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On Dihedral Configurations and their Coxeter Geometries

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Within the theory of homogeneous coherent configurations, the dihedral configurations play the role which is played by the finite dihedral groups in the theory of finite groups. Imitating Tits' construction of a geometry from a set of subgroups of a given group, we assign a geometry of rank 2 to each dihedral configuration, its 'Coxeter geometry'. (Each finite generalized polygon is a Coxeter geometry in this sense.)

Apart from general results on the relationship between dihedral configurations and their Coxeter geometries, we settle completely the (ordinary) representation theory of the dihedral configurations of rank 7. We obtain three major classes. The Coxeter geometries of the first class are exactly the non-symmetric 2-designs with $\lambda = 1$. The other two classes lead to questions which require a further combinatorial treatment.

1. INTRODUCTION

Let (X, G) be a homogeneous coherent configuration [2].

For all $E, F \subseteq G$, we define

$$EF := \bigcup_{e \in E} \bigcup_{f \in F} \{g \in G \mid a_{efg} \neq 0\},$$

where a_{efg} denotes the intersection number of e, f and g given by (X, G).

A subset F of G is said to be *closed* if $FF \subseteq F \neq \emptyset$.

We shall denote by $\mathscr{C}(G)$ the set of all closed subsets of G.

For each subset F of G, we define

$$\langle F \rangle := \bigcap_{F \subseteq H \in \mathscr{C}(G)} H.$$

An element $g \in G$ will be called a *generalized involution* if $|\langle g \rangle| = 2$.†

The set of generalized involutions of G will be denoted by $\mathrm{Inv}(G)$.

The pair (X, G) will be called *dihedral* if there exists $L \subseteq \mathrm{Inv}(G)$ such that $|L| = 2$ and $\langle L \rangle = G$.

For each $x \in X$ and, for each $F \subseteq G$, we define

$$xF := \bigcup_{f \in F} \{y \in X \mid (x, y) \in f\}.$$

It is easy to prove that, for each $H \in \mathscr{C}(G)$,

$$X/H := \{xH \mid x \in X\}$$

is a partition of X; see [5, (1.1)]. In particular, for each $\mathscr{H} \subseteq \mathscr{C}(G)$,

$$(X, \{X/H \mid H \in \mathscr{H}\})$$

is a chamber system in the sense of [4, Section 2.1]. We shall denote by $\mathscr{G}(X, \mathscr{H})$ the geometry associated with this chamber system via [4, Section 2.2].

† For each $g \in G$, we abbreviate $\langle g \rangle := \langle \{g\} \rangle$.

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Now assume that there exists $L \subseteq \mathrm{Inv}(G)$ such that $|L| = 2$ and $\langle L \rangle = G$. We shall say that (X, G) is *degenerate* if, for all $h, k \in L$, $hk = kh$.† We shall call $\mathscr{G}(X, \{\langle l \rangle \mid l \in L\})$ the *Coxeter geometry of* (X, G) (*with respect to* L).

It is easy to see that the rank of a non-degenerate dihedral configuration is at least 6. (This follows immediately from [2, (4.1)].) But, already, the rank 6 case seems to be hard. The Coxeter geometry of a non-degenerate dihedral configuration of rank 6 is a generalized triangle of order n if n denotes the subdegree corresponding to one (and hence both) of the generating generalized involutions; see Theorem 3.3. Conversely, a straightforward computation shows that each projective plane is the Coxeter geometry of such a dihedral configuration. Therefore it is impossible to classify the non-degenerate dihedral configurations of rank 6.

On the other hand, the intersection numbers of a non-degenerate dihedral configuration of rank 6 are uniquely determined by the (identical) subdegrees corresponding to the two generating generalized involutions.

For non-degenerate dihedral configurations of rank 7, we obtain the same result if the subdegrees in question are different; see Theorem 4.4. The case in which these subdegrees are equal will be the subject of Theorem 4.5 and Theorem 4.6. It raises a number of interesting questions, which cannot be answered with the help of the representation-theoretical approach of this paper.

The notation of this paper is essentially that of [2]; but, for each $g \in G$, we define

$$g^* := \{(y, z) \mid (z, y) \in g\},$$

and we set

$$1 := \{(x, x) \mid x \in X\}.$$

Recall that, for all $d, e, f \in G$,

$$a_{def} = a_{e^*d^*f^*}$$

and

$$a_{def} n_f = a_{fe^*d} n_d.$$

These fundamental equations will frequently be applied without explicit reference. For a proof, see [2, (2.15)(b)] and [2, (2.16)(b)].

2. Preliminaries

In this section, (X, G) will be a homogeneous coherent configuration.

Let V denote the free $\mathbb{C}$-module based on X. For each $g \in G$, we shall denote by σ_g the (unique) vector-space endomorphism of V such that, for each $x \in X$,

$$x\sigma_g = \sum_{y \in xg} y.$$

We shall denote by $\mathbb{C}[G]$ the subalgebra of $\mathrm{End}_{\mathbb{C}}(V)$ generated by $\{\sigma_g \mid g \in G\}$.

Lemma 2.1. (i) *For all* $e, f \in G$,

$$\sigma_e \sigma_f = \sum_{g \in G} a_{efg} \sigma_g.$$

(ii) $\{\sigma_g \mid g \in G\}$ *is a* $\mathbb{C}$-*basis of* $\mathbb{C}[G]$.

For a proof of Lemma 2.1, see [2, (2.6)].

† For all $e, f \in G$, we abbreviate $ef := \{e\}\{f\}$.

