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10 Points in Dimension 4 not Projectively Equivalent to the Vertices of a Convex Polytope

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Using oriented matroids, and with the help of a computer, we have found a set of 10 points in $\mathbb{R}^4$ not projectively equivalent to the vertices of a convex polytope. This result confirms a conjecture of Larman [6] in dimension 4.

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PROBLEM (McMullen [6]). Determine the largest integer $n = f(d)$ such that for any given n points in general position in $\mathbb{R}^d$ there is an admissible projective transformation mapping these points onto the vertices of a convex polytope.

Here admissible means that none of the n points is sent to infinity by the projective transformation.

For dimension two and three the numbers $f(d)$ are known: $f(2) = 5$ and $f(3) = 7$. For $d \geq 2$, Larman has established in [6] the bounds $2d + 1 \leq f(d) \leq (d + 1)^2$, and conjectured that $f(d) = 2d + 1$. The upper bound has been improved to $f(d) \leq (d + 1)(d + 2)/2$ by Las Vergnas [7], as a corollary of Redei's theorem for tournaments. Recently, Ramírez Alfonsín [8] has proven the linear upper bound $f(d) \leq 5d/2 + 1$, by a construction using Lawrence oriented matroids (unions of rank 1 oriented matroids).

In the context of oriented matroids the problem can be conveniently restated in terms of hyperplanes. We refer the reader to [1] for information regarding oriented matroid theory. As easily seen, the oriented matroids of the images of a given configuration of points by admissible projective tranformations are all the acyclic reorientations of the oriented matroid defined by the affine dependencies of the configuration. The dual of a configuration of points is an arrangement of hyperplanes, and the regions defined by this arrangement are in 1–1 correspondence with the acyclic reorientations of the oriented matroid. We say that a region which meets all hyperplanes in dimension $d - 1$ is *complete*. It is almost immediate to verify that a region is complete if and only if all corresponding admissible projective transformations maps the given n points onto the set of vertices of convex polytopes (note that these convex polytopes necessarily have the same oriented matroid).

Hence the McMullen problem is equivalent to: *determine the largest integer $n = f(d)$ such that any arrangement of n hyperplanes in general position in $\mathbb{R}^d$ contains a complete region.*

The same problem for general oriented matroids has been considered by Cordovil and Da Silva [4]: *determine the largest integer $n = g(r)$ such that any uniform rank r oriented matroid M with n elements has a complete region.* A *region* (or *tope*) of an oriented matroid is a region determined by the pseudohyperplanes of its topological representation. The regions of an oriented matroid are in 1–1 correspondence with its maximal covectors, and a region is complete if and only if changing the sign of any element in the corresponding maximal covector produces another maximal covector. Obviously $g(r) \leq f(r + 1)$. Cordovil and Da Silva have shown in [4] that $2r - 1 \leq g(r)$, generalizing Larman's lower bound.

In this paper, we construct uniform rank 5 oriented matroids on 10 elements without complete region, hence, $g(5) = 9$. One of these oriented matroids has a realization in R^4, hence $f(4) = 9$.

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As a preliminary step for the rank 5 case, using a computer, we have gone through the complete list of all 2628 reorientation classes of uniform rank 4 oriented matroids on eight elements, as per the work of Bokowski and Richter-Gebert [2].

PROPOSITION 1. *There are precisely* 114 *non-isomorphic reorientation classes of uniform rank* 4 *oriented matroids on eight elements without complete region. One such reorientation class has only mutants without complete region. Two of them are not realizable.*

The unique realizable uniform rank 4 oriented matroid on eight elements without complete region, such that all its mutants are also without complete region, has the following base signature (or chirotope):

$$+++++++++++++++++++++----+--------++++$$
$$-+-----+--------++--------++++++$$

THEOREM 2. *There is a set of* 10 *points of* $\mathbb{R}^4$ *in general position such that:*

- *there is no admissible projective transformation mapping these points onto the vertices of a convex polytope, or, equivalently,*
- *the corresponding uniform oriented matroid has no complete region.*

The theorem means that $f(4) = g(5) = 9$.

PROOF. Using a computer, it can be checked that the oriented matroid of affine dependencies of the following 10 points of $\mathbb{R}^4$ has no complete region.

1	0.7702	0.2217	−6.3645	0
2	0.7426	0.2284	−6.3977	0
3	0.6	1.01	−5.44	0
4	1.75	7.07	−0.45	0
5	−2	2	2	1
6	2	−2	2	1
7	2	2	−2	1
8	−2	−2	−2	1
9	−2.44	−2.13	1.4	1.71
10	0.35	1.77	−0.38	1.011

□

The signature of the 252 bases of this uniform rank 5 oriented matroid on 10 elements is:

$$++++++--+---+--+++---------+--+++----+--++$$
$$+---+---+-+++-+--+-----++++--+-+++--+-++++$$
$$---++++++++-----++++---+++++++---++++---+-$$
$$+++++-----++++-+-++---++++++---++++++++---$$
$$-+++++---+++++++---++++---+-++++----+++++-$$
$$--+++++++---++++---+-+++---++++----+--+-+-$$

Its face lattice is that of a stacked 4-cross-polytope with 19 facets (we recall that a *stacked* polytope is obtained by the addition of new vertices building shallow pyramids over facets):

1234	1238	1247	1278	1346	1368	1467	1678	2345	2358
2457	2578	3456	3568	4567	5679	5689	5789	6789	

The point 9 is stacked on the 4-cross-polytope by the vertices $1, \ldots, 8$ and the point 10 lies inside the convex hull of the points 5, 6, 7, 8 and 9. The vertices 5, 6, 7 and 8 form a regular tetrahedron. The computer program provides the number of regions adjacent to each of the 256 regions of the oriented matroid: there are 16 with five neighbours, 57 with six neighbours, 72 with seven neighbours, 65 with eight neighbours, 46 with nine neighbours and 0 with 10 neighbours.

We now explain how we arrived to our example. Since a list of all reorientation classes of uniform rank 5 oriented matroids on 10 elements does not exist we cannot use exhaustion as in the rank 4 case.

We start with the list of 135 reorientation classes of uniform rank 5 oriented matroids on eight elements [2, 3]. From this list we can generate the 3501 non-isomorphic matroid polytopes of rank 5 with eight vertices. The face lattices of these matroid polytopes are the 37 3-spheres with eight vertices described by Grünbaum and Sreedharan [5].

For any such matroid polytope P and any disjoint pair of facets f_1, f_2 of its face lattice we generate a partial uniform rank 5 oriented matroid M on 10 elements as follows. The face lattice of M is a stacked 3-sphere where the vertex 9 is stacked on f_1 in the 3-sphere P. The element 10 of M is an interior element with a special relationship to some of its combinatorial hyperplanes. The facets f_1 and f_2 each have four elements. Let H_{31} resp. H_{13} be a combinatorial hyperplane with three elements of f_1 and 1 of f_2 resp. three elements of f_2 and one of f_1. Then the element 10 lies on the same side of H_{31} as the element of $f_1 \setminus H_{31}$ and on the same side of H_{13} as the element of $f_2 \setminus H_{13}$. In this way we can construct 18 872 partial oriented matroids. Starting from these partial oriented matroids, we generate 1112 uniform rank 5 oriented matroids on 10 elements without complete region. They lie in 414 reorientation classes. If we build the mutants of these oriented matroids, we come up to 465 non-isomorphic reorientation classes of oriented matroids without complete region. None of them has all its mutants without complete region.

THEOREM 3. *There are at least* 465 *non-isomorphic reorientation classes of uniform rank* 5 *oriented matroids on* 10 *elements without complete region.*

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