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Nonoscillation and oscillation of second-order impulsive differential equations with periodic coefficients

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ABSTRACT

In this paper, we give a nonoscillation criterion for half-linear equations with periodic coefficients under fixed moments of impulse actions. The method is based on the existence of positive solutions of the related Riccati equation and a recently obtained comparison principle. In the special case when the equation becomes impulsive Hill equation new oscillation criteria are also obtained.

1. Introduction

Consider the second-order half-linear impulsive equation

$$\begin{aligned} &(\Phi_\alpha(x'))' + p(t)\Phi_\alpha(x') + q(t)\Phi_\alpha(x) = 0, \quad t \neq \theta_i;\\ &\Delta\Phi_\alpha(x') + \beta_i\Phi_\alpha(x) = 0, \quad t = \theta_i, \end{aligned} \tag{1.1}$$

where $p, q \in \mathrm{PLC}(\mathbb{R}_+, \mathbb{R}) := \{f \in C(\theta_i, \theta_{i+1}), f(\theta_i^{\pm}) \text{ exist}, f(\theta_i) = f(\theta_i^-), i \in \mathbb{N}\}$; $\{\beta_i\}$ is a sequence of real numbers; $\Phi_\alpha(x) = |x|^{\alpha-2}x$, $\alpha > 1$; $\Delta f(t) := f(t^+) - f(t^-)$ with $f(t^{\pm}) = \lim_{\tau\to t^{\pm}} f(\tau)$.

We will assume that (1.1) is ω-periodic, which means that there exist a positive real number ω and a positive integer r such that

(i) $p(t+\omega) = p(t)$, $q(t+\omega) = q(t)$ for all $t \in \mathbb{R}_+ \setminus \{\theta_i : i \in \mathbb{N}\}$.
(ii) $\theta_i + \omega = \theta_{i+r}$ for all $i \in \mathbb{N}$.
(iii) $\beta_{i+r} = \beta_i$ for all $i \in \mathbb{N}$.

By a solution of (1.1) defined on $\mathbb{R}_+$ we mean a nontrivial continuous function x such that $x', \Phi_\alpha(x') \in \mathrm{PLC}(\mathbb{R}_+, \mathbb{R})$ and that $x(t)$ satisfies (1.1) for all $t \in \mathbb{R}_+$. Such a solution $x(t)$ of (1.1) is called oscillatory if it has arbitrarily large zeros; nonoscillatory otherwise. (1.1) is called oscillatory (nonoscillatory) if all of its solutions are oscillatory (nonoscillatory). By a Sturm type comparison theorem [1] we know that (1.1) is oscillatory if and only if it has an oscillatory solution.

If $\alpha = 2$, then (1.1) is said to be an impulsive Hill equation

$$\begin{aligned} &x'' + p(t)x' + q(t)x = 0, \quad t \neq \theta_i;\\ &\Delta x' + \beta_i x = 0, \quad t = \theta_i, \end{aligned} \tag{1.2}$$

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as it reduces to the well-known Hill equation

$$x'' + p(t)x' + q(t)x = 0, \tag{1.3}$$

when the impulses are absent. Actually, the original Hill equation does not contain a damping term and has many applications in engineering and physics, including problems in mechanics, astronomy, and metal conductivity of electricity; see [2].

It is well known that if q is nontrivial and ω-periodic of mean value zero, i.e.,

$$\int_0^\omega q(t)\mathrm{d}t = 0,$$

then every solution of

$$x'' + q(t)x = 0 \tag{1.4}$$

is oscillatory. However, the same is not true for (1.3) when p and q are ω-periodic of mean value zero as pointed out in [3], where the authors observe that

$$x'' + (\sin t)x' + (\cos t)x = 0 \tag{1.5}$$

has a nonoscillatory solution $x(t) = \exp(\cos t)$, while every solution of

$$x'' + (\cos t)x' + (\sin t)x = 0 \tag{1.6}$$

is oscillatory [4]. Related to this problem for (1.3) the following theorems are obtained in [3]. For a time scale extension, see [5].

Theorem 1.1. *Let p, q be ω-periodic and $Q(t)$ be an indefinite integral of $q(t)$. If q is of mean value zero, then*

$$[p(t) - Q(t)]Q(t) \geq 0, \quad 0 \leq t \leq \omega \tag{1.7}$$

implies that (1.3) is nonoscillatory.