The next five results are consequences of [2, (3.2)], [5, (4.5)], [5, (4.10)], [5, (4.7)] and [5, (4.8)], respectively.

LEMMA 2.2. *There exists an algebra homomorphism* $\chi_1: \mathbb{C}[G] \to \mathbb{C}$ *such that, for each* $g \in G$,

$$\chi_1(\sigma_g) = n_g.$$

The linear character χ_1 of $\mathbb{C}[G]$ is usually called the *principal* character of $\mathbb{C}[G]$.

LEMMA 2.3. *For each non-principal irreducible character* χ *of* $\mathbb{C}[G]$,

$$\sum_{g \in G} \chi(\sigma_g) = 0.$$

LEMMA 2.4. *Let* $F \subseteq G$ *be such that* $\langle F \rangle = G$, *and let* λ *be a linear character of* $\mathbb{C}[G]$ *such that, for each* $f \in F$, $\lambda(\sigma_f) = n_f$. *Then* $\lambda = \chi_1$.

Let $H \in \mathscr{C}(G)$ be given. For each $g \in G$, we shall write gH instead of $\{g\}H$. We set

$$G/H := \{gH \mid g \in G\}.$$

The elements of G/H will be called *left cosets* of H in G. It is easy to prove that G/H is a partition of G; see [5, (1.1)].

LEMMA 2.5. *Let* $H \in \mathscr{C}(G)$ *be given, and define*

$$\mathbb{C}_H := \bigcap_{h \in H} \{\sigma \in \mathbb{C}[G] \mid \sigma\sigma_h = n_h\sigma\}.$$

Then $|G/H| = \dim_{\mathbb{C}}(\mathbb{C}_H)$.

LEMMA 2.6. *Let* $l \in \mathrm{Inv}(G)$ *be given. Then we have the following.*

(i) $\sigma_l^2 = n_l 1 + (n_l - 1)\sigma_l$.

(ii) *Let* W *be a* $\mathbb{C}[G]$*-module. Then* W *is the sum of* $\{w \in W \mid w\sigma_l = -w\}$ *and* $\{w \in W \mid w\sigma_l = n_l w\}$.

3. GENERAL RESULTS AND THE RANK 6 CASE

Let (X, G) be a homogeneous coherent configuration. For each $H \in \mathscr{C}(G)$, we define

$$G//H := \{HgH \mid g \in G\}.$$

(Note that, for all $D, E, F \subseteq G$, $(DE)F = D(EF)$. In particular, HgH is a well-defined subset of G.)

THEOREM 3.1. *Let* (X, G) *be a homogeneous coherent configuration. Let* $h, k \in \mathrm{Inv}(G)$ *be such that* $\langle h, k \rangle = G$ *and* $hk \neq kh$.†

Then $\mathscr{G}(X, \{\langle h \rangle, \langle k \rangle\})$ *is a tactical configuration of type* $(n_h + 1, n_h + 1)$ *or a 2-design with point set* $X/\langle k \rangle$ *if, and only if* $|G//\langle k \rangle| = 2$.‡

† For all $e, f \in G$, we abbreviate $\langle e, f \rangle := \langle \{e, f\} \rangle$.

‡ The geometric terminology here is borrowed from [1].

In this case, if $J := hk \cap kh$, *then*

$$1 + \sum_{j \in J} a_{jkh}$$

is the number of blocks on two points.

Proof. The equivalence is obvious. Therefore, we assume that $|G//\langle k\rangle| = 2$. Let $(y, z) \in h$ be given.

Let $j \in J$ be given, and let $w \in yj \cap zk$ be given.† Then, as $\langle h\rangle \cap \langle k\rangle = \{1\}$, $|yk \cap wh| = 1$.

Conversely, for each $(v, w) \in h$ with $v \in yk$ and $w \in zk$, there exists $j \in J$ such that $(y, w) \in j$. □

Proposition 3.2. *Let* (X, G) *be a homogeneous coherent configuration. Let* $h, k \in \mathrm{Inv}(G)$ *be such that* $hk \setminus kh \neq \emptyset$. *Let* $m \in hk \setminus kh$ *be given, and define*

$$J := G \setminus \{1, h, k, m, m^*\}.$$

Assume that $J \subseteq hk$. *Then we have the following.*

(i) *The left cosets of* $\langle h\rangle$ *in G are*

$$\langle h\rangle, \qquad \{k, m^*\} \cup J, \qquad \{m\}.$$

(ii) $2 \leqslant |\{a_{hmj} \mid j \in J\}|$.

Proof. (i) Since $m \in hk$, $a_{hkm} \neq 0$. Therefore, $a_{khm^*} \neq 0$, whence

$$k\langle h\rangle = m^*\langle h\rangle.$$

Similarly, as $m \notin kh$,

$$k\langle h\rangle \neq m\langle h\rangle.$$

Also, since $J \subseteq hk$,

$$k\langle h\rangle = j\langle h\rangle$$

for all $j \in J$. (Note that, for each $j \in J$, $j^* \in J$.)

(ii) Let us assume, by way of contradiction, that there exists $a \in \mathbb{N}$ such that, for each $j \in J$,

$$a = a_{hmj}.$$

Set

$$\sigma := \sum_{j \in J} \sigma_j.$$

From (i) we know that $m \notin m^*\langle h\rangle$. Therefore, $a_{m^*hm} = 0$. Thus, by Lemma 2.1(i),

$$\sigma_h\sigma_m = a_{hmk}\sigma_k + a_{hmm}\sigma_m + a\sigma$$

and

$$\sigma_{m^*}\sigma_h = a_{hmk}\sigma_k + a_{hmm}\sigma_{m^*} + a\sigma. \tag{1}$$

It follows that

$$\sigma_h\sigma_m - \sigma_{m^*}\sigma_h = a_{hmm}(\sigma_m - \sigma_{m^*}). \tag{2}$$

† For each $x \in X$ and, for each $g \in G$, we abbreviate $xg := x\{g\}$.

