Kernels and Equisummation Properties of Uniformly Elliptic Operators

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In this note we study eigenfunction expansions associated with a general class of elliptic operators on $\mathbb{R}^n$, and in particular, equisummation properties for expansions in eigenfunctions of pairs of operators. This analysis depends on modern techniques of operator calculus, rather than the hard classical analysis traditionally associated with equisummability. We apply our results to classical equisummability involving Sturm–Liouville theory, and to equiconvergence for parabolic equations [Be2].

This study originated in a desire to understand better the behavior of differences of resolvents of elliptic operators, or, equivalently, of their semigroups. This effort has been particularly useful in index theory for Dirac and other operators (see, e.g., [Ca]). Specifically, if $D_0$ is a differential operator on vector-valued functions and $D$ is a perturbation, differences $e^{-tD} - e^{-tD_0}$ have been of interest. In [Ca], the ergodic limit $t \to \infty$ is studied (see [HP]). This paper effectively considers the opposite ergodic limit, $t \to 0$, which has applications to equisummability theory of elliptic operators.

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Ordinary summation of multidimensional elliptic eigenfunction expansions is known to fail in $L^p$ ($p \neq 2$) [F, KST], and the Riesz method is limited in the range of $p$ values for which it is effective [KST]. It is therefore natural to study summability with respect to the widely studied class of analytical methods.

The motivation for the study of equisummability is old and rather simple: it is easier to study expansions in eigenfunctions of unperturbed operators (like the Laplacian) than of perturbed ones (like Sturm–Liouville or Schrödinger operators). Classical work in this area began early in this century with the work of Haar [H], and Walsh [W], who showed that the difference in expansions with respect to eigenfunctions of a Sturm–Liouville operator and the ordinary Fourier series tends to zero everywhere (equiconvergence). Stone [St1] and Tamarkin [Ta] studied analogous questions for so-called Birkhoff series, i.e., expansions in eigenfunctions of higher order differential operators on intervals.

Such questions have been studied more recently as well. Levitan [LS] has showed uniform equiconvergence on finite intervals (with Fourier transforms) of singular Sturm–Liouville expansions on $[0, \infty)$. Komornik [K] has generalized this to non-self-adjoint operators (Schrödinger operators with complex potentials).

In some one-dimensional cases where such equiconvergence fails, Stone [St2] showed that equisummability is its appropriate replacement. Expansions with respect to two orthogonal systems are equisummable if the termwise differences of the two expansions converge in some topology to 0 under a given summability method. Questions of equisummability have also been studied by Titschmarsh [Ti], Levitan [LS], and more recently by Benzinger [Be1]. The latter showed equisummability for Riesz typical means of expansions in eigenfunctions of ordinary differential operators. He also used equisummability [Be2] to study convergence to initial values in the heat equation.

In higher dimensions, Il'in [II] studied Riesz equisummability of expansions in eigenfunctions of the Dirichlet Laplacian in a domain, with Fourier integral expansions on $\mathbb{R}^n$. Equisummability of certain trigonometric series containing irregular frequencies with Fourier series was studied by Sedletskii. A survey of equisummability results in the Soviet literature up to 1982 is given in [Go]. Gurarie and Kon [GK2] began the study of this in a more abstract and general way by use of the operator calculus formulation of summability. Raphael [Ra] has given a simple proof of Gurarie and Kon's equisummability results.

Let $X$ be a measure space, and $A, B$ be linear operators on $L^p(X)$, $1 \leq p \leq \infty$. Let $f \in L^p(X)$, and $\phi$ be a function analytic on the spectrum of $A, B$, with $\phi(0) = 1$. Note that under the Dunford operator calculus, $\phi(A)f$ represents the $\phi$-summability means of the expansion of $f$ with respect to
The question of equisummability is: When does convergence (as \( \varepsilon \to 0 \)) of \( \phi(\varepsilon A)f \) imply that of \( \phi(\varepsilon B)f \)? Specifically, we say that eigenfunction expansions with respect to \( A \) and \( B \) are \( \phi \)-equisummable from \( L^p \) to \( L^q \) if for \( f \in L^p \)