Theorem 1.2. *In addition to the assumptions in Theorem 1.1, if $q(t) \not\equiv 0$, $p(t)$ and $Q(t)$ are ω-periodic of mean value zero and satisfy*

$$[p(t) - Q(t)]Q(t) \leq 0, \quad 0 \leq t \leq \omega, \tag{1.8}$$

and furthermore

$$\text{measure}\,\{t \in [0, \omega] : [p(t) - Q(t)]Q(t) < 0\} > 0, \tag{1.9}$$

then (1.3) is oscillatory.

Next, let us consider (1.1) without impulses

$$(\Phi_\alpha(x'))' + p(t)\Phi_\alpha(x') + q(t)\Phi_\alpha(x) = 0. \tag{1.10}$$

In [6], Došlý and Elbert proved that if $p \equiv 0$ and q is nontrivial and periodic of mean value zero, then (1.10) is oscillatory. An extension of Theorem 1.1 to (1.10) is given by Sugie and Matsumura [7] as follows, where α^* denotes the conjugate exponent of α, i.e.,

$$\frac{1}{\alpha} + \frac{1}{\alpha^*} = 1.$$

Theorem 1.3. *Let p and q be ω-periodic and $Q(t)$ be an indefinite integral of $q(t)$, where $q(t)$ is of mean value zero, then*

$$[p(t) - (\alpha - 1)\Phi_{\alpha^*}(Q(t))]Q(t) \geq 0, \quad 0 \leq t \leq \omega \tag{1.11}$$

implies that (1.10) is nonoscillatory.

The proof of the above theorem is based on the fact that (1.10) is nonoscillatory if and only if there exist a $t_0 \geq 0$ and a continuously differentiable function $z : [t_0, \infty) \to \mathbb{R}$ such that the Riccati inequality

$$z' \geq (\alpha - 1)|z|^{\alpha^*} - p(t)z + q(t), \quad t \geq t_0 \tag{1.12}$$

holds; see [8].

It is a natural question to ask for similar results for impulsive differential equations of the form (1.1). In this paper, by using the tools of impulsive differential equations we provide some answers to these questions. It turns out that there is a substantial difference due to impulse effects.

2. Main results

First we recall that a sequence $\{a_i\}$ is called r-periodic if

$$a_{i+r} = a_i \quad \text{for all } i \in \mathbb{N},$$

and of mean value zero if

$$\sum_{i=1}^{r} a_i = 0.$$

Let $\{\gamma_i\}$ be an r-periodic sequence, and define

$$Q(t) := \int_0^t q(s)\mathrm{d}s + \sum_{0\le\theta_i<t} \gamma_i.$$

It is clear that

$$Q'(t) = q(t) \quad \text{for } t \neq \theta_i; \quad \Delta Q = \gamma_i \quad \text{for } t = \theta_i. \tag{2.1}$$

We will assume in the sequel that $Q(\omega) = 0$, i.e.,

$$\int_0^{\omega} q(t)\mathrm{d}t + \sum_{i=1}^{r} \gamma_i = 0. \tag{2.2}$$

Note that if q and $\{\gamma_i\}$ are of mean value zero, then (2.2) holds. However, the converse is in general not true.

Denote by

$$J_{imp} = \{\theta_1, \theta_2, \ldots, \theta_r\}$$

the impulse points in $[0, \omega]$.

The main results of this paper are as follows.

Theorem 2.1. *Let* (i)–(iii), *and* (2.2) *hold.*

If

$$[p(t) - (\alpha - 1)\Phi_{\alpha^*}(Q(t))]Q(t) \ge 0, \quad t \in [0, \omega] \setminus J_{imp} \tag{2.3}$$

and

$$\gamma_i \ge \beta_i, \quad i = 1, 2, \ldots, r, \tag{2.4}$$

then (1.1) *is nonoscillatory.*

Theorem 2.2. *In addition to the assumptions in Theorem 2.1, suppose that p and Q are of mean value zero.*

