cross-section for M. Thus, there is a polynomial p such that $|w| \leq p(|\min_S(w)|)$ holds for all $w \in C$. Hence, the mapping which sends each $x \in \Sigma^*$ to its representative $x_0 \in C$ induces an injective mapping from $\{w \in \mathrm{MIN}(S) \mid |w| \leq n\}$ into the set $\{w \in C \mid |w| \leq p(n)\}$ for all $n \geq 0$. Thus, for all $n \geq 0$, $g_M(n) = g_{\mathrm{MIN}(S)}(n) \leq g_C(p(n))$. Therefore, if g_C is of polynomial growth, then so is g_M. □

Actually, we can strengthen Proposition 5.3 as follows.

Corollary 5.4. *Let M be a finitely generated monoid, let $(\Sigma; S)$ be a finitely generated presentation of M, and let $C \subseteq \Sigma^*$ be a linearly mild cross-section for M. Then $[g_C] = [g_M]$.*

Proof. As C is a cross-section, we have $g_C(n) \leq g_M(n)$ for all $n \in \mathbb{N}$. On the other hand, if C is linearly mild, then there exist positive constants α and β such that $|w| \leq \alpha \cdot |\min_S(w)| + \beta$ holds for all $w \in C$. Now the proof of Proposition 5.3 shows that $g_M(n) \leq g_C(\alpha \cdot n + \beta)$ holds for all $n \in \mathbb{N}$. Hence, $[g_C] = [g_M]$, that is, the growth of C coincides with the growth of M.□

A string-rewriting system R on Σ is called *polynomially mild* if there exists a polynomial p such that $|v| \leq p(|u|)$ holds for all $u \in \Sigma^*$ and all descendants $v \in \Delta_R^*(u)$. If p is a polynomial of degree 1, then the system R is called *linearly mild*.

Proposition 5.5. *Let M be a finitely generated monoid, and let $(\Sigma; R)$ be a finitely generated presentation of M such that the string-rewriting system R is convergent. If R is polynomially mild, then* $\mathrm{IRR}(R)$ *is a polynomially mild, s-closed cross-section for M.*

Proof. As R is convergent, $\mathrm{IRR}(R)$ is an s-closed cross-section for M. Assume that there exists a polynomial p such that $|v| \leq p(|u|)$ holds for all $u \in \Sigma^*$ and all $v \in \Delta_R^*(u)$, and let $w \in \mathrm{IRR}(R)$. Then $w_0 := \min_R(w)$ satisfies $w_0 \to_R^* w$, and hence, $|w| \leq p(|w_0|)$. Thus, $\mathrm{IRR}(R)$ is indeed a polynomially mild cross-section for M. □

From the proof of Proposition 5.5 we see that $\mathrm{IRR}(R)$ is a linearly mild, s-closed cross-section for M, if the system R is convergent and linearly mild.

If the string-rewriting system R is length-reducing or weight-reducing, then it is linearly mild. Actually, we can generalize this observation.

Let R be a noetherian string-rewriting system on Σ. For $w \in \Sigma^*$, $d_R(w)$ denotes the length of the longest reduction sequence starting with w. The function $D_R : \mathbb{N} \to \mathbb{N}$, which is defined by $D_R(n) := \max\{d_R(w) \mid w \in \Sigma^* \text{ and } |w| \leq n\}$, is then called the *derivational complexity* of R.

Proposition 5.6. *Let R be a finite noetherian string-rewriting system on Σ.*

(a) If the derivational complexity D_R of R is bounded from above by some polynomial, then the system R is polynomially mild.
(b) If the derivational complexity D_R of R is bounded from above by some linear function, then the system R is linearly mild.

Proof. (a) Assume that $d_R(w) \leq p(|w|)$ holds for all $w \in \Sigma^*$, where p is a polynomial. If $w \to_R v$, then $w = x\ell y$ and $v = xry$ for some strings $x, y \in \Sigma^*$ and a rule $(\ell, r) \in R$. Let $\alpha := \max(\{|r| - |\ell| \mid (\ell, r) \in R\} \cup \{0\})$. Then $|v| \leq |w| + \alpha$. It follows inductively that $w \to_R v_1 \to_R v_2 \to_R \cdots \to_R v_m$ implies that $|v_m| \leq |w| + m \cdot \alpha$. Thus, for all $v \in \Delta_R^*(w)$, we have $|v| \leq |w| + d_R(w) \cdot \alpha \leq |w| + \alpha \cdot p(|w|)$. Hence, R is indeed a polynomially mild system.

(b) If p is a linear function, that is, $p(n) = \beta \cdot n + \gamma$ for some constants β and γ, then the proof above shows that $|v| \leq |w| + \alpha \cdot \beta \cdot |w| + \alpha \cdot \gamma = (\alpha \cdot \beta + 1) \cdot |w| + \alpha \cdot \gamma$ holds for all $w \in \Sigma^*$ and all $v \in \Delta_R^*(w)$. Thus, in this case R is a linearly mild system. □

Observe that the length-reducing and weight-reducing systems have linearly bounded derivational complexity. However, the converse of Proposition 5.6 is not true in general.

Example 5.7. Let $\Sigma = \{0, 1, 0', 1', \$, ¢, \#\}$, and let R be the following finite, length-preserving string-rewriting system on Σ:

$$R := \{\$1' \to 1\$, \$0' \to 0\$, \$¢ \to \#¢, 1\# \to \#0', 0\# \to 1\$\}.$$

As $|\ell| = |r|$ for all rules $(\ell, r) \in R$, the system R is obviously linearly mild.

Let $>_{\ell\ell}$ be the lexicographical ordering on Σ^* that is induced by the linear ordering $\$ > 0 > 1 > \# > 0' > 1' > ¢$. Then $\ell >_{\ell\ell} r$ holds for each rule $(\ell, r) \in R$, and hence, the system R is noetherian. In fact, as R has no critical pairs, it is also confluent. Thus, IRR(R) is a linearly mild, s-closed cross-section for the monoid M presented by $(\Sigma; R)$.

But now let us look at the derivational complexity D_R of R.
Claim: $d_R(\$0'^n¢) = 2^{n+2} - 3$ for all $n \geq 0$.

Proof. As R is noetherian, and as R has no critical pairs at all, all reduction sequences from $\$0'^n¢$ to its normal form $\#0'^n¢$ have exactly the same length. By induction on n we now prove that the string $\$0'^n¢$ is reduced to $\#0'^n¢$ in $2^{n+2} - 3$ steps. If $n = 0$, then $\$¢ \to \#¢$, that is, $d_R(\$¢) = 1 = 2^2 - 3$. For all $n \geq 0$,

$$\$0'0'^n¢ \to 0\$0'^n¢ \to^{2^{n+2}-3} 0\#0'^n¢ \to 1\$0'^n¢ \to^{2^{n+2}-3} 1\#0'^n¢ \to \#0'0'^n¢,$$

that is, $d_R(\$0'^{n+1}¢) = 2 \cdot (2^{n+2} - 3) + 3 = 2^{n+3} - 3$.