\[
\phi(\varepsilon A)f - \phi(\varepsilon B)f \xrightarrow[\varepsilon \to 0]{} 0
\]

in \( L^q \). If \( q = \infty \), the expansions are pointwise equisummable. Our strategy is to study first the case when \( \phi(\varepsilon A) \) represents the resolvent, i.e., \( \phi(\varepsilon A) = \zeta (\xi - A)^{-1}, (\zeta = -1/\varepsilon) \). In this case we employ a sharp bound on the integral kernel of the difference \( (\xi - A)^{-1} - (\xi - B)^{-1} \). The rest of the argument follows from the integral representation

\[
\phi(\varepsilon A) = \frac{1}{2\pi i} \int_{\Gamma} \phi(\varepsilon \xi)(\xi - A)^{-1} d\xi,
\]

where \( \Gamma \subset C \) is a contour enclosing the spectrum \( \sigma(A) \) of \( A \).

We briefly remark on the connection between \( \phi(\varepsilon A)f \) and the spectral expansion of \( f \). Formally, if \( \{u_\gamma \}_{\gamma \in \sigma(A)} \) is a complete set of eigenfunctions of \( A \), then if

\[
f(x) \sim \int_{\sigma(A)} u_\gamma(x) F(\gamma) d\rho(\gamma)
\]

we have

\[
\phi(\varepsilon A)f(x) \sim \int_{\sigma(A)} \phi(\varepsilon \gamma) u_\gamma(x) F(\gamma) d\rho(\gamma).
\]

The advantage of the operator calculus approach to equisummability is its insensitivity to detailed spectral considerations, or even to the existence of a complete spectrum. For example, the approach addresses non-self-adjoint problems as readily as self-adjoint ones. The method is a strict generalization of the classical one.

The construction of eigenfunction expansions is in general quite difficult. Two approaches to this are recently detailed by Agmon [Ag]; they are based on the treatment of the Schrödinger operator as a perturbation of the Laplacian. Such techniques lend themselves to the explicit verification of the results below, at least for Schrödinger operators with short range potentials.

Some of our results have appeared (for more restricted operators) in [GK2]. The authors would like to thank P. Gilkey for his encouragement to study this problem.
1. **The Class of Operators**

We begin by presenting basic definitions and notation. Let $a = (a_1, \ldots, a_n)$ be a multiindex and

$$D^a = i^{-|a|} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}, \quad \xi^a = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}, \quad |a| = \sum_i \alpha_i.$$

An operator $A = \sum_{|\alpha| \leq m} a_{\alpha}(x) D^\alpha$ is uniformly elliptic if its leading symbol $a(x, \xi) = \sum_{|\alpha| = m} a_{\alpha}(x) \xi^\alpha$ satisfies

$$c_1 |\xi|^m \leq a(x, \xi) \leq c_2 |\xi|^m \quad (\xi \in \mathbb{R}^n)$$

with constants $c_1, c_2 > 0$ independent of $x \in \mathbb{R}^n$. Let $A_0 = \sum_{|\alpha| = m} a_{\alpha}(x) D^\alpha$ be a uniformly elliptic homogeneous operator. Let $C^\infty(\mathbb{R}^n)$ be the class of $C^\infty$ functions $f$ on $\mathbb{R}^n$ which are bounded with all their partial derivatives, i.e., $\sup_{x \in \mathbb{R}^n} |\partial^\alpha f(x)| < \infty$ for all $\alpha$. We assume the coefficients $a_\alpha$ of $A_0$ are in $C^\infty$.

The operators we consider are perturbations of $A_0$. Namely, we study $A = A_0 + B$, where $B = \sum_{|\alpha| < m} b_{\alpha}(x) D^\alpha$, with $b_{\alpha} \in L^\infty + L^\infty$ on $\mathbb{R}^n$. We must require (see [GK2]) that

$$d = \max_x \left( \frac{n}{r_x} + |x| \right) < m.$$  

More generally $\{b_{\alpha}\}$ can be defined on quotient spaces $\mathbb{R}^n/U_x$, where $U_x$ is a linear subspace of $\mathbb{R}^n$, with $b_{\alpha} \in (L^\infty + L^\infty)(\mathbb{R}^n/U_x)$ (see [GK1]). The allowable singularities in $b_{\alpha}$ are important in applications involving Schrödinger operators.