If

$$[p(t) - Q(t)]Q(t) \le 0, \quad t \in [0, \omega] \setminus J_{imp}, \tag{2.5}$$

$$\gamma_i \le \beta_i, \quad i = 1, 2, \ldots, r \tag{2.6}$$

and furthermore

$$\text{measure}\,\{t \in [0, \omega] : [p(t) - Q(t)]Q(t) < 0\} + \max\{\beta_i - \gamma_i : i = 1, 2, \ldots, r\} > 0, \tag{2.7}$$

then (1.2) *is oscillatory.*

Theorem 2.3. *Let the assumptions in Theorem 2.2 hold, and denote*

$$k(t) = \exp \int_0^t (p(s) - 2Q(s))\mathrm{d}s. \tag{2.8}$$

If

$$\int_0^{\omega} k(t)[Q(t) - p(t)]Q(t)\mathrm{d}t + \sum_{i=1}^{r} k(\theta_i)\,(\beta_i - \gamma_i) > 0, \tag{2.9}$$

then (1.2) *is oscillatory.*

Remark 1. If the impulses are dropped, then by taking $\beta_i = 0$ and $\gamma_i = 0$ in Theorem 2.1 we recover Theorem 1.3 and hence Theorem 1.1 when $\alpha = \alpha^* = 2$. Similarly, Theorems 2.2 and 2.3 are the extensions of the oscillation criteria given in Theorem 1.2 and [7, Theorem 3.1].

Remark 2. We see from Theorem 2.2 that if q is nontrivial ω-periodic and $\{\beta_i\}$ is r-periodic, and (2.2) holds, then every solution of

$$\begin{aligned} &x'' + q(t)x = 0, \quad t \neq \theta_i; \\ &\Delta x' + \beta_i x = 0, \quad t = \theta_i, \end{aligned} \tag{2.10}$$

is oscillatory. This means that the well-known oscillation criterion for (1.4) is also true for the impulsive equation (2.10). In fact, we may allow $q \equiv 0$ by replacing (2.2) with an r-periodic $\{\gamma_i\}$ of mean value zero such that $\gamma_i \geq \beta_i$ and $\gamma_i \not\equiv \beta_i$. The problem for half-linear impulsive equation (1.1) when $p \equiv 0$ and $\alpha \neq 2$ is open.

Remark 3. It is seen from the results that the nonoscillation (oscillation) behavior of solutions can be altered by imposing impulse conditions. We see from Theorem 2.1 that the nonoscillation of the differential equation without impulses is necessary for the nonoscillation of solutions of the related impulsive differential equation. However, the same is not true for the oscillation of the equations as seen from Theorems 2.2 and 2.3.

Remark 4. Unfortunately, the extension of Theorem 2.2 to (1.1) is not possible by the same technique since Riccati type inequality (1.12) is not of the same type under a linear transformation unless $\alpha = 2$; see the proof of Theorem 2.2.

3. Proofs

First we need a lemma which is analogous to the one given in [8]. The proof of the lemma is similar but require a Sturm type comparison theorem for half-linear impulsive differential equations, which is available in [1]. In the case $\alpha = 2$ and $\beta_i \equiv 0$, see also [9] or [10, Theorem 7.2].

Lemma 3.1. *Eq. (1.1) is nonoscillatory on $[0, \infty)$ if and only if there exist a $t_1 \in [0, \infty)$ and a function $u \in \mathrm{PLC}[t_1, \infty)$ such that*

$$\begin{aligned} &u' \geq (\alpha - 1)|u|^{\alpha^*} - p(t)u + q(t), \quad t \neq \theta_i; \\ &\Delta u \geq \beta_i, \quad t = \theta_i \end{aligned} \tag{3.1}$$

for all $t \geq t_1$.

Proof. Let $x(t)$ be a solution of Eq. (1.1) having no zero in $[t_1, \infty)$. It is easy to see that the function u defined by $u(t) = -\Phi_\alpha(x'(t)/x(t))$ for $t \geq t_1$ satisfies the Riccati type impulsive equation

$$\begin{aligned} &u' = (\alpha - 1)|u|^{\alpha^*} - p(t)u + q(t), \quad t \neq \theta_i; \\ &\Delta u = \beta_i, \quad t = \theta_i. \end{aligned} \tag{3.2}$$

Conversely, let there exist a function $u \in \mathrm{PLC}[t_1, \infty)$ satisfying (3.1). Define

$$\begin{aligned} &f(t) := u'(t) - (\alpha - 1)|u(t)|^{\alpha^*} + p(t)u(t) - q(t), \quad t \neq \theta_i; \\ &f_i := \Delta u - \beta_i, \quad t = \theta_i. \end{aligned}$$