Thus, $D_R(n+2) \geq 2^{n+2} - 3$ for all $n \geq 0$, that is, D_R is not bounded from above by any polynomial. □

As shown by Kobayashi (1995), $[g_{\mathrm{IRR}(R(1))}(n)] = [2^{\sqrt{n}}]$ for the left-context-free, convergent string-rewriting system $R(1)$ of Example 4.6. Similarly, it can be shown that $[g_{\mathrm{IRR}(R(t))}(n)] = [2^{\sqrt{n}}]$ for all $t \geq 0$. For all $u, v \in \Sigma^*$, if $u \to_{R(t)}^* v$, then $|v| \leq |u| + |u|_b \leq 2 \cdot |u|$, because occurrences of the symbol b are not generated by applications of rules of $R(t)$, and $(bc \to aca)$ is the only length-increasing rule. Thus, $R(t)$ is a linearly mild system, and so $\mathrm{IRR}(R(t))$ is a linearly mild, s-closed cross-section for the monoid M_t presented by $(\Sigma; S(t))$. From Corollary 5.4 we see that $[g_{M_t}(n)] = [g_{\mathrm{IRR}(R_t)}(n)] = [2^{\sqrt{n}}]$, that is, the monoid M_t has intermediate growth. From Example 4.6 we know that M_t has no regular cross-section.

Now consider the monoid $M_1 * (f, g; \emptyset)$, that is, the free product of the monoid M_1 and the free monoid F_2 generated by the set $\{f, g\}$. By Corollary 4.8 this monoid does not have a regular cross-section, either. On the other hand, its word problem is decidable in polynomial time, as M_1 and F_2 have word problems decidable in polynomial time. Further, it is finitely presented, and it has exponential growth.

Finally, let us return to the example monoids $N_t (t \geq 3)$ of Example 4.7. As shown there N_t is a finitely presented monoid with a word problem that is decidable in polynomial time, but N_t has no regular cross-section. On the other hand, $|v| \leq |u| + |u|_b \leq 2 \cdot |u|$ for all $u, v \in \Sigma_t^*$ satisfying $u \to_{R(t)}^* v$, which is shown in exactly the same way as above. Thus, $R(t)$ is a linearly mild system, and so $\mathrm{IRR}(R(t))$ is a linearly mild, s-closed cross-section for the monoid N_t. However,

$$\begin{array}{ll}
\multicolumn{2}{l}{\mathrm{IRR}(R(t))} \\
= & \{a_i^{k_i} c_i a_{i+1}^{k_{i+1}} \ldots c_{j-1} a_j^{k_j} b^k \mid 1 \leq i \leq j \leq t-1, 0 \leq k_i < \cdots < k_{j-1}, 0 \leq k_j, 0 \leq k\} \\
\cup & \{a_i^{k_i} c_i a_{i+1}^{k_{i+1}} \ldots a_{t-1}^{k_{t-1}} c_{t-1} a_t^{k_t} \mid 1 \leq i \leq t, 0 \leq k_i < k_{i+1} < \cdots < k_{t-1}, 0 \leq k_t\} \\
\cup & \{a_i^{k_i} c_i a_{i+1}^{k_{i+1}} \ldots a_{t-1}^{k_{t-1}} c_{t-1} b^k \mid 1 \leq i \leq t, 0 \leq k_i < k_{i+1} < \cdots < k_{t-1}, 0 \leq k\} \\
\cup & \{a_i^{k_i} c_i a_{i+1}^{k_{i+1}} \ldots a_{t-1}^{k_{t-1}} c_{t-1} a_t^{k_t} c_t \mid 1 \leq i \leq t, 0 \leq k_i < k_{i+1} < \cdots < k_t\} \\
\cup & \{d\}.
\end{array}$$

Hence, $[g_{\mathrm{IRR}(R(t))}(n)] = [n^t]$, and $\mathrm{IRR}(R(t))$ has polynomial growth of degree t. By Corollary 5.4 the monoid N_t has polynomial growth of degree t.

Thus, for each of the three possible cases, polynomial growth, intermediate growth, and exponential growth, we have found a finitely presented monoid with that rate of growth such that this monoid has no regular cross-section, although its word problem is decidable in polynomial time. This answers the question raised as Problem 3 in Kobayashi (1995).

6. Context-free Cross-sections

The class of context-free languages is not a complementary class, as it is not closed under complement. Thus, Proposition 3.3 does not apply to this class of languages. Nevertheless, the question remains whether or not a corresponding result can be established for the class of context-free languages. Further, we have seen in Section 4 that there exist finitely presented monoids with easily decidable word problems that have no regular cross-sections. Do all finitely presented monoids with easily decidable word problems have at least context-free cross-sections? In this section we will answer both these questions. We begin with an easy extension of Lemma 4.2.

Lemma 6.1. *Let M be a finitely generated monoid given by a presentation $(\Sigma; S)$, and let $C \subseteq \Sigma^*$ be a context-free cross-section for M. Then the monoid M is infinite if and only if it contains an element of infinite order.*

Proof. If M contains an element of infinite order, then M is infinite. So assume conversely that M is infinite. Then the context-free cross-section $C \subseteq \Sigma^*$ is an infinite language. Let $k(C)$ denote the constant that can be associated with C according to the pumping lemma for context-free languages, and let $w \in C$ such that $|w| > k(C)$. Then w can be factored as $w = uvxyz$ such that $1 \leq |vy| \leq k(C)$ and $uv^nxy^nz \in C$ for all $n \in \mathbb{N}$. As C is a cross-section for M, this means that $[uv^nxy^nz]_S \neq_M [uv^mxy^mz]_S$ for all $n \neq m$. Now assume that both v and y represent elements of finite order of M, that is, there are integers $n_1, n_2 \geq 0$ and $k_1, k_2 \geq 1$ such that $[v^{n_1+k_1}]_S =_M [v^{n_1}]_S$ and $[y^{n_2+k_2}]_S =_M [y^{n_2}]_S$. But then $[uv^{n_1+n_2}xy^{n_1+n_2}z]_S =_M [uv^{n_1+n_2+k_1\cdot k_2}xy^{n_1+n_2+k_1\cdot k_2}z]_S$, contradicting the above statement. Thus, v or y represents an element of infinite order of M. □

Thus, the infinite Burnside monoids $B(m,n)$ and the infinite Burnside groups $G(m)$ do not even have context-free cross-sections. In order to state a more general technical result we need the following generalization of the pumping lemma for context-free languages.

Proposition 6.2. (Ogden's Lemma (Berstel, 1979)) *Let $L \subseteq \Sigma^*$ be a context-free language. Then there exists a positive integer $k := k(L) \geq 2$ such that each string $w \in L$ containing at least k marked positions can be factored as $w = uvxyz$ such that the following conditions are satisfied:*

(1) u, and v, and x or x, and y, and z contain each at least one marked position,
(2) vy contains at most k marked positions, and
(3) $uv^nxy^nz \in L$ for each $n \geq 0$.

In addition, we need the following notion. Let M be a monoid, and let $\alpha \in M$. A triple $(\beta, \gamma, \delta) \in M^3$ is a *bi-torsional* factor of α, if $\beta\gamma\delta$ is a factor of α, that is, $\alpha =_M \varphi\beta\gamma\delta\eta$ for some $\varphi, \eta \in M$, and there are positive integers m, n, $m \neq n$, such that $\varphi\beta^m\gamma\delta^m\eta =_M \varphi\beta^n\gamma\delta^n\eta$.

Proposition 6.3. *Let M be a finitely generated monoid given by a presentation $(\Sigma; S)$. If M contains a subset A that satisfies the following condition $(*)$, then there is no context-free cross-section $C \subseteq \Sigma^*$ for M:*

$(*)$ For each $n \in \mathbb{N}$, there is an element $\alpha_n \in A$ such that, for all $w \in \Sigma^*$ satisfying $[w]_S = \alpha_n$, there is a marking of n positions in w such that, whenever $w = uvxyz$ is a decomposition of w satisfying the conditions (1) and (2) of Proposition 6.2, then $([v]_S, [x]_S, [y]_S)$ is a bi-torsional factor of α_n.