From $J \subseteq hk$, $m \in hk$ and $m^* \notin hk$ we conclude that

$$\sigma_h \sigma_k = \sigma_m + \sigma$$

and that

$$\sigma_k \sigma_h = \sigma_{m^*} + \sigma. \tag{3}$$

It follows that

$$\sigma_h \sigma_k - \sigma_k \sigma_h = \sigma_m - \sigma_{m^*}. \tag{4}$$

Let W be an irreducible $\mathbb{C}[G]$-module on which σ_h and σ_k do not commute. Then, by Lemma 2.6(ii), there exists $w \in W \setminus \{0\}$ such that

$$w\sigma_h = -w. \tag{5}$$

From (i) and (5) we obtain that

$$a_{hm^*m^*} w\sigma_{m^*} = w\sigma_h \sigma_{m^*} = -w\sigma_{m^*}.$$

But $a_{hm^*m^*} \neq -1$. Therefore, $w\sigma_{m^*} = 0$. Now, by (1),

$$0 = a_{hmk} w\sigma_k + aw\sigma.$$

Thus, by (3),

$$0 = a_{hmk} w\sigma_k + aw\sigma_k \sigma_h.$$

However, $a_{hmk} \geqslant 0 \leqslant a$. Thus, by Lemma 2.6(ii), we conclude that

$$(w\sigma_k)\sigma_h = -w\sigma_k.$$

From $|G/\langle h\rangle| = 3$ and Lemma 2.5 we deduce that $\dim_{\mathbb{C}}(W) = 2$. Therefore, by (5), $\mathbb{C}w$ is σ_k-invariant. Now we conclude that W possesses a basis $\{u, v\}$, say, such that

$$u\sigma_k = n_k u, \qquad v\sigma_h = n_h v.$$

Let δ denote the matrix representation of $\mathbb{C}[G]$ afforded by $\{u, v\}$. Then, without loss of generality, we may assume that

$$\delta(\sigma_h) = \begin{pmatrix} -1 & e \\ 0 & n_h \end{pmatrix}, \qquad \delta(\sigma_k) = \begin{pmatrix} n_k & 0 \\ 1 & -1 \end{pmatrix},$$

where $e \in \mathbb{C}$ has to be chosen suitably.

Let $c_1, c_2, c_3, c_4 \in \mathbb{C}$ be such that

$$\delta(\sigma_m) = \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix}.$$

Then, by (i) and Lemma 2.2,

$$n_h \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix} = \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix} \begin{pmatrix} -1 & e \\ 0 & n_h \end{pmatrix} = \begin{pmatrix} -c_1 & c_1 e + c_3 n_h \\ -c_2 & c_2 e + c_4 n_h \end{pmatrix}.$$

Since $n_h \neq -1$, this implies that $c_1 = 0 = c_2$. Interchanging the roles of h and k in (i), we obtain, similarly, that

$$n_k \begin{pmatrix} 0 & c_3 \\ 0 & c_4 \end{pmatrix} = \begin{pmatrix} n_k & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & c_3 \\ 0 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & n_k c_3 \\ 0 & c_3 - c_4 \end{pmatrix}.$$

Thus $c_3 = (n_k + 1)c_4$, whence

$$\delta(\sigma_m) = \begin{pmatrix} 0 & (n_k + 1)c_4 \\ 0 & c_4 \end{pmatrix}.$$

Similarly, we find $d \in \mathbb{C}$ with

$$\delta(\sigma_{m^*}) = \begin{pmatrix} ed & 0 \\ (n_h+1)d & 0 \end{pmatrix}.$$

Now (4) yields

$$\begin{pmatrix} e & -(n_k+1)e \\ n_h+1 & -e \end{pmatrix} = \begin{pmatrix} -ed & (n_k+1)c_4 \\ -(n_h+1)d & c_4 \end{pmatrix}.$$

Thus, $c_4 = -e$ and $d = -1$. It follows that

$$\delta(\sigma_m) = \begin{pmatrix} 0 & -(n_k+1)e \\ 0 & -e \end{pmatrix}, \qquad \delta(\sigma_{m^*}) = \begin{pmatrix} -e & 0 \\ -(n_h+1) & 0 \end{pmatrix}.$$

Now, by (2),

$$\begin{pmatrix} -e & (n_k+1)e \\ -(n_h+1) & e \end{pmatrix} = a_{hmm}\begin{pmatrix} e & -(n_k+1)e \\ n_h+1 & -e \end{pmatrix},$$

contrary to $0 \leq a_{hmm}$. □

THEOREM 3.3. *Let (X, G) be a homogeneous coherent configuration. Let $h, k \in \mathrm{Inv}(G)$ be such that $\langle h, k\rangle = G$ and $hk \neq kh$.*

Then $6 \leq |G|$ and, if $|G| = 6$, then $n_h = n_k$ and $\mathcal{G}(X, \{\langle h\rangle, \langle k\rangle\})$ is a generalized triangle of order n_h.

PROOF. Since $hk \neq kh$, $hk \setminus kh \neq \emptyset$. Let $m \in hk \setminus kh$ be given. Note that $m^* \neq m$. Obviously, $\{1, h, k, m, m^*\} \neq G$. Therefore, $6 \leq |G|$.