From (3.1), $f(t) \geq 0$ for $t \geq t_1$ and $f_i \geq 0$ for all i for which $\theta_i \geq t_1$. Thus we have the Riccati equation with impulses

$$\begin{aligned} &u' = (\alpha - 1)|u|^{\alpha^*} - p(t)u + q(t) + f(t), \quad t \neq \theta_i; \\ &\Delta u = \beta_i + f_i, \quad t = \theta_i. \end{aligned} \tag{3.3}$$

The corresponding impulsive differential equation is

$$\begin{aligned} &(\Phi_\alpha(x'))' + p(t)\Phi_\alpha(x') + \{q(t) + f(t)\}\Phi_\alpha(x) = 0, \quad t \neq \theta_i; \\ &\Delta\Phi_\alpha(x') + \{\beta_i + f_i\}\Phi_\alpha(x) = 0, \quad t = \theta_i. \end{aligned} \tag{3.4}$$

Let

$$k(t) = \exp \int_0^t p(\tau)d\tau, \quad t \geq t_1.$$

We may write from (1.1) and (3.4),

$$\begin{aligned} &(k(t)\Phi_\alpha(x'))' + k(t)q(t)\Phi_\alpha(x) = 0, \quad t \neq \theta_i; \\ &\Delta k(t)\Phi_\alpha(x') + k(t)\beta_i\Phi_\alpha(x) = 0, \quad t = \theta_i \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&(k(t)\Phi_\alpha(x'))' + k(t)\{q(t)+f(t)\}\Phi_\alpha(x) = 0, \quad t \neq \theta_i;\\ &\Delta k(t)\Phi_\alpha(x') + k(t)\{\beta_i + f_i\}\Phi_\alpha(x) = 0, \quad t = \theta_i\end{aligned} \tag{3.6}$$

respectively. Clearly, $x(t) = \exp \int^t \Phi_{\alpha^*}(-u(\tau))\mathrm{d}\tau$ is a nonoscillatory solution of (3.6).

Since

$$q(t) + f(t) \geq q(t)$$

and

$$\beta_i + f_i \geq \beta_i,$$

by the Sturm type comparison theorem [1, Corollary 2.2] for half-linear impulsive differential equations, we may conclude that (3.5) and hence (1.1) is also nonoscillatory. □

Proof of Theorem 2.1. We first claim that the function Q is ω-periodic. Indeed, since $Q(\omega) = 0$ we have

$$\begin{aligned} Q(t+\omega) - Q(t) &= \int_0^{t+\omega} q(s)\mathrm{d}s + \sum_{0\leq\theta_i<t+\omega}\gamma_i - \int_0^t q(s)\mathrm{d}s - \sum_{0\leq\theta_i<t}\gamma_i\\ &= Q(\omega) + \int_\omega^{t+\omega} q(s)\mathrm{d}s - \int_0^t q(s)\mathrm{d}s + \sum_{\omega\leq\theta_i<t+\omega}\gamma_i - \sum_{0\leq\theta_i<t}\gamma_i\\ &= \int_0^t q(s+\omega)\mathrm{d}s - \int_0^t q(s)\mathrm{d}s + \sum_{0\leq\theta_i<t}\gamma_{i+r} - \sum_{0\leq\theta_i<t}\gamma_i\\ &= 0.\end{aligned}$$

In view of (2.1), (2.3), and (2.4), we obtain

$$\begin{aligned}&Q'(t) \geq (\alpha-1)|Q(t)|^{\alpha^*} - p(t)Q(t) + q(t), \quad t \neq \theta_i;\\ &\Delta Q(t) \geq \beta_i, \quad t = \theta_i,\end{aligned} \tag{3.7}$$

which by Lemma 3.1 gives us that (1.1) is nonoscillatory. □

Proof of Theorem 2.2. Suppose on the contrary that (1.2) is nonoscillatory. We may assume without loss of generality that there exists a positive solution $x(t)$ defined on $[t_0, \infty)$ for some $t_0 \geq 0$.

Let $u(t) = -x'(t)/x(t)$ for $t \geq t_0$. It is easy to see that $u(t)$ satisfies the Riccati type impulsive equation

$$\begin{aligned}&u' = u^2 - p(t)u + q(t), \quad t \neq \theta_i;\\ &\Delta u = \beta_i, \quad t = \theta_i.\end{aligned} \tag{3.8}$$

Define $z(t) = u(t) - Q(t)$, $t \geq t_0$. The function $z(t)$ solves

$$\begin{aligned}&z' = z^2 + [2Q(t) - p(t)]z + Q^2