Proof. Assume that $C \subseteq \Sigma^*$ is a context-free cross-section for M, and let $n := k(C) \geq 2$ be the constant of Proposition 6.2 corresponding to the language C. Let α_n be the element of A given by $(*)$ for the number n. As C is a cross-section for M, there is an element $w \in C$ such that $[w]_S = \alpha_n$. By $(*)$ there is a marking of n positions in w such that, whenever w is decomposed as $w = uvxyz$ satisfying (1) and (2) of Proposition 6.2, then

$([v]_S, [x]_S, [y]_S)$ is a bi-torsional factor of α_n. On the other hand, Proposition 6.2 asserts that also for this particular marking of w, there exists a decomposition $w = uvxyz$ that satisfies all the conditions (1), (2), and (3) for C. As C is a cross-section for M, $[uv^ixy^iz]_S \neq_M [uv^jxy^jz]_S$ for all $i \neq j$, and hence, for this particular decomposition $([v]_S, [x]_S, [y]_S)$ cannot be a bi-torsional factor of $[w]_S$. This contradiction shows that M does not have a context-free cross-section $C \subseteq \Sigma^*$. □

If $(\Sigma_1; S_1)$ and $(\Sigma_2; S_2)$ are two finitely generated presentations of the same monoid M, then there exists a context-free cross-section $C_1 \subseteq \Sigma_1^*$ for M if and only if there exists a context-free cross-section $C_2 \subseteq \Sigma_2^*$ for M. Hence, if a finitely generated monoid M satisfies the hypothesis of Proposition 6.3, then no finitely generated presentation of M contains a context-free cross-section for M.

Next we present some examples of finitely presented monoids that satisfy the hypothesis of Proposition 6.3.

Example 6.4. Let $\Sigma = \{a, b, c, d\}$ and $S = \{ba \to ab, bc \to aca, cc \to d\} \cup \{de \to d, ed \to d \mid e \in \Sigma\}$, that is, S is the system $S(0)$ of Example 4.6. Then S is equivalent to the following left-context-free convergent system:

$$R := S \cup \{a^nca^nc \to d \mid n \geq 1\}.$$

Assume that $C \subseteq \Sigma^*$ is a context-free cross-section for the monoid M presented by $(\Sigma; S)$, and let $A := \{\alpha_n \mid n \geq 1\}$, where $\alpha_n := [w_n]_S$, and w_n denotes the string $w_n := a^nca^{n+1}ca^{n+2}ca^{n+3}ca^{n+4}c$. It can be shown that this set A does indeed satisfy the condition $(*)$ of Proposition 6.3. However, in order to reduce the number of cases that have to be considered, we proceed in a slightly different way.

The strings w_n are irreducible modulo R, and hence, $[w_n]_S \subseteq (\{a, b\}^+ \cdot c)^5$. Thus, the subset $C_0 := C \cap (\{a, b\}^+ \cdot c)^5$ of C contains the representatives of all the elements α_n, $n \geq 1$.

Let $n := k(C_0)$ be the constant of Proposition 6.2 corresponding to the context-free language C_0, and let $p_n \in C_0$ be the representative of α_n. Then $p_n = q_1cq_2cq_3cq_4cq_5c$ for some $q_1, \ldots, q_5 \in \{a, b\}^+$, and $|q_1| = n$. In p_n we now mark the first n letters, that is, we mark the letters of the prefix q_1. By Proposition 6.2 p_n can be factored as $p_n = uvxyz$ such that

(1) u, v, and x or x, y, and z each contain at least one marked position,
(2) vy contains at most n marked positions, and
(3) $uv^\ell xy^\ell z \in C_0$ for all $\ell \geq 0$.

We shall show that $([v]_S, [x]_S, [y]_S)$ is a bi-torsional factor of $[p_n]_S = \alpha_n$. From (3) above we see that $|v|_c = |y|_c = 0$. Analyzing the two factorizations $p_n = q_1cq_2cq_3cq_4cq_5c$ and $p_n = uvxyz$, we have to consider various cases.

Case 1: x, y, and z each contain at least one marked position. Then $uvxy$ is a prefix of q_1, that is, $q_1 = uvxyq_1'$ for some $q_1' \in \{a, b\}^*$.

If $|v|_a + |y|_a = \mu \geq 1$, then $vy \leftrightarrow_S^* b^\nu a^\mu$ for some $\nu \geq 0$. Hence, for all $\ell \geq 2$,

$$uv^\ell xy^\ell z \leftrightarrow_S^* b^{\nu\cdot(\ell-1)}a^{\mu\cdot(\ell-1)}uvxyz \leftrightarrow_S^* b^{\nu\cdot(\ell-1)}\underline{a^{\mu\cdot(\ell-1)}a^nca^{n+1}c}a^{n+2}ca^{n+3}ca^{n+4}c \to_R^* d,$$

that is, $([v]_S, [x]_S, [y]_S)$ is a bi-torsional factor of α_n.

If $|v|_a + |y|_a = 0$, then $vy \leftrightarrow_S^* b^\mu$ for some $\mu \geq 1$. Hence, for all $\ell \geq 2$,

$$\begin{aligned} uv^\ell xy^\ell z &\leftrightarrow_S^* a^n b^{\mu\cdot(\ell-1)} ca^{n+1} ca^{n+2} ca^{n+3} ca^{n+4} c \\ &\leftrightarrow_S^* a^{n+\mu\cdot(\ell-1)} c\underline{a^{\mu\cdot(\ell-1)+n+1} ca^{n+2} c}a^{n+3} ca^{n+4} c \to_R^* d, \end{aligned}$$

and so, $([v]_S, [x]_S, [y]_S)$ is again a bi-torsional factor of α_n.

This completes the analysis of Case 1.

Case 2: u, v, and x each contain at least one marked position. Then uv is a prefix of q_1, and y is a substring of one of the substrings q_1 to q_5, as $|y|_c = 0$. If y is empty, or if y is a substring of q_1, then it follows as in Case 1 that $([v]_S, [x]_S, [y]_S)$ is a bi-torsional factor of α_n. So, let us assume that $y \neq \lambda$, and that y is a substring of q_i for some $i \in \{2, \ldots, 5\}$, that is, $uvxy$ is a prefix of $q_1cq_2c \cdots q_{i-1}cq_i$, and $q_1cq_2c \cdots q_{i-1}c$ is a prefix of uvx.

(i) Let y be a substring of q_2. If $|y|_a = \mu \geq 1$, then, for all $\ell \geq 2$,

$$\begin{aligned} uv^\ell xy^\ell z &\leftrightarrow_S^* v^{\ell-1} q_1 c q_2 y^{\ell-1} c q_3 c q_4 c q_5 c \\ &\leftrightarrow_S^* v^{\ell-1} a^n c b^{\nu\cdot(\ell-1)} \underline{a^{n+1+\mu\cdot(\ell-1)} ca^{n+2} c}a^{n+3} ca^{n+4} c \to_R^* d, \end{aligned}$$

where $\nu := |y|_b$. If $y = b^\mu$ for some $\mu \geq 1$, then for all $\ell \geq 2$,

$$\begin{aligned} uv^\ell xy^\ell z &\leftrightarrow_S^* v^{\ell-1} a^n ca^{n+1} b^{\mu\cdot(\ell-1)} ca^{n+2} ca^{n+3} ca^{n+4} c \\ &\leftrightarrow_S^* v^{\ell-1} a^n ca^{n+1+\mu\cdot(\ell-1)} c\underline{a^{n+2+\mu\cdot(\ell-1)} ca^{n+3} c}a^{n+4} c \to_R^* d. \end{aligned}$$

(ii) Let y be a substring of q_3. Then, for all $\ell \geq 2$,

$$uv^\ell xy^\ell z \leftrightarrow_S^* v^{\ell-1} a^n ca^{n+1} ca^{n+2} y^{\ell-1} ca^{n+3} ca^{n+4} c,$$

and it follows analogously that $uv^\ell xy^\ell z \to_R^* d$.