Assume that $|G| = 6$. Then there exists $j \in G$ such that

$$\{1, h, k, m, m^*, j\} = G.$$

From Proposition 3.2(ii) we obtain that $j \notin hk$. Therefore, we have

$$hk = \{m\}, \qquad kh = \{m^*\}, \qquad hkh = \{j\} = khk.$$

Now the result follows from Theorem 3.1. □

4. ON DIHEDRAL CONFIGURATIONS OF RANK 7

For the first three lemmata of this section, (X, G) will denote a non-degenerate dihedral configuration of rank 7.

Let $h, k \in \mathrm{Inv}(G)$ be such that $\langle h, k\rangle = G$ and $hk \neq kh$. Let $m \in hk \setminus kh$ be given. Since $|G| = 7$, there exist $j, l \in G$ such that

$$\{1, h, k, m, m^*, j, l\} = G.$$

The non-principal linear characters of $\mathbb{C}[G]$ will be denoted by χ_2 and χ_3.

LEMMA 4.1. *We have $\{j, l\} \not\subseteq hk$.*

PROOF. Assume, by way of contradiction, that $\{j, l\} \subseteq hk$. Then, by Proposition 3.2(i) and Lemma 2.2,

$$\sigma_m\sigma_h = n_h\sigma_m, \qquad \sigma_h\sigma_{m^*} = n_h\sigma_{m^*}. \tag{6}$$

Let $i \in \{2, 3\}$ be given. By Proposition 3.2(i), $|G/\langle h\rangle| = 3$. Thus, by Lemma 2.5 and Lemma 2.6(ii), we must have

$$\chi_i(\sigma_h) = -1 = \chi_i(\sigma_k).$$

Since $n_h \neq -1 = \chi_i(\sigma_h)$, (6) yields

$$\chi_i(\sigma_m) = 0 = \chi_i(\sigma_{m^*}).$$

But now we may use Proposition 3.2(i) to obtain

$$a_{hmk} = a_{hmj}\chi_i(\sigma_j) + a_{hml}\chi_i(\sigma_l).$$

On the other hand, by Lemma 2.3,

$$1 = \chi_i(\sigma_j) + \chi_i(\sigma_l).$$

Since $\chi_2 \neq \chi_3$, we conclude that $a_{hmj} = a_{hml}$, contrary to Proposition 3.2(ii). □

LEMMA 4.2. *Assume that $\{j, l\} \cap hk = \varnothing$. Then we have the following.*
(i) $\sigma_h\sigma_k = \sigma_m$ *and* $\sigma_k\sigma_h = \sigma_{m^*}$.
(ii) *Assume that $n_h \leqslant n_k$. Then $\sigma_h\sigma_k\sigma_h \in \{\sigma_j, \sigma_l, \sigma_j + \sigma_l\}$ and $\sigma_k\sigma_h\sigma_k = \sigma_j + \sigma_l$.*

PROOF. (i) Follows from the choice of m.
(ii) From (i) we deduce that

$$\sigma_h\sigma_k\sigma_h = \sigma_{mhj}\sigma_j + a_{mhl}\sigma_l, \qquad \sigma_k\sigma_h\sigma_k = a_{kmj}\sigma_j + \sigma_{kml}\sigma_l \tag{7, 8}$$

and

$$\{a_{mhj}, a_{mhl}, a_{kmj}, a_{kml}\} \subseteq \{0, 1\}. \tag{9}$$

Assume first that

$$a_{mhj} = 1 = a_{kmj}, \qquad a_{mhl} = 0 = a_{kml}.$$

Then, by (7) and (8),

$$a_h\sigma_k\sigma_h = \sigma_j = \sigma_k\sigma_h\sigma_k.$$

In particular, for each $i \in \{2, 3\}$, $\chi_i(\sigma_h) = \chi_i(\sigma_k)$. But $\langle h, k\rangle = G$. Therefore, by Lemma 2.6(ii) and Lemma 2.4, we must have $\chi_i(\sigma_h) = -1 = \chi_i(\sigma_k)$ for each $i \in \{2, 3\}$. But then Lemma 2.3 and Lemma 2.1(ii) lead to the contradiction $\chi_2 = \chi_3$.

Assume next that

$$a_{mhj} = 1 = a_{kml}, \qquad a_{mhl} = 0 = a_{kmj}.$$

Then, by (7) and (8),

$$\sigma_h\sigma_k\sigma_h = \sigma_j, \qquad \sigma_k\sigma_h\sigma_k = \sigma_l.$$

In particular,

$$\chi_2(\sigma_j) = \chi_2(\sigma_h)^2\chi_2(\sigma_k), \qquad \chi_2(\sigma_l) = \chi_2(\sigma_h)\chi_2(\sigma_k)^2.$$

Using (i), Lemma 2.6(ii) and Lemma 2.3, it is easy to obtain a contradiction from these two equations.

The claim now follows from (9). □

LEMMA 4.3. *Assume that $j \in hk$. Then we have the following.*
(i) $\sigma_h\sigma_k = \sigma_m + \sigma_j$ *and* $\sigma_k\sigma_h = \sigma_{m^*} + \sigma_j$.
(ii) *Set*

$$c := \frac{n_m}{n_k}, \qquad d := \frac{n_j}{n_k}.$$

Then

$$\sigma_{m^*}\sigma_h = c\sigma_k + (c-1)\sigma_{m^*} + c\sigma_j,$$
$$\sigma_j\sigma_h = d\sigma_k + d\sigma_{m^*} + (d-1)\sigma_j,$$
$$\sigma_m\sigma_h = d\sigma_m + (d+1)\sigma_l$$

and

$$\sigma_l\sigma_h = c\sigma_m + (c-1)\sigma_l.$$

(iii) $n_h = n_k$.