For the sequel we define the exponentially decaying radial function

$$H_{s, \gamma}(|z|) = \begin{cases} 1 + |z|^{-s} (-\ln|z|) & \text{if } s = 0; \\ |z|^{-\gamma} e^{-\gamma|z|}, & |z| > 1 \end{cases} \quad (z \in \mathbb{R}^n).$$

2. **The Main Result**

Let $A = A_0 + B$ be as above. Denote

$$R_\xi \equiv (\xi - A)^{-1}; \quad R_0^\varphi \equiv (\xi - A_0)^{-1},$$

when the resolvents exist. Define the parabolic domain (about $\mathbb{R}^+ \in C$)

$$\Omega_{k, \gamma} \equiv \{\xi = \rho e^{i\theta}; \rho^{1/m} < k |\theta|^{-\gamma} \},$$
for \( k, \tau > 0 \), and

\[
C(\zeta) = C|\theta|^{-\tau}(1 - C|\theta|^{-\tau} \rho^{-1/m})^{-1}
\]

(5)

for some \( C > 0 \).

The fundamental characteristics of uniformly elliptic operators \( A \) which we require are in \([G]\)).

**Lemma 1** (Gurarie \([G]\)). Let \( A_0 \) be a uniformly elliptic homogeneous differential operator of order \( m \) with real positive symbol, and \( D \) be any homogeneous differential operator of order \( m' < m \). Assume that \( A_0 \) and \( D \) both have coefficients in \( CB^\infty \). Then there exists a parabolic domain

\[ Q_{k,\tau} = \{ \zeta = \rho e^{i\theta} : \rho^{1/m} < k|\theta|^{-\tau} \} \]

\( k, \tau > 0 \), in the complement of which the kernel of \( K_\zeta = D(\zeta - A_0)^{-1} \) is bounded by \( L^1 \)-dilations of a radial function \( H_{x,t,\gamma} \):

\[
|K_\zeta(x,y)| \leq C_1(\zeta) \rho^{-1+(m'+m)/m}H_{x,t,\gamma}(\rho^{1/m}|x-y|) \quad (\zeta = \rho e^{i\theta}),
\]

(6)

where \( C_1(\zeta) \) has the form (5) for some \( C, s = n - m + m', t > n \) and \( \gamma = \gamma_0 \sin \theta/m \).

The proof follows a perturbation approach. The resolvent \( (\zeta - A_0)^{-1} \) of the leading term is first constructed and bounded by construction of a first order parametrix for \( \zeta - A_0 \), and solving iteratively an equation involving the full resolvent and the parametrix. The pseudodifferential calculus together with analytic continuation of symbols to tube domains about \( \mathbb{R}^n \subset C^n \) (to obtain exponential decay) is then used to complete the argument.

The main tool of this paper is

**Theorem 2.** Let \( A = A_0 + B \) be as above. There exist \( k, \tau > 0 \) depending on \( A \) such that outside \( Q_{k,\tau} \), the kernel of \( L^{(1)}_\zeta = R_\zeta - R_\zeta^0 \) is estimated by

\[
|L^{(1)}_\zeta(x,y)| \leq C(\zeta) \rho^{(d+n)/m-2}H_{x,t,\gamma}(\rho^{1/m}|x-y|) \quad (\zeta = \rho e^{i\theta}),
\]

(7)

where \( C(\zeta) \) as in (5), \( s = n - 2m + d \), \( t > n \), and \( \gamma = \gamma_0 \sin \theta/m \).

We require a lemma. For two non-negative functions \( f_1 \) and \( f_2 \) on \( \mathbb{R}^n \), define their \( p \)-convolution (\( 1 \leq p \leq \infty \)) by

\[ f_1 *_p f_2 = (f_1^p * f_2^p)^{1/p} \]

where \( * \) denotes ordinary convolution, with the sup norm instead of integral when \( p = \infty \).