(iii) Let y be a substring of q_4. If $|v|_a = \mu \geq 1$, then, for all $\ell \geq 2$,

$$uv^\ell xy^\ell z \leftrightarrow_S^* b^{\nu\cdot(\ell-1)} \underline{a^{n+\mu\cdot(\ell-1)} ca^{n+1} c}a^{n+2} cy^{\ell-1} a^{n+3} ca^{n+4} c \to_R^* d,$$

where $\nu := |v|_b$, and if $v = b^\mu$ for some $\mu \geq 1$, then, for all $\ell \geq 2$,

$$\begin{aligned} uv^\ell xy^\ell z &\leftrightarrow_S^* a^n b^{\mu\cdot(\ell-1)} ca^{n+1} ca^{n+2} cy^{\ell-1} a^{n+3} ca^{n+4} c \\ &\leftrightarrow_S^* a^{n+\mu\cdot(\ell-1)} c\underline{a^{\mu\cdot(\ell-1)+n+1} ca^{n+2} c}y^{\ell-1} a^{n+3} ca^{n+4} c \to_R^* d. \end{aligned}$$

(iv) Finally, if y is a substring of q_5, $uv^\ell xy^\ell z \to_R^* d$ follows as in the previous case for all $\ell \geq 2$.

Thus, $([v]_S, [x]_S, [y]_S)$ is a bi-torsional factor of α_n. This contradicts the assumption that $uv^\ell xy^\ell z \in C_0$ for all $\ell \geq 0$ as in the proof of Proposition 6.3. Hence, the monoid M does not have a context-free cross-section.

Example 6.4 shows that a finitely presented monoid that can be presented by some left-context-free convergent string-rewriting system need not have any context-free cross-section. Hence, Proposition 3.3 cannot be extended to the class of context-free languages. Observe that the set $\mathrm{dom}(R)$ of left-hand sides of the convergent string-rewriting R of Example 6.4 is even a deterministic context-free language. Thus, Proposition 3.3 cannot even be extended to the class of deterministic context-free languages, either.

Notice that the proof given above can be adopted to all the systems $S(t)(t \in \mathbb{N})$ of Example 4.6. Hence, none of the monoids M_t considered there has a context-free cross-section. The same proof idea also works for the systems $S(t)(t \geq 5)$ of Example 4.7. Thus, none of the monoids $N_t(t \geq 5)$ has a context-free cross-section, either.

Finally, let us consider the system $S(3)$ of Example 4.7 in more detail. Notice that the set $\mathrm{IRR}(R(3))$ is not context-free as can be easily verified using Proposition 6.2. Nevertheless, we have the following result.

Proposition 6.5. *The monoid N_3 has the following properties:*

(1) it is finitely presented;
(2) its word problem is solvable in polynomial time;
(3) it does not have a regular cross-section; but
(4) it has a context-free, s-closed cross-section.

Proof. (1)–(3) were proved in Example 4.7. Thus, it remains to prove (4). Consider the string-rewriting system S' that is defined as follows:

$$\begin{array}{rcl} S' & := & \{a_1a_2 \to d, a_1a_3 \to d, a_1c_2 \to d, a_1c_3 \to d, \\ & & a_2a_1 \to d, a_2a_3 \to d, a_2c_1 \to d, a_2c_3 \to d, \\ & & a_3a_1 \to d, a_3a_2 \to d, a_3b \to d, a_3c_1 \to d, a_3c_2 \to d, \\ & & ba_3 \to d, bc_3 \to d, c_1a_1 \to d, c_1a_3 \to d, c_1c_1 \to d, \\ & & c_1c_2 \to d, c_1c_3 \to d, c_2a_1 \to d, c_2a_2 \to d, c_2c_1 \to d, \\ & & c_2c_2 \to d, c_2c_3 \to d, c_3a_1 \to d, c_3a_2 \to d, c_3a_3 \to d, \\ & & c_3b \to d, c_3c_1 \to d, c_3c_2 \to d, c_3c_3 \to d\} \\ & \cup & \{de \to d, ed \to d \mid e \in \Sigma_3\} \\ & \cup & \{a_1b \to ba_1, ba_2 \to a_2b, a_1c_1a_2 \to bc_1, bc_2 \to a_2c_2a_3\}. \end{array}$$

Then, S' is equivalent to the system $S(3)$, that is, $(\Sigma_3; S')$ also presents the monoid N_3. If $>$ denotes the syllable ordering induced by the precedence $a_1 > c_1 > b > a_3 > a_2 > c_2 > c_3 > d$, where each letter has the status 'right', then $\to_{S'} \subseteq >$, and hence, S' is noetherian.

Based on this ordering completion yields the following infinite string-rewriting system from S':

$$\begin{array}{rcl} R' & := & S' \cup \{a_2b^na_1 \to d, c_1b^na_1 \to d, c_2b^na_1 \to d, \\ & & \quad a_2b^nc_1 \to d, c_1b^nc_1 \to d, c_2b^nc_1 \to d \mid n \geq 1\} \\ & & \cup \{a_2^nc_2a_3^nc_3 \to d \mid n \geq 1\} \\ & & \cup \{b^mc_1a_2^nc_2a_3^{m+n}c_3 \to d \mid n, m \geq 1\}. \end{array}$$

Then R' is a left-context-free convergent string-rewriting system that is equivalent to S', and hence, to $S(3)$.

Let $R'' := R' \smallsetminus (\{a_2^nc_2a_3^nc_3 \to d \mid n \geq 1\} \cup \{b^mc_1a_2^nc_2a_3^{m+n}c_3 \to d \mid n, m \geq 1\})$. Then R'' is a left-regular system, and so $\mathrm{IRR}(R'')$ is a regular language. It can be checked easily that a string $w \in \Sigma_3^*$ belongs to the set $\mathrm{IRR}(R'')$ if and only if it is a substring of a string belonging to the following regular set:

$$C'' := b^* \cdot c_1 \cdot a_2^+ \cdot c_2 \cdot b^* \cup b^* \cdot c_1 \cdot a_2^+ \cdot c_2 \cdot a_3^+ \cdot c_3 \cup b^* \cdot a_1^* \cdot c_1 \cdot b^* \cup b^* \cdot c_1 \cdot a_2^+ \cdot b^* \cup d.$$

Now a string w is irreducible modulo R' if and only if w is irreducible modulo R'' and ($|w|_{c_3} = 0$ or w is a suffix of a string of the form $b^{m_1}c_1a_2^{m_2}c_2a_3^{m_3}c_3$ with $m_1 \geq 0, m_2 > 0,$

and $m_3 > m_1 + m_2$). This characterization shows that the set IRR(R') is a context-free, s-closed cross-section for N_3. □

We close this section by considering yet another example.

Example 6.6. Let $\Sigma = \{a, b, c, u, x, d\}$, let $S := \{uaac \to d, xu \to ubx, xb \to bx, xa \to abx, xc \to c\} \cup \{de \to d, ed \to d \mid e \in \Sigma\}$, and let M be the monoid that is presented by $(\Sigma; S)$. Using the syllable ordering $>$ that is induced by the precedence $x > u > c > b > a > d$ with status 'right', we see that S is noetherian. Completion yields the system $R := S \cup \{ub^nab^nab^nc \to d \mid n \geq 1\}$, because

$$\begin{array}{lllllll} \underline{x\overline{u}}\,\overline{aac} & \to & xd \to d & & \underline{x\overline{u}}\,\overline{b^nab^nab^nc} & \to & xd \to d \\ \downarrow & & & \text{and} & \downarrow & & \\ ubxaac & \to^* & ubababc & & ubxb^nab^nab^nc & \to^* & ub^{n+1}ab^{n+1}ab^{n+1}c \end{array}$$

for all $n \geq 1$. The system R is left-context-sensitive and convergent, but it is obviously not left-context-free. However, it can be used to solve the word problem for M in polynomial time.