Proof. (i) follows from the choice of m and from Lemma 4.1.

(ii) From (i) we deduce that $\{k, m^*, j\}$ is a left coset of $\langle h\rangle$ in G and that $j^* = j$. It follows that $l^* = l$. Thus,

$$\sigma_{m^*}\sigma_h = c\sigma_k + a_{hmm}\sigma_{m^*} + a_{hmj}\sigma_j, \tag{10}$$
$$\sigma_j\sigma_h = d\sigma_k + a_{hjm}\sigma_{m^*} + a_{hjj}\sigma_j, \tag{11}$$
$$\sigma_m\sigma_h = a_{mhm}\sigma_m + a_{mhl}\sigma_l \tag{12}$$

and

$$\sigma_l\sigma_h = a_{lhm}\sigma_m + a_{lhl}\sigma_l. \tag{13}$$

By (i) and Lemma 2.6(i), we have

$$\sigma_k\sigma_h^2 = n_h\sigma_k + (n_h-1)(\sigma_{m^*} + \sigma_j)$$

and

$$(\sigma_k\sigma_h)\sigma_h = \sigma_{m^*}\sigma_h + \sigma_j\sigma_h.$$

Thus, by (10) and (11),

$$a_{hmm} + a_{hjm} = n_h - 1 = a_{hmj} + a_{hjj}. \tag{14}$$

From (i), (12) and (11) we obtain that

$$\sigma_h(\sigma_k\sigma_h) - (\sigma_h\sigma_k)\sigma_h = \sigma_h\sigma_{m^*} + \sigma_h\sigma_j - \sigma_m\sigma_h - \sigma_j\sigma_h = (a_{hjm} - a_{mhm})(\sigma_m - \sigma_{m^*}),$$

whence

$$a_{hjm} = a_{mhm}. \tag{15}$$

From (11), (i) and (12) we obtain that

$$\begin{aligned}\sigma_h(\sigma_j\sigma_h) &= d\sigma_h\sigma_k + a_{hjm}\sigma_h\sigma_{m^*} + a_{hjj}\sigma_h\sigma_j \\ &= d(\sigma_m + \sigma_j) + a_{hjm}(a_{mhm}\sigma_{m^*} + a_{mhl}\sigma_l) + a_{hjj}(d\sigma_k + a_{hjm}\sigma_m + a_{hjj}\sigma_j)\end{aligned}$$

and that

$$\begin{aligned}(\sigma_h\sigma_j)\sigma_h &= d\sigma_k\sigma_h + a_{hjm}\sigma_m\sigma_h + a_{hjj}\sigma_j\sigma_h \\ &= d(\sigma_{m^*} + \sigma_j) + a_{hjm}(a_{mhm}\sigma_m + a_{mhl}\sigma_l) + a_{hjj}(d\sigma_k + a_{hjm}\sigma_{m^*} + a_{hjj}\sigma_j).\end{aligned}$$

Thus, we have

$$d + a_{hjj}a_{hjm} = a_{hjm}a_{mhm},$$

whence, by (15),

$$d = a_{hjm}(a_{hjm} - a_{hjj}) \tag{16}$$

and so

$$c = a_{hmj}(a_{hjm} - a_{hjj}). \tag{17}$$

From (16), (17) and (14) we obtain that

$$n_h = c + d = (a_{hmj} + a_{hjm})(a_{hjm} - a_{hjj}) = (a_{hmj} + a_{hjm})(a_{hmj} - a_{hmm}).$$

In particular, we have

$$a_{hmj} + a_{hjm} \leqslant n_h, \qquad a_{hmm} + 1 \leqslant a_{hmj},$$

which, by (14), yields

$$n_h = a_{hmm} + 1 + a_{hjm} \leqslant a_{hmj} + a_{hjm} \leqslant n_h.$$

It follows that

$$a_{hmm} + 1 = a_{hmj},$$

and therefore, by (14),

$$a_{hjj} + 1 = a_{hjm}.$$

Now, by (17) and (16),

$$c = a_{hmj}, \qquad d = a_{hjm}. \tag{18}$$

This establishes the first two of the equations of (ii).

From (12) and Lemma 2.2 we obtain that

$$n_h n_m = a_{mhm} n_m + a_{lhm} n_m,$$

which, by (15), yields that $n_h = a_{hjm} + a_{lhm}$. Thus, by the the second equation of (18),

$$c = a_{lhm}. \tag{19}$$

Similarly, by (13) and Lemma 2.2,

$$n_h n_l = a_{mhl} n_l + a_{lhl} n_l,$$

which yields

$$n_h = a_{mhl} + a_{lhl}. \tag{20}$$

Finally, by (13), (19) and (10),

$$\sigma_h(\sigma_l\sigma_h) = c\sigma_h\sigma_m + a_{lhl}\sigma_h\sigma_l = c(c\sigma_k + (c-1)\sigma_m + c\sigma_j) + a_{lhl}(c\sigma_{m^*} + a_{lhl}\sigma_l)$$

and

$$\begin{aligned}(\sigma_h\sigma_l)\sigma_h &= c\sigma_{m^*}\sigma_h + a_{lhl}\sigma_l\sigma_h \\ &= c(c\sigma_k + (c-1)\sigma_{m^*} + c\sigma_j) + a_{lhl}(c\sigma_m + a_{lhl}\sigma_l).\end{aligned}$$

Thus, we have

$$c = a_{lhl} + 1.$$

Now (ii) follows from (19) and (20).