**Lemma 3.** Let \( H_{x_1, t_1, \gamma_1} \) and \( H_{x_2, t_2, \gamma_2} \) be a pair of functions of type (2). Let
$1 \leq p \leq \infty$ and for some $\delta > 0$, $s_i \leq n/p - \delta$, $t_i \geq n/p + \delta$, $\gamma_i \geq \delta$ ($i = 1, 2$). Then if $s_1 + s_2 \geq n/p + \delta$,
\begin{equation}
H_{s_1, t_1, \gamma_1} *_p H_{s_2, t_2, \gamma_2} \leq C H_{s, t, \gamma}, \tag{8}
\end{equation}
where $s = s_1 + s_2 - n/p$, $t = \min(t_1, t_2)$ and $\gamma = \min(\gamma_1, \gamma_2)$. Furthermore, $C$ is independent of $s_i, t_i, \gamma_i$ for fixed $\delta > 0$.

The proof of this lemma follows along the same lines as Lemma 1 in [GK1] and Lemma 2 in [G], and is omitted.

**Proof of Theorem 2.** Some of the arguments in this proof which also appear in [GK2] are only sketched. The resolvent $R_\zeta$ (see [GK1, G]) is given by
\begin{equation}
R_\zeta = R_\zeta^0 \sum_{k=0}^{\infty} (BR_\zeta^0)^k, \tag{9}
\end{equation}
where $R_\zeta^0 = (\zeta - A_0)^{-1}$, so that
\begin{equation}
R_\zeta - R_\zeta^0 = R_\zeta^0 \sum_{k=1}^{\infty} (BR_\zeta^0)^k. \tag{10}
\end{equation}
Consider the term $R_\zeta^0 (BR_\zeta^0)^k$, $k \geq 1$. When multiplied out, this involves terms of the form
\begin{equation}
R_\zeta^0 b_{x'} D^a R_\zeta^0 b_{x'} D^a \cdots R_\zeta^0 b_{x'} D^a R_\zeta^0, \tag{11}
\end{equation}
with kernel $L_{x'}^{1, \ldots, \alpha}(x, y)$, where $\{\alpha^i\}_{i=1}^{k}$ are multiindices appearing in $B$. The term $D^a R_\zeta^0 = K_x$ has kernel bounded by
\begin{equation}
|K_x(x, y)| \leq C_x(\rho, \theta) \rho^{n/m-1} H_{s, t, \gamma}(\rho^{1/m}|x-y|), \tag{12}
\end{equation}
where $\zeta = \rho e^{i\theta}$, and $C_x, s, t, \gamma$ are as in Lemma 1. We assume
\begin{equation}
b_x(x) \in L^{r_x} \quad (|x| < m); \tag{13}
\end{equation}
this involves no loss, since the same arguments hold for $b_x \in L^\infty$, and hence for $b_x \in L^{r_x} + L^\infty$. Let $1/p_x + 1/r_x = 1$. Using a multiple Hölder's inequality on the kernel of (11), we have
\begin{equation}
|L_{x'}^{1, \ldots, \alpha'}(x, y)| \leq \prod_{i=1}^{k} \|b_{x'}\|_{r_x} |K_{\alpha'}| \ast \cdots \ast |K_{\alpha'}|, \tag{14}
\end{equation}
where the $i$th $\ast$ denotes a $p_{x'}$-convolution. By (12), Lemma 1, and a scaling identity for $p$-convolutions,
\begin{equation}
|L_{x'}^{1, \ldots, \alpha'}| \leq \prod_{i=1}^{k} \|b_{x'}\|_{r_x} C_2(\zeta)^{k+1} \rho^{n/m-1}(C_1 \rho^{d/m-1})^k H_{s, t, \gamma}(\rho^{1/m}|x-y|), \tag{15}
\end{equation}
where \( t > n \), \( y = y_0 \sin \theta/m \), and \( s \leq n - 2m + d \). The function \( C_2(\rho, \theta) \) is of the form (5), and is independent of the collection \( x^1, \ldots, x^k \). The condition on \( s \) is obtained from repeated use of Lemma 1. We sum (15) over all collections of multiindices and then all values of \( k \), obtaining a geometric series; summing the series according to (10) gives

\[
|L^{(1)}(x, y)| \leq \frac{C_2(\zeta) C_4(\zeta) \rho^{d/m - 1}}{1 - C_4(\zeta) \rho^{d/m - 1}} \rho^{(d+n)/m - 2} H_{s, r}(\rho^{1/m}|x - y|) \tag{16}
\]

where

\[
C_4(\zeta) \equiv C_3 C_2(\zeta) = C_2(1 - C_5 |\theta|^{-t} \rho^{-1/m})^{-1}.
\]