Claim: IRR(R) is a context-free language.

Proof. Obviously, $\mathrm{IRR}(R) = \mathrm{IRR}(S) \cap \mathrm{IRR}(R \smallsetminus S)$. As IRR($S$) is regular, it suffices to show that the set $\mathrm{IRR}(R \smallsetminus S)$ is context-free. Now a string $w \in \Sigma^*$ is irreducible modulo $R \smallsetminus S$ if and only if each substring $y \in u \cdot b^* \cdot a \cdot b^* \cdot a \cdot b^* \cdot c$ of w belongs to the set $C_1 := \{ub^iab^jab^kc \mid i \neq j, \text{ or } j \neq k, \text{ or } i \neq k\}$. Obviously, C_1 is a context-free language, and hence, it is easily seen that $\mathrm{IRR}(R \smallsetminus S)$ is a context-free language. □

Thus, there are some left-context-free convergent string-rewriting systems that present monoids that are finitely presented, which have easily decidable word problems, but which do not have any context-free cross-sections (Example 6.4), but there also exist convergent string-rewriting systems that are normalized but not left-context-free, but which yield s-closed context-free cross-sections (Example 6.6). Finally, let us again point out that Proposition 6.5 has established the fact that there exist finitely presented monoids with easily decidable word problems such that these monoids have no regular cross-sections, although they admit context-free, s-closed cross-sections.

7. Church–Rosser Languages as Cross-sections

Here we will prove that the Church–Rosser languages provide cross-sections for all finitely generated monoids with decidable word problems. This result will follow from the fact that the Church–Rosser languages form a basis for the recursively enumerable languages.

A family $\mathbb{C}$ of formal languages is said to be a *basis* for the recursively enumerable (r.e.) languages if the r.e. languages can be characterized as follows:

A language $L \subseteq \Sigma^*$ is r.e. if and only if there exists a language $B \in \mathbb{C}$ on some alphabet $\Delta \supsetneq \Sigma$ such that $\pi_\Sigma(B) = L$.

Here $\pi_\Sigma : \Delta^* \to \Sigma^*$ denotes the canonical projection, that is, π_Σ is the morphism that is defined through $b \mapsto b$ ($b \in \Sigma$) and $c \mapsto \lambda$ ($c \in \Delta \smallsetminus \Sigma$).

It is well known that the class CSL of context-sensitive languages is a basis for the r.e.

languages, while the class CFL of context-free languages is not a basis, as it is closed under morphisms. Buntrock (1996) has shown that the class GCSL of growing context-sensitive languages is also a basis for the r.e. languages. Here we will improve upon his result.

A language $L \subseteq \Sigma^*$ is called a *Church–Rosser language* (CRL) if there exist an alphabet $\Gamma \supsetneq \Sigma$, a finite, length-reducing, and confluent string-rewriting system R on Γ, two strings $t_1, t_2 \in (\Gamma \smallsetminus \Sigma)^* \cap \mathrm{IRR}(R)$, and a letter $Y \in (\Gamma \smallsetminus \Sigma) \cap \mathrm{IRR}(R)$ such that, for all $w \in \Sigma^*$, $t_1 w t_2 \rightarrow_R^* Y$ if and only if $w \in L$ (McNaughton *et al.*, 1988).

By admitting weight-reducing string-rewriting systems we obtain the class of *generalized Church–Rosser languages* (GCRL), which can be interpreted as the class of deterministic growing context-sensitive languages (Buntrock and Otto, 1998). It has only been observed recently that the class of Church–Rosser languages coincides with this class of languages (Niemann and Otto, 1998). Thus, we have the following chain of inclusions:

$$\mathrm{CRL} = \mathrm{GCRL} \subsetneq \mathrm{GCSL} \subsetneq \mathrm{CSL}.$$

In this section we will establish the following result.

Theorem 7.1. *The class* CRL *of Church–Rosser languages is a basis for the recursively enumerable languages.*

This result has the following consequence.

Corollary 7.2. *Let $(\Sigma; R)$ be a finitely generated presentation of a monoid M with a decidable word problem. Then $(\Sigma \cup \{¢, e\}; R \cup \{¢ \rightarrow \lambda, e \rightarrow \lambda\})$ is another finitely generated presentation of M such that there exists a cross-section $C \subseteq (\Sigma \cup \{¢, e\})^*$ for M which is a Church–Rosser language.*

Proof. Let $\Delta := \Sigma \cup \{¢, e\}$, where ¢ and e are two additional letters, and let $S := R \cup \{¢ \rightarrow \lambda, e \rightarrow \lambda\}$. Then $(\Delta; S)$ is another finitely generated presentation of M. Let $>_{\ell\ell}$ be the length-lexicographical ordering on Σ^*. As M has a decidable word problem, the set $MIN(R, >_{\ell\ell}) := \{w \in \Sigma^* \mid w$ is minimal in $[w]_R$ with respect to the ordering $>_{\ell\ell}\}$ is a recursive cross-section for M (see the proof of Proposition 3.6). As the class CRL is a basis for the r.e. languages, there exist an alphabet $\Gamma \supsetneq \Sigma$ and a Church–Rosser language $C \subseteq \Gamma^*$ such that $\pi_\Sigma(C) = MIN(R, >_{\ell\ell})$. Actually, we will see in the proof of Theorem 7.1 that we can choose $\Gamma := \Delta$, and that C can be chosen in such a way that π_Σ is actually a bijection from C onto $MIN(R, >_{\ell\ell})$. Thus, $C \subseteq \Delta^*$ is a cross-section for the presentation $(\Delta; S)$. □

Observe that in the above corollary $(\Delta; S)$ is a finite presentation if the given presentation $(\Sigma; R)$ is finite. We now turn to the proof of Theorem 7.1.

Let $L \subseteq \Sigma^*$ be a r.e. language, and let $M = (Q, \Sigma, q_0, q_n, \delta)$ be a deterministic single-tape Turing machine (TM) accepting L. Here $Q = \{q_0, q_1, \ldots, q_n\}$ is the set of states, q_0 is the initial state, q_n is the final state, and $\delta : (Q \smallsetminus \{q_n\}) \times \Sigma_b \rightarrow Q \times (\Sigma \cup \{\ell, r\})$ is the transition function of M, where $\Sigma_b := \Sigma \cup \{b\}$ (b denotes the blank symbol), and ℓ (r) denotes the operation of moving M's read-/write-head to the left (right). Observe that M cannot print the blank symbol, and that M halts if and only if it enters the state q_n.