(iii) From (ii) we may easily conclude that

$$|G//\langle h\rangle| = 2 = |G//\langle k\rangle|.$$

Thus, by Theorem 3.1, $\mathcal{G}(X, \{\langle h\rangle, \langle k\rangle\})$ is a tactical configuration of type $(n_h + 1, n_h + 1)$ or a symmetric 2-design. Therefore, $n_h = n_k$. □

THEOREM 4.4. *Let (X, G) be a homogeneous coherent configuration of rank 7. Let h, $k \in \mathrm{Inv}(G)$ be such that $\langle h, k\rangle = G$ and $hk \neq kh$.*

Assume that $n_h + 1 \leqslant n_k$. Then the intersection numbers of (X, G) are uniquely determined by n_h and n_k, and $\mathcal{G}(X, \{\langle h\rangle, \langle k\rangle\})$ is a 2-$(n_h n_k + n_h + 1, n_h + 1, 1)$-design (with point set $X/\langle k\rangle$).

PROOF. Since $hk \neq kh$, $hk \setminus kh \neq \varnothing$. Let $m \in hk \setminus kh$ be given. Since $|G| = 7$, there exist j, $l \in G$ such that

$$\{1, h, k, m, m^*, j, l\} = G. \tag{21}$$

By Lemma 4.3(iii) and Lemma 4.2(i),

$$\sigma_h\sigma_k = \sigma_m, \qquad \sigma_k\sigma_h = \sigma_{m^*}. \tag{22}$$

From Lemma 4.2(ii) we obtain

$$\sigma_k\sigma_h\sigma_k = \sigma_j + \sigma_l$$

and, without loss of generality,

$$\sigma_h\sigma_k\sigma_h = \sigma_j.$$

Thus, $\mathbb{C}[G]$ is generated (as a $\mathbb{C}$-algebra) by $\{\sigma_h, \sigma_k\}$; see (21) and Lemma 2.1(ii). In particular, the structure constants of $\mathbb{C}[G]$ with respect to the basis $\{\sigma_g \mid g \in G\}$ are uniquely determined by n_h and n_k, so that the first claim follows from Lemma 2.1(i).

The second claim follows from Theorem 3.1 and (22). □

THEOREM 4.5. *Let (X, G) be a homogeneous coherent configuration of rank 7. Let $h, k \in \mathrm{Inv}(G)$ be such that $\langle h, k\rangle = G$ and $hk \neq kh$.*

Assume that $n_h = n_k$ and that $|hk| = 1$. Then we have the following.

(i) *Let $m, j, l \in G$ be such that $hk = \{m\}$ and $\{1, h, k, m, m^*, j, l\} = G$. Then there exist $a, b \in \mathbb{N}\setminus\{0\}$ with $a + b = n_h$ and $x, y \in \mathbb{C}$ such that $\mathbb{C}[G]$ has the following irreducible representations*:

	1	σ_h	σ_k	σ_m	σ_{m^*}	σ_j	σ_l
χ_1	1	n_h	n_h	n_h^2	n_h^2	an_h^2	bn_h^2
χ_2	1	-1	-1	1	1	x	$-x-1$
χ_3	1	-1	-1	1	1	y	$-y-1$
δ	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & n_h\\ 0 & n_h\end{pmatrix}$	$\begin{pmatrix}n_h & 0\\ 1 & -1\end{pmatrix}$	$\begin{pmatrix}0 & -n_h\\ n_h & -n_h\end{pmatrix}$	$\begin{pmatrix}-n_h & n_h^2\\ -1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -an_h\\ -a & 0\end{pmatrix}$	$\begin{pmatrix}0 & -bn_h\\ -b & 0\end{pmatrix}$

(ii) *$\mathscr{G}(X, \{\langle h\rangle, \langle k\rangle\})$ is a projective plane of order n_h.*

PROOF. (i) Since $hk = \{m\}$,

$$\sigma_h\sigma_k = \sigma_m, \qquad \sigma_k\sigma_h = \sigma_{m^*}. \tag{23}$$

By Lemma 4.2(ii) and Lemma 2.2,

$$\sigma_h\sigma_k\sigma_h = \sigma_j + \sigma_l = \sigma_k\sigma_h\sigma_k. \tag{24}$$

Let χ_2 and χ_3 denote the non-principal linear characters of $\mathbb{C}[G]$, and set

$$x := \chi_2(\sigma_j), \qquad y := \chi_3(\sigma_j).$$

Then the values of χ_2 and χ_3 in the table of (i) are easily obtained from (23), (24) and Lemma 2.3. (Note that, by (24), Lemma 2.4 and Lemma 2.6(ii), $\chi_i(\sigma_h) = -1 = \chi_i(\sigma_k)$ for each $i \in \{2, 3\}$.)

The second equation of (23) implies that $\{k, m^*\}$ is a left coset of $\langle h\rangle$ in G. Thus,

$$\sigma_h\sigma_j = a_{hjm^*}\sigma_{m^*} + a_{hjj}\sigma_j + \sigma_{hjl}\sigma_l.$$

In particular, for each $z \in \{x, y\}$,

$$z(a_{hjj} + 1 - a_{hjl}) = a_{hjl} - a_{hjm^*}.$$

Since $x \neq y$, this implies that

$$a_{hjj} + 1 = a_{hjl} = a_{hjm^*}.$$

Thus, for $a := a_{hjl}$, we have

$$(\sigma_h + 1)\sigma_j = a(\sigma_{m^*} + \sigma_j + \sigma_l). \tag{25}$$

Similarly, we conclude that

$$(\sigma_k + 1)\sigma_j = a(\sigma_m + \sigma_j + \sigma_l). \tag{26}$$

From (25), Lemma 2.2, (23) and (24) we obtain that n_h^2 divides n_j, so that we have all the values of χ_1.