Equation (16) holds, of course, only for \( \rho \) in the complement of a parabolic domain, defined by \( \Omega_5 = \{ \zeta = \rho e^{i\theta}; C_5 |\theta|^{-t} \rho^{-1/m} < 1 \} \). Equation (16) together with its domain of validity yield, with some reorganization of constants, (7).

Remark. It should be noted that although the operator \( R \) (as defined by its perturbation series) is bounded in all \( L^p \) \((1 \leq p \leq \infty)\), it is a true resolvent of \( A \) only in \( L^p \), \( p \leq \min x_r \). For \( p > \min x_r \), \( R \) remains the left inverse of \( \zeta - A \), but the \( L^p \)-domain of \( A \) may be trivial (see remark after proof of Theorem 2 in [GK1]).

3. Equisummability under Analytic Multipliers

Let \( A \) be a uniformly elliptic operator described above. We now examine analytic functions \( \phi(A) \), defined by the Dunford calculus (see [DS]),

\[
\phi(A) = \frac{1}{2\pi i} \int_{\Gamma} R \phi(\zeta) \, d\zeta. \tag{17}
\]

The unbounded contour \( \Gamma \) encloses the spectrum \( \sigma(A) \subset \Omega_{k, r} \) and we require

\[
\int_{\Gamma} |\phi(\zeta)| \, d\mu(\zeta) < \infty, \tag{18a}
\]

where the measure is \( d\mu(\zeta) = (1/\rho) C(\zeta) \, d\zeta \), with \( C(\zeta) \) given by (5). We also assume for well-definedness that

\[
\int_{|\zeta| = C} |\phi(\zeta)| \, d\mu \xrightarrow{C \to \infty} 0 \tag{18b}
\]

where \( \Gamma^\circ \) is the interior of \( \Gamma \).
For $\theta > 0$, let $W_\theta = \{ \zeta \in C : |\arg \zeta| < \theta \}$; note $W_\theta$ contains the positive real axis. Given operators $A_0, A$ and $1 \leq p, q \leq \infty$, the eigenfunction expansions with respect to $A$ and $A_0$ are \textit{resolvent equisummable} from $L^p$ to $L^q$ (in $\Omega \subset C$) if for $f \in L^p(\mathbb{R}^n)$

$$\| \zeta (\zeta - A)^{-1} - \zeta (\zeta - A_0)^{-1} f \|_q \to 0$$

as $\zeta \to \infty$ (in $\Omega$). If $q = \infty$, they are \textit{pointwise} equisummable.

We prove an optimal resolvent equisummability theorem.

**Theorem 4.** Let $A$ be a uniformly elliptic operator as above, with $A_0$ its leading term. Let $\theta > 0$, and $W_\theta$ be as above; let $1/p - 1/q < (m - d)/n$, and $f \in L^p + L^\infty$. Then the expansions of $f$ with respect to $A_0$ and $A$ are $L^p + L^q$ resolvent equisummable in $\sim W_\theta$, i.e., in the complement of an arbitrary cone about the positive real axis. In particular, if $p > n/(m - d)$, the expansions are pointwise resolvent equisummable in $\sim W_\theta$.

\textbf{Proof.} We may assume $f \in L^p$, since the argument for $f \in L^\infty$ is similar. If $\zeta \in \sim W_\theta$ is sufficiently large, and $R_\zeta, R_0^\zeta$ are as in (3), then for $s, t, \gamma$ as in (7),

$$\| \zeta (R_\zeta - R_0^\zeta) f \|_q \leq C(\zeta) \rho^{(d + n)/m - 1} \| H_{s,t,\gamma}(\rho^{1/m} x) \ast f(x) \|_q$$

$$\leq C(\zeta) \rho^{(d + n)/m - 1} \| H_{s,t,\gamma}(\rho^{1/m} x) \|_r \| f \|_p$$