We will simulate M through a finite string-rewriting system $R(M)$ that is length-reducing and confluent. To this end we introduce five additional letters \$, ¢, d, e, and Y, and take $\Gamma := \Sigma_b \cup Q \cup \{\$, ¢, d, e, Y\}$. The system $R(M)$ consists of the following three groups of rules:

(1) Rules to simulate the stepwise behaviour of M:

$$\left.\begin{array}{lcllcl}
q_i a_k dd & \to & q_j a_\ell & \text{if } \delta(q_i, a_k) & = & (q_j, a_\ell) \\
q_i ¢ ee & \to & q_j a_\ell ¢ & \text{if } \delta(q_i, b) & = & (q_j, a_\ell) \\
q_i a_k dd & \to & a_k q_j & \text{if } \delta(q_i, a_k) & = & (q_j, r) \\
q_i ¢ ee & \to & b q_j ¢ & \text{if } \delta(q_i, b) & = & (q_j, r) \\
a_\ell q_i a_k dd & \to & q_j a_\ell a_k & \text{if } \delta(q_i, a_k) & = & (q_j, \ell) \\
a_\ell q_i ¢ ee & \to & q_j a_\ell ¢ & \text{if } \delta(q_i, b) & = & (q_j, \ell) \\
\$ q_i a_k dd & \to & \$ q_j b a_k & \text{if } \delta(q_i, a_k) & = & (q_j, \ell) \\
\$ q_i ¢ ee & \to & \$ q_j b ¢ & \text{if } \delta(q_i, b) & = & (q_j, \ell)
\end{array}\right\} \text{ for all } a_\ell \in \Sigma_b.$$

(2) Rules to shift occurrences of the letter d to the left:

$$\left.\begin{array}{lcl}
a_i ¢ ee & \to & a_i d ¢ \\
a_i d ¢ ee & \to & a_i dd ¢ \\
a_i a_j dd & \to & a_i d a_j \\
a_i d a_j dd & \to & a_i dd a_j
\end{array}\right\} \text{ for all } a_i, a_j \in \Sigma_b.$$

(3) Rules to erase halting configurations:

$$\left.\begin{array}{lcl}
q_n a_i dd & \to & q_n \\
a_i q_n ¢ ee & \to & q_n ¢
\end{array}\right\} \text{ for all } a_i \in \Sigma_b.$$
$$\$ q_n ¢ \to Y$$

The system $R(M)$ has the following properties.

PROPOSITION 7.3.

(a) The string-rewriting system $R(M)$ is finite, length-reducing, and confluent.
(b) For $w \in \Sigma^*$, the following two statements are equivalent:

(i) $w \in L$, and
(ii) $\exists m \in \mathbb{N} : \$q_0 w ¢ e^m \to^*_{R(M)} Y$.

(c) $\forall w \in \Sigma^* \forall m, n \in \mathbb{N} : \$q_0 w ¢ e^m \to^*_{R(M)} Y$ and $\$q_0 w ¢ e^n \to^*_{R(M)} Y$ imply that $m = n$.

PROOF. (a) Obviously the system $R(M)$ is finite and length-reducing. As the Turing machine M is deterministic, $R(M)$ is an orthogonal system, and hence, it is confluent.

(b) If $w \in L$, then M has an accepting computation of the form $q_0 w \vdash_M u_1 q_1 v_1 \vdash_M \cdots \vdash_M u_n q_n v_n$, where $u_i, v_i \in \Sigma_b^*$ and $q_i \in Q$. Thus, we see from the form of the rules of groups (1) and (2) of $R(M)$ that $\$q_0 w ¢ e^{m_1} \to^*_{R(M)} \$u_n q_n v_n ¢$ holds for some $m_1 \in \mathbb{N}$. In fact, this integer m_1 is uniquely determined. Further, there is a unique integer m_2 such that $\$u_n q_n v_n ¢ e^{m_2} \to^*_{R(M)} \$q_n ¢ \to_{R(M)} Y$. Thus, $m := m_1 + m_2$. This also proves part (c).

Conversely, if $\$q_0 w ¢ e^m \to^*_{R(M)} Y$ for some integer m, then we see from the form of the rules of $R(M)$ that the Turing machine M must accept on input w. Thus, $w \in L$. This completes the proof of (b). $\square$

Now we define the language $B := \{w¢e^{m(w)} \mid w \in L\}$, where $m(w)$ denotes the unique integer m satisfying $\$q_0 w¢e^m \to^*_{R(M)} Y$. Let $\Delta := \Sigma \cup \{¢, e\}$. Then $\pi_\Sigma : \Delta^* \to \Sigma^*$ satisfies $\pi_\Sigma(B) = L$. In fact, π_Σ induces a bijection from B onto L. It remains to prove the following lemma.

Lemma 7.4. *B is a Church–Rosser language.*

Proof. Take $t_1 := \$q_0$ and $t_2 := \lambda$. Then $t_1 w¢e^{m(w)} t_2 = \$q_0 w¢e^{m(w)} \to^*_{R(M)} Y$ for all $w¢e^{m(w)} \in B$. Conversely, let $u \in \Delta^*$ such that $t_1 u t_2 = \$q_0 u \to^*_{R(M)} Y$. We must verify that $u \in B$, that is, $u = w¢e^{m(w)}$ for some $w \in L$.

As there is only one rule containing the symbol Y, we have $\$q_0 u \to^*_{R(M)} \$q_n¢ \to_{R(M)} Y$. As this last rule is the only one that deletes \$- and ¢-symbols, we see that $|u|_¢ = 1$, that is, $u = u_1¢u_2$ for some $u_1, u_2 \in (\Sigma \cup \{e\})^*$. Hence, the above reduction sequence can be written as $\$q_0 u = \$q_0 u_1¢u_2 \to_{R(M)} \$x_1¢y_1 \to_{R(M)} \$x_2¢y_2 \to_{R(M)} \cdots \to_{R(M)} \$x_m¢y_m = \$q_n¢$, where $|x_i|_Q = 1$ for all $i = 1, \ldots, m$. If $|u_1|_e > 0$, then we can conclude from the form of the rules of $R(M)$ that $|x_i|_e > 0$ for all $i = 1, \ldots, m$. As $x_m = q_n$, we see that $|u_1|_e = 0$, that is, $u_1 = w \in \Sigma^*$. Analogously, we obtain that $u_2 = e^k$ for some $k \in \mathbb{N}$. Hence, $\$q_0 u = \$q_0 w¢e^k \to^*_{R(M)} Y$, which implies that $w \in L$ and $k = m(w)$ by Proposition 7.3 (b) and (c). Thus, $u = u_1¢u_2 = w¢e^{m(w)} \in B$.

This proves that B is the Church–Rosser language specified by t_1, t_2, Y, and $R(M)$. $\square$

With this lemma we have completed the proof of Theorem 7.1.

From Corollary 7.2 we see that each finitely presented monoid M with a decidable word problem has a finite presentation $(\Sigma; S)$ which contains a cross-section for M that is a Church–Rosser language. However, in contrast to the situation for regular and context-free cross-sections, it appears that the existence of a cross-section that is a Church–Rosser language actually depends on the chosen finite presentation.

8. Left-regular Systems that are Tractable

For a finite convergent string-rewriting system R the complexity of the normal form algorithm for solving the word problem for R is closely related to the lengths of the reduction sequences that this algorithm generates. Thus, the derivational complexity of R (cf. Section 5) induces an upper bound for the complexity of the word problem for R, although the derivational complexity of R can be far worse than the actual degree of complexity of the word problem (Bauer and Otto, 1984). For left-regular convergent string-rewriting systems in general, the situation is quite different, however.