Let W be an irreducible $\mathbb{C}[G]$-module on which σ_h and σ_k do not commute.

Let $w \in W$ be such that

$$w\sigma_h = n_h w = w\sigma_k.$$

By (23) and (24), Cw is fixed by σ_m, σ_{m^*} and $\sigma_j + \sigma_l$. Thus, by (25), $\mathbb{C}w$ is $\mathbb{C}[G]$-invariant. Since W is irreducible, $w = 0$.

It follows that W possesses a basis $\{u, w\}$, say, such that

$$u\sigma_k = n_h u, \qquad v\sigma_h = n_h v.$$

Let δ denote the matrix representation of $\mathbb{C}[G]$ afforded by $\{u, v\}$. Then, without loss of generality, we may assume that

$$\delta(\sigma_h) = \begin{pmatrix} -1 & e \\ 0 & n_h \end{pmatrix}, \qquad \delta(\sigma_k) = \begin{pmatrix} n_k & 0 \\ 1 & -1 \end{pmatrix},$$

where $e \in \mathbb{C}$ has to be chosen suitably.

From (24) we obtain that

$$\begin{pmatrix} n_h - e & e(e - 2n_h) \\ -n_h & n_h(e - n_h) \end{pmatrix} = \delta(\sigma_j) + \delta(\sigma_l) = \begin{pmatrix} n_h(e - n_h) & -n_h e \\ e - 2n_h & n_h - e \end{pmatrix}.$$

Thus, $e = n_h$ and

$$\delta(\sigma_j) + \delta(\sigma_l) = \begin{pmatrix} 0 & -n_h^2 \\ -n_h & 0 \end{pmatrix}.$$

Let $c_1, c_2, c_3, c_3 \in \mathbb{C}$ be such that

$$\delta(\sigma_j) = \begin{pmatrix} c_1 & c_3 \\ c_2 & c_4 \end{pmatrix}.$$

Then, by (26),

$$\begin{pmatrix} (n_h + 1)c_1 & (n_h + 1)c_3 \\ c_1 & c_3 \end{pmatrix} = a\begin{pmatrix} 0 & -n_h(n_h + 1) \\ 0 & -n_h \end{pmatrix},$$

whence $c_1 = 0$ and $c_3 = -an_h$. Similarly, by (25), $c_4 = 0$ and $c_2 = -a$.

(ii) follows from Theorem 3.1 and $hk \cap kh = \varnothing$. □

THEOREM 4.6. *Let (X, G) be a homogeneous coherent configuration of rank 7. Let $h, k \in \mathrm{Inv}(G)$ be such that $\langle h, k\rangle = G$ and $hk \neq kh$.*

Assume that $n_h = n_k$ and that $|hk| \neq 1$. Then we have the following.

(i) *Let $m, j \in hk$ be such that $m^* \neq m$, and let $l \in G$ be such that $\{1, h, k, m, m^*, j, l\} = G$. Then there exist $c, d \in \mathbb{N}\backslash\{0\}$ with $c + d = n_h$ and $x, y \in \mathbb{C}$ such that $\mathbb{C}[G]$ has the following irreducible representations:*

	1	σ_h	σ_k	σ_m	σ_{m^*}	σ_j	σ_l
χ_1	1	n_h	n_h	cn_h	cn_h	dn_h	$c^2n_h/(d+1)$
χ_2	1	-1	-1	x	x	$1-x$	$-x$
χ_3	1	-1	-1	y	y	$1-y$	$-y$
δ	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & c \\ 0 & n_h \end{pmatrix}$	$\begin{pmatrix} n_h & 0 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & -c(d+1) \\ c & -c \end{pmatrix}$	$\begin{pmatrix} -c & c^2 \\ -(d+1) & 0 \end{pmatrix}$	$\begin{pmatrix} -d & cd \\ d & -d \end{pmatrix}$	$\begin{pmatrix} 0 & -c^2 \\ -c & 0 \end{pmatrix}$

(ii) *Either* $c = 1$ *or* $\mathscr{G}(X, \{\langle h\rangle, \langle k\rangle\})$ *is a* (*symmetric*) $2-((n_h(n_h+1)/(d+1))+1, n_h+1, d+1)$*-design.*

PROOF. (i) From Lemma 4.3(ii) we conclude that

$$|G/\langle h\rangle| = 3 = |G/\langle k\rangle|.$$

Let χ_2 and χ_3 denote the non-principal linear characters of $\mathbb{C}[G]$. Then, by Lemma 2.5, we have

$$\chi_i(\sigma_h) = -1 = \chi_i(\sigma_k)$$

for each $i \in \{2, 3\}$. We set

$$x := \chi_2(\sigma_m), \qquad y := \chi_3(\sigma_m).$$

Then the values of χ_2 and χ_3 are obtained from Lemma 4.3(i) and Lemma 2.3.

The values of χ_1 are given by Lemma 4.3 and Lemma 2.2.

Let W be an irreducible $\mathbb{C}[G]$-module on which σ_h and σ_k do not commute.