(19)

where $1/r + 1/p = 1 + 1/q$. We have $H_{s,t,\gamma} \in L^r$ if $1/r > (n - 2m + d)/n$; this is clearly satisfied if $p > n/(m - d)$. Using

$$\| g(ax) \|_r = a^{-n/r} \| g \|_r,$$

(20)

we have

$$\| \zeta (R_\zeta - R_0^\zeta) f \|_q \leq C(\zeta) \rho^{(d + n - n/r)/m - 1} \| H_{s,t,\gamma} \|_r \| f \|_p.$$  

(21)

Note that

$$\left( d + n - \frac{n}{r} \right)^{1/m - 1} - (d + n - (n - m + d))^{1/m - 1} = 0,$$

and $C(\zeta)$ is bounded in $\Omega_{k,\epsilon}$ (Thm. 2), so the right side of (21) vanishes as $\zeta \to \infty$ in $W_\theta$.

For $r > 0$, let $D_r$ be the open unit disk of radius $r$. We can now apply Theorem 4 directly to (17) (noting (18)) to obtain

**Theorem 5.** Let $A_0$ and $A$ be as in Theorem 2, and let $\theta > 0$. Let
\[ \sigma(A) \cup \sigma(A_0) \subset D_r \cup W_\theta \equiv D_{r, \theta}, \text{ and } \phi \text{ be analytic in } D_{r, \theta}, \text{ with } \phi(0) = 1 \text{ and } \phi(z) = O(z^{-\delta}) \text{ for some } \delta > 0. \]

Let

\[ \frac{1}{p} - \frac{1}{q} < \frac{m - d}{n} \]

(22)

and \( f \in L^p + L^\infty \). Then the expansions of \( f \) with respect to \( A \) and \( A_0 \) are \( \phi \)-equisummable, that is,

\[ \phi(\varepsilon A) f - \phi(\varepsilon A_0) f \to 0 \]

in \( L^q \) as \( \varepsilon \to 0 \).

We will now show the scale of \( L^p \)-spaces given in (22) is the best possible, by finding \( A_0 \) and \( A \) for which equisummability fails when (22) is violated. Thus, assume \( 1/p - 1/q > (m - d)/n \). Let \( \chi_c(x), c > 0 \) denote the characteristic function of the ball in \( \mathbb{R}^n \) of radius \( c \). We select operators \( A_0 \) and \( A \) which are not \( p, q \) equisummable as follows. Let \( A_0 \) be any constant coefficient operator of order \( m \), with positive homogeneous symbol \( a_0(\xi) \).

Let \( A = A_0 + B \), where \( B = \sum_{|\alpha| < m} b_\alpha(x) D^\alpha \) is a lower order perturbation of \( A \) as before. Recall \( b_\alpha(x) \in L^{p} + L^\infty \), and \( d = \max_{|\alpha| < m} (n/r_\alpha + |\alpha|) \). Assume the latter maximum is assumed for only one multiindex \( \alpha \), and that \( b_\alpha = \chi_1|\xi|^{-n/r_\alpha + \delta} \), for a \( \delta > 0 \); thus \( b_\alpha \in L^{m} \).

Choose \( \phi(\varepsilon A) = \zeta R_{\zeta} \), with \( \zeta = -1/\varepsilon \), and \( f(x) = \chi_1|x|^d - m n/q - \varepsilon \), with \( \varepsilon > 0 \) sufficiently small that \( f \in L^p \). Let \( K_\zeta = D^{x} R_0^\zeta \). Since \( K_\zeta \) is translation invariant, its convolution kernel obtains easily via Fourier transform \( \mathcal{F} \) (in \( \zeta \)):

\[ K_\zeta(x) = \mathcal{F}_\zeta \left[ \frac{\zeta^x}{\zeta - a_0(\xi)} \right] (x) = |\xi|^{m - 1} \mathcal{F}_\zeta \left[ \frac{(|\xi|^{-1/m} \xi)^x}{e^{ix} - a_0(|\xi|^{-1/m} \xi)} \right] (x) \]

\[ \equiv |\xi|^{m - 1} \left[ \xi K_{x, 0}(\xi^{1/m} x) \right] \]