Define a left-regular string-rewriting system R_0 as follows:

$$R_0 := \{w \to w_0 \mid w \in \mathrm{RED}(R)\},$$

where $w_0 \in \mathrm{IRR}(R)$ denotes the unique normal form of the string w with respect to the system R. Then R_0 is equivalent to R, and it is convergent. Further, if $w \,{}_L{\to}_{R_0}\, w_1 \,{}_L{\to}_{R_0}\, w_2 \,{}_L{\to}_{R_0} \cdots {}_L{\to}_{R_0}\, w_0$ is a left-most reduction sequence from w to w_0, then it is of length at most $|w|$. Thus, the complexity of the normal form algorithm that is based on computing left-most reduction sequences with respect to R_0 does not really depend on the lengths of the sequences computed. Here the task of determining the right-hand side of a rule given its left-hand side majorizes the complexity of the normal form algorithm. Actually, this very task is equivalent to solving the word problem. Thus, a convergent string-rewriting system of this form is not really useful for solving the word problem. For that, we need convergent string-rewriting systems for which this task can be solved easily. Accordingly, we will call a left-regular string-rewriting system R on Σ *tractable*, if there exists a polynomial-time algorithm for solving the following task:

INPUT: A string $\ell \in \mathrm{dom}(R)$.
OUTPUT: A string $r \in \mathrm{range}(R)$ such that $(\ell, r) \in R$.

Actually, we consider the following classes of left-regular systems. A left-regular string-rewriting system R on Σ is called

1-regular, if $R = R_0 \cup \{\ell \to r \mid \ell \in L\}$, where R_0 is a finite system, $r \in \Sigma^*$, and $L \subseteq \Sigma^*$ is a regular language,

f-regular, if $R = \bigcup_{i=1}^{n} (L_i \times \{r_i\})$, where $r_1, \ldots, r_n \in \Sigma^*$, and $L_1, \ldots, L_n \subseteq \Sigma^*$ are regular languages,

c-regular, if the language $c_R = \{\ell \# r \mid (\ell, r) \in R\} \subseteq (\Sigma \cup \{\#\})^*$ is a regular language, where $\#$ is an additional symbol not in Σ,

gsm-regular, if there exists a deterministic generalized sequential machine (gsm) that accepts the set $\mathrm{dom}(R)$, and that, for each $\ell \in \mathrm{dom}(R)$, produces as output the string $r \in \Sigma^*$ satisfying $(\ell, r) \in R$.

The f-regular systems were considered by Ó'Dúnlaing (1983), where he proved that confluence is undecidable for f-regular systems that are length-reducing. The gsm-regular systems were proposed by Benninghofen *et al.* (1987). Obviously, all these systems are tractable. Also each 1-regular system is f-regular, and each f-regular system is c-regular. However, not every c-regular system is gsm-regular. This changes when we restrict attention to systems that are unambiguous. Here a string-rewriting system R is called *unambiguous* if no two different rules of R have identical left-hand sides. Obviously, each gsm-regular system is unambiguous.

Proposition 8.1. *An unambiguous string-rewriting system is f-regular if and only if it is c-regular.*

Proof. As each f-regular system is c-regular, it remains to prove the converse inclusion. So let R be a c-regular system that is unambiguous. Then the language $L := \{\ell \# r \mid (\ell, r) \in R\}$ is regular, and hence, there exists a deterministic finite state acceptor (dfsa) $A := (Q, \Sigma \cup \{\#\}, q_0, F, \delta\}$ that accepts this language.

For each $\ell \in \mathrm{dom}(R)$, there is a unique state $q \in Q$ such that $\delta(q_0, \ell\#) = q$, and there is a unique string $r \in \Sigma^*$ such that $(\ell, r) \in R$, as R is unambiguous. Hence, r is the only string satisfying $\delta(q, r) \in F$. As Q is finite, this means that $\mathrm{range}(R)$ must be finite. For each $r \in \mathrm{range}(R)$, the language $\{\ell \mid \ell \# r \in L\}$ is regular. Thus, R is indeed f-regular. □

What is the expressive power of these various classes of tractable left-regular string-rewriting systems? Obviously, the unambiguous 1-regular string-rewriting systems that are convergent have more expressive power than the finite convergent string-rewriting systems. Just look at Squier's example $S_1 = \{ab \to \lambda, xa \to atx, xb \to bx, xt \to tx, xy \to \lambda\} \cup \{at^n b \to \lambda \mid n \geq 1\}$, which is a normalized, convergent, 1-regular system, but which presents a finitely presented monoid that cannot be presented by any finite convergent string-rewriting system (Squier *et al.*, 1994). But are the other classes of tractable, left-regular string-rewriting systems more powerful with respect to their expressive power? As far as finitely generated monoids are concerned this is indeed the case as shown by the following simple examples.

Example 8.2. Let $\Sigma_1 := \{a, b, c, d, e\}$, and let $R_1 := \{ab^{2n}c \to d \mid n \geq 0\} \cup \{ab^{2n+1}c \to e \mid n \geq 0\}$. Then R_1 is a normalized f-regular system that is easily seen to also be convergent.

Claim 1: There is no 1-regular string-rewriting system S on Σ_1 that is equivalent to R_1.

Proof. Let S be a string-rewriting system on Σ_1 that is equivalent to R_1. We have $[d]_S = \{d\} \cup \{ab^{2n}c \mid n \geq 0\}$ and $[e]_S = \{e\} \cup \{ab^{2n+1}c \mid n \geq 0\}$. The set $I := \{d, e\} \cup \{ab^n c \mid n \geq 0\}$ is an infix code, that is, no element of this set is a substring of any other element. Also, for each proper substring w of any element $u \in I$, $[w]_S = \{w\}$. Hence, for each $u \in I$, S must contain a rule with left-hand side u or with right-hand side u. Thus, S is certainly not 1-regular. $\square$

Based on this observation we can now establish the following stronger result.

Claim 2: The monoid M_{R_1} presented by $(\Sigma_1; R_1)$ does not have a finitely generated presentation of the form $(\Gamma; S)$ with a 1-regular system S.

Proof. Assume that $(\Gamma; S)$ is a finitely generated presentation of M_{R_1}. Then there exists a mapping $\varphi : \Gamma \to \Sigma_1^*$ such that this mapping induces an isomorphism from $\Gamma^*/\leftrightarrow_S^*$ onto M_{R_1}. Also there exists a mapping $\psi : \Sigma_1 \to \Gamma^*$ such that this mapping induces an isomorphism from M_{R_1} onto $\Gamma^*/\leftrightarrow_S^*$, and for all $a \in \Sigma_1$, $\varphi(\psi(a))$ represents the same monoid element as the letter a. Further, for all $(\ell, r) \in S$, $\varphi(\ell)$ and $\varphi(r)$ represent the same monoid element. Hence, the string-rewriting system $S_1 := \{(a, \varphi(\psi(a))) \mid a \in \Sigma\} \cup \{(\varphi(\ell), \varphi(r)) \mid (\ell, r) \in S\}$ is equivalent to the system R_1. If S were 1-regular, then S_1 would be 1-regular, contradicting Claim 1. $\square$

Thus, the monoid M_{R_1} cannot be presented by any 1-regular string-rewriting system, although it is presented by the normalized and convergent f-regular system R_1.

Example 8.3. Let $\Sigma_2 := \{a, b, c, d\}$, and let $R_2 := \{ab^n c \to ad^n c \mid n \geq 0\}$. Then R_2 is a gsm-regular system that is not f-regular. Obviously, R_2 is normalized and convergent. Arguing as in the previous example one can show that the monoid M_{R_2} presented by $(\Sigma_2; R_2)$ cannot be presented by any f-regular string-rewriting system.

Example 8.4. Let $\Sigma_3 := \{a, b, c, d\}$, and let $R_3 := \{ab^n c \to ad^{n^2} c \mid n \geq 0\}$. Then R_3 is a tractable left-regular system that is not gsm-regular, and R_3 is normalized and convergent. Again it can be shown that the monoid M_{R_3} presented by $(\Sigma_3; R_3)$ cannot be presented by any gsm-regular string-rewriting system.