Let $w \in W$ be such that

$$w\sigma_h = n_h w = w\sigma_k.$$

Then, by Lemma 4.3(i),

$$w\sigma_m = w\sigma_{m^*}. \tag{27}$$

On the other hand, the last equation of Lemma 4.3(ii) yields that

$$\sigma_l\sigma_h - \sigma_h\sigma_l = c(\sigma_m - \sigma_{m^*}).$$

Thus,

$$n_h w\sigma_l = w\sigma_h\sigma_l = w\sigma_l\sigma_h.$$

This implies that $w\sigma_l \in \mathbb{C}w$. Therefore, we may conclude from Lemma 4.3(ii) and (27) that $\mathbb{C}w$ is $\mathbb{C}[G]$-invariant. But W is irreducible. Thus, $w = 0$.

It follows that W possesses a basis $\{u, v\}$, say, such that

$$u\sigma_k = n_h u, \qquad v\sigma_h = n_h v.$$

Let δ denote the matrix representation of $\mathbb{C}[G]$ afforded by $\{u, v\}$. Then, without loss of generality, we may assume that

$$\delta(\sigma_h) = \begin{pmatrix} -1 & e \\ 0 & n_h \end{pmatrix}, \qquad \delta(\sigma_k) = \begin{pmatrix} n_h & 0 \\ 1 & -1 \end{pmatrix},$$

where $e \in \mathbb{C}$ has to be chosen suitably.

From Lemma 4.3(i) we obtain that

$$\delta(\sigma_m) + \delta(\sigma_j) = \begin{pmatrix} e - n_h & -e \\ n_h & -n_h \end{pmatrix} \tag{28}$$

and that

$$\delta(\sigma_m) - \delta(\sigma_{m^*}) = \begin{pmatrix} e & -e(n_h+1) \\ n_h+1 & -e \end{pmatrix}. \tag{29}$$

Let $a_1, a_2, a_3, a_4 \in \mathbb{C}$ be such that

$$\delta(\sigma_m) = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}.$$

Then we obtain from the first equation of Lemma 4.3(ii) that

$$\begin{pmatrix} ea_2 - a_1 & ea_4 - a_3 \\ n_h a_2 & n_h a_4 \end{pmatrix} = c\begin{pmatrix} n_h & 0 \\ 1 & -1 \end{pmatrix} + (c-1)\begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} + c\begin{pmatrix} e - n_h - a_1 & -e - a_3 \\ n_h - a_2 & -n_h - a_4 \end{pmatrix},$$

whence $a_2 = c = -a_4$. Therefore, by (29),

$$\delta(\sigma_{m^*}) = \begin{pmatrix} a_1 - e & a_3 + e(n_h+1) \\ -(d+1) & e - c \end{pmatrix}.$$

If we interchange the roles of h and k in Lemma 4.3(ii), then the first equation there yields

$$\begin{pmatrix} n_h(a_1-e) & n_ha_3+n_he(n_h+1) \\ a_1-e+d+1 & a_3+e(n_h+1)+c-e \end{pmatrix} = c\begin{pmatrix} -1 & e \\ 0 & n_h \end{pmatrix} + (c-1)\begin{pmatrix} a_1-e & a_3+e(n_h+1) \\ -(d+1) & e-c \end{pmatrix} + c\begin{pmatrix} e-n_h-a_1 & -e-a_3 \\ d & -d \end{pmatrix};$$

use (28). It follows that $a_1 = e - c$ and $a_3 = -e(d+1)$. Now we have

$$\delta(\sigma_m) = \begin{pmatrix} e-c & -e(d+1) \\ c & -c \end{pmatrix},$$

$$\delta(\sigma_{m^*}) = \begin{pmatrix} -c & ec \\ -(d+1) & e-c \end{pmatrix}$$

and

$$\delta(\sigma_j) = \begin{pmatrix} -d & ed \\ d & -d \end{pmatrix}.$$

Similarly, let $b_1, b_2, b_3, b_4 \in \mathbb{C}$ be such that

$$\delta(\sigma_l) = \begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix}.$$

Then we obtain from the third equation of Lemma 4.3(ii) that

$$\begin{pmatrix} c-e(d+1) & e^2-2ec \\ -n_h(d+1) & n_h(e-c) \end{pmatrix} = d\begin{pmatrix} -c & ec \\ -(d+1) & e-c \end{pmatrix} + (d+1)\begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix},$$

whence $b_1 = c - e$ and $b_2 = -c$. This time, we interchange the roles of h and k in the third equation of Lemma 4.3(ii) to obtain that

$$\begin{pmatrix} n_h(e-c) & -n_he(d+1) \\ e-2c & c-ed-e \end{pmatrix} = d\begin{pmatrix} e-c & -e(d+1) \\ c & -c \end{pmatrix} + (d+1)\begin{pmatrix} c-e & b_3 \\ -c & b_4 \end{pmatrix}.$$

Thus, we have $e = c$, $b_3 = -c^2$ and $b_4 = 0$. The proof of (i) is complete.

(ii) follows from Lemma 4.3(ii) and Theorem 3.1. □

We finish this section with a few remarks.

A little more can be said in the situation of Theorem 4.5. First of all, a lengthy computation shows that (in the notation of Theorem 4.5) $j^* = j$ and $l^* = l$. By a shorter argument we obtain, then, that $n_j = n_l$ or $x, y \in \mathbb{Z}$. E. Shult has constructed examples in which $n_j = n_l$.

The situation of Theorem 4.6 has been discussed to a certain extent in [3]. Examples (of coherent configurations satisfying the hypotheses of Theorem 4.6) are provided by Hadamard designs.

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