where \( \zeta = |\xi| \ e^{ix} \), and \( K_{x, 0} \) depends on \( |\xi| \) only through its argument. We now consider the first term in the expression (10) of \( R_\zeta - R_0^\zeta = (\zeta - A)^{-1} - (\zeta - A_0)^{-1} \). Writing out \( \zeta R_0^\zeta B R_0^\zeta f \) as a sum of multiplications and convolutions by expanding \( B \), we have

\[ \zeta R_0^\zeta B R_0^\zeta f = \sum_{|\alpha| < m} \zeta |\xi|^{m - 2} K_{x, 0}(y) * (b_\alpha(|\xi|^{-1/m} y)(K_{x, 0}(y) * f(|\xi|^{-1/m} y))) |_{y = |\xi|^{1/m} x} \]

(23)
The $L^q$ norm of the $x$th term in (23) is

$$\left| \zeta \right|^{(1/m)(n/r_x + |z|) - d/m + (e - \delta)/m} \| M_\zeta(y) \|_q,$$

where

$$M_\zeta(y) = K_{0,0}(y)^\ast(\chi_{|\zeta|^{1/m}}(y)|y|^{-(n/r_x + \delta)}(K_{x,0}(y)^\ast(\chi_{|\zeta|^{1/m}}(y)|y|^{d/m - n/q - \epsilon}))).$$

Let $M_\infty(y) = \lim_\infty M_\zeta(y)$, for $\zeta \in \sim W_\theta$, $\theta > 0$. A calculation shows this limit exists in $L^q$ for $1/p - 1/q < (2m - d)/n$, which we can assume is the case. Since $\| M_\infty \|_q \not= 0$, $\| M_\zeta \|_q \not= 0$ for $\zeta \in \sim W_\theta$ sufficiently large. We now choose $\epsilon = 2\delta$, with $\delta = \inf_{x \not= x_1} (d - (n/r_x + |z|))$. By (24), every term in (23) vanishes in $L^q$ as $\zeta \to \infty$, except for the $x = x_1$ term, so that (23) diverges in $L^q$ as $\zeta \to \infty$ in $\sim W_\theta$. Note that $f \in L'$ if $1/r > (m - d + \epsilon)/n + 1/q$. Let $1/r + 1/r^* = 1 + 1/q$. Using the technique of Theorem 2 we can bound the remainder $\sum_{k=2}^\infty R_k^0(BR^0_\zeta)^k$ of the expansion (10) by a convolution kernel whose $L^r$ norm is of order $\left| \zeta \right|^{(1/m)(2d - 3m + n/r - n/q)}$. If $\epsilon$ and $1/r - ((m - d)/n + 1/q)$ are sufficiently small, the exponent is negative, so that

$$\left\| \sum_{k=2}^\infty R_k^0(BR^0_\zeta)^k f \right\|_q \xrightarrow{\zeta \to \infty} 0,$$

while, since (23) diverges in $L^q$,

$$\left\| (R_\zeta - R_0^0) f \right\|_q \xrightarrow{\zeta \to \infty} \infty,$$

so equisummability fails.

**Remarks.** Our results extend to uniformly elliptic systems, such as the Dirac operator

$$D = \left[ i \sum_{j=1}^3 \gamma_j(\partial^j + A^j(x)) - \beta \Phi(x) \right],$$

where $\gamma_j$, $\beta$ are $4 \times 4$ gamma matrices, $\partial^j = \partial/\partial x_j$, and $A$, $\Phi$ denote vector and electromagnetic potential. This gives nice bounds, for example, on the (ergodic) $t \to 0$ limit of

$$e^{-tD} - e^{-tD_0},$$

where $D_0 = i \sum_{j=1}^3 \gamma_j \partial^j$. For a study of the other ergodic limit ($t \to \infty$, useful in index theory of the Dirac operator), see [Ca].

The above statements on equisummability probably hold for a more general class of strictly elliptic operators, i.e., those for whose leading sym-
bol only the first half of the inequality (1) holds. However, kernel bounds of the type in Theorem 2 are generally much more complicated for such operators, and such results are beyond the scope of the present techniques.

4. Some Applications

(a) Heat Diffusion with Non-uniform Conductivity

Consider the heat equation

\[
Au = (-a(x) A + \vec{b}(x) \cdot \vec{\nabla} + q(x)) u = -\frac{\partial u}{\partial t}, \tag{26a}
\]

\[
u(x, 0) = u_0(x) \in (L^p + L^\infty)(\mathbb{R}^n) \tag{26b}
\]

where \(a(x) > 0\) represents a position dependent on heat conductivity, and \(\vec{b}(x)\) and \(q(x)\) are drift and dissipation terms. We assume \(a(x) \in C^B(\mathbb{R}^n), \\vec{b}(x) \in L^{n+\varepsilon} + L^\infty,\) and \(q(x) \in L^{n+1+\varepsilon} + L^\infty,\) for some \(\varepsilon > 0.\) Theorem 6 extends some results of Benzinger \([\text{Be}2]\).

**Theorem 6.** If \(u_0 \in L^{n+\varepsilon} + L^\infty,\) then \(u(x, t) \to t \to 0\) \(u_0(x)\) at exactly the same set of points as the solution of the unperturbed problem

\[-a(x) Au = -\frac{\partial u}{\partial t}.\]

The theorem follows from the fact that \(e^{-tAu}u_0\) solves (26), and the semigroup falls into the class of multipliers \(\phi\) in Theorem 5.

(b) Recovery of Transformed Functions

Consider the Sturm–Liouville operator \(A = -d^2/dx^2 + q(x),\) where \(q \in (L^1 + L^\infty)(\mathbb{R})\) is real and continuous. For real \(\lambda,\) let \(u_1, u_2\) solve \(Au_i = \lambda u_i\) with \((u_1(0), u_1'(0)) = (0, 1)\) and \((u_2(0), u_2'(0)) = (1, 0).\) Then we can expand \(f(x)\) in the Sturm–Liouville expansion

\[
f \sim \int_{-\infty}^{\infty} u_1(x, \lambda) d\rho_1(\lambda) + \int_{-\infty}^{\infty} u_2(x, \lambda) d\rho_2(\lambda), \tag{27}\]

where \(\rho_1\) and \(\rho_2\) combine the spectral functions and generalized Fourier transform of \(f;\) the equality holds in \(L^2.\) If \(q = 0,\) we have the Fourier transform:

\[
u_1(x, \lambda) = \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}}, \quad \nu_2(x, \lambda) = \frac{\cos \sqrt{\lambda} x}{\sqrt{\lambda}},
\]

\[
d\rho_1(\lambda) = i(\hat{f}(\sqrt{\lambda}) - \hat{f}(-\sqrt{\lambda})) \sqrt{\lambda} d\sqrt{\lambda},
\]

\[
d\rho_2(\lambda) = (\hat{f}(\sqrt{\lambda}) + \hat{f}(-\sqrt{\lambda})) \sqrt{\lambda} d\sqrt{\lambda} \quad (\lambda \geq 0),
\]
where \( \hat{f} \) is the Fourier transform of \( f \). If \( \phi \) is an analytic multiplier satisfying the conditions of Theorem 5, then \( \phi \) can be used according to standard summability techniques to recover \( f \) pointwise from \( \hat{f} \); in fact, standard harmonic analysis shows that

\[
\int_{-\infty}^{\infty} \phi(\varepsilon k^2) \hat{f}(k) e^{-ikx} \, dk \xrightarrow{\varepsilon \to 0} f(x)
\]

almost everywhere. Theorem 5 provides a relation between (27) and (28), the latter simply being an eigenfunction expansion with respect to the leading term \(-d^2/dx^2\): the summation

\[
\int_{-\infty}^{\infty} \phi(\varepsilon \lambda) [u_1(x, \lambda) d\rho_1(\lambda) + u_2(x, \lambda) d\rho_2(\lambda)] \xrightarrow{\varepsilon \to 0} f(x) \tag{29}
\]

at exactly the same points as the Fourier transform (28) \( \phi \)-sums to \( x \). This is an extension to analytic multipliers of some classical and more recent results \([St1, St2, LS]\) on equisummability of expansions in eigenfunctions of ordinary differential operators, and Fourier series.

REFERENCES