Thus, if $\mathbb{M}_{1-\mathrm{REG}}$ ($\mathbb{M}_{\mathrm{f-REG}}, \mathbb{M}_{\mathrm{gsm-REG}}, \mathbb{M}_{\mathrm{t-REG}}$) denotes the class of finitely generated monoids that can be presented through 1-regular (f-regular, gsm-regular, tractable left-regular) string-rewriting systems that are normalized and convergent, then we have the following chain of proper inclusions:

$$\mathbb{M}_{\mathrm{fin}} \subsetneq \mathbb{M}_{1-\mathrm{REG}} \subsetneq \mathbb{M}_{\mathrm{f-REG}} \subsetneq \mathbb{M}_{\mathrm{gsm-REG}} \subsetneq \mathbb{M}_{\mathrm{t-REG}} \subsetneq \mathbb{M}_{\mathrm{CF}},$$

where $\mathbb{M}_{\mathrm{fin}}$ denotes the class of monoids that can be presented by finite convergent string-rewriting systems, and $\mathbb{M}_{\mathrm{CF}}$ denotes the class of monoids that can be presented by left-context-free convergent string-rewriting systems.

9. Conclusion

Every finitely presented monoid with a decidable word problem can be presented through a left-recursive convergent string-rewriting system. On the other hand, the left-regular convergent string-rewriting systems do not suffice to present all these monoids. Hence, the following question arises:

Question 1: Does the class of left-context-free (left-context-sensitive) convergent string-rewriting systems suffice to present all finitely presented monoids with decidable word problems?

Observe that although we have seen that each finitely presented monoid with a decidable word problem has a context-sensitive cross-section, we do not know whether or not each monoid of this form admits a presentation through some left-context-sensitive convergent string-rewriting system.

If a monoid does not admit a regular cross-section, then it cannot be presented by any left-regular convergent system. Is the following converse implication valid?

Question 2: Does a finitely presented monoid have a presentation through a left-regular convergent string-rewriting system if it has a regular cross-section?

As convergent string-rewriting systems yield s-closed cross-sections, also the following restricted version of Question 2 is of interest:

Question 3: Does a finitely presented monoid have a presentation through a left-regular convergent string-rewriting system if it has an s-closed, regular cross-section?

Obviously, these questions can be asked for every complementary class $\mathbb{C}$ of languages, that is, does a finitely presented monoid have a presentation through a left-$\mathbb{C}$ convergent string-rewriting system if it has a (s-closed) cross-section from $\mathbb{C}$? If Question 2 had an affirmative answer for the class CSL of context-sensitive languages, then also Question 1 had an affirmative answer for left-context-sensitive systems due to Corollary 7.2.

Finally, it should be pointed out that none of our example monoids is a group. Indeed, our technique is not easily applicable to groups, and in fact, not even to cancellative monoids. Hence, the following question remains open:

Question 4: Does each finitely presented group with a decidable word problem have a regular cross-section?

With respect to the finitely generated monoids they present we have obtained a proper hierarchy from the various classes of tractable, left-regular, and convergent string-rewriting systems considered in Section 8.

However, it remains the question of whether this result remains valid when we consider only monoids that are finitely presented. Does there exist a finitely presented monoid that is presented by some normalized and convergent f-regular (gsm-regular, or tractable left-regular, respectively) string-rewriting system, but that cannot be presented through any 1-regular (f-regular, gsm-regular) string-rewriting system that is normalized and convergent?

Finally, instead of restricting attention to left-regular systems one could also consider left-context-free or even left-context-sensitive systems that are tractable in the sense described above. Observe that all our example systems considered in Section 4 are 1-context-free.

References

Adian, A.I. (1979). *The Burnside Problem and Identities in Groups.* Berlin, Springer-Verlag.

Avenhaus, J. (1986). On the descriptive power of term rewriting systems. *J. Symb. Comput.*, **2**, 109–122.

Avenhaus, J., Madlener, K. (1990). Term rewriting and equational reasoning. In R.B. Banerji, ed., *Formal Techniques in Artificial Intelligence—A Sourcebook*, pp. 1–43. Amsterdam, North-Holland.

Bauer, G., Otto, F. (1984). Finite complete rewriting systems and the complexity of the word problem. *Acta Informatica*, **21**, 521–540.

Benninghofen, B., Kemmerich, S., Richter, M.M. (1987). *Systems of Reductions*, LNCS **277**. Berlin, Springer-Verlag.

Berstel, J. (1979). *Transductions and Context-free Languages.* Teubner Studienbücher. Stuttgart, Teubner-Verlag.

Book, R.V., Otto, F. (1993). *String-Rewriting Systems.* New York, Springer-Verlag.

Borho, W., Kraft, H. (1976). On the Gelfond–Kirillov-dimension. *Math. Annalen*, **220**, 1–24.

Buntrock, G. (1996). Wachsende kontext-sensitive Sprachen. Habilitationsschrift, Fakultät für Mathematik und Informatik, Universität Würzburg.

Buntrock, G., Otto, F. (1998). Growing context-sensitive languages and Church–Rosser languages. *Inf. Comput.*, **141**, 1–36.

Dershowitz, N., Jouannaud, J.P. (1990). Rewrite systems. In J. van Leeuwen, ed., *Handbook of Theoretical Computer Science, Vol. B.: Formal Models and Semantics*, pp. 243–320. Amsterdam, Elsevier.

Gilman, R.H. (1987). Groups with a rational cross-section. In S.M. Gersten and J.R. Stallings, eds, *Computational Group Theory and Topology*, pp. 175–183. Princeton, Princeton University Press.

Grzegorczyk, A. (1953). Some classes of recursive functions. *Rozprawy Matematycne*, **4**, 1–45.

Jantzen, M. (1988). *Confluent String Rewriting.* Berlin, Springer-Verlag.

Kobayashi, Y. (1995). A finitely presented monoid which has solvable word problem but has no regular complete presentation. *Theoret. Comput. Sci.*, **146**, 321–329.

Madlener, K., Otto, F. (1988). Pseudo-natural algorithms for finitely generated presentations of monoids and groups. *J. Symb. Comput.*, **5**, 339–358.

McNaughton, R., Narendran, P., Otto, F. (1988). Church–Rosser Thue systems and formal languages. *J. Assoc. Computing Machinery*, **35**, 324–344.

Niemann, G., Otto, F. (1998). The Church–Rosser languages are the deterministic variants of the growing context-sensitive languages. In M. Nivat, ed., *Foundations of Software Science and Computation Structures, Proceedings FoSSaCS'98*, LNCS **1378**, pp. 243–257. Berlin, Springer-Verlag.

Ó'Dúnlaing, C. (1983). Infinite regular Thue systems. *Theoret. Comput. Sci.*, **25**, 171–192.

Sattler-Klein, A. (1991). Divergence phenomena during completion. In R.V. Book, ed., *Rewriting Techniques and Applications*, LNCS **488**, pp. 374–385. Berlin, Springer-Verlag.

Sattler-Klein, A. (1996). A systematic study of infinite canonical systems generated by Knuth-Bendix completion and related problems, Doctoral Dissertation, Fachbereich Informatik, Universität Kaiserslautern.

Squier, C.C. (1987). Word problems and a homological finiteness condition for monoids. *J. Pure Appl. Algebra*, **49**, 201–217.

Squier, C.C., Otto, F., Kobayashi, Y. (1994). A finiteness condition for rewriting systems. *Theoret. Comput. Sci.*, **131**, 271–294.

Tourlakis, G.J. (1984). *Computability.* Reston, Reston Publ. Co..

Weihrauch, K. (1974). *Teilklassen primitiv-rekursiver Wortfunktionen.* Bericht Nr. 91, GMD Bonn.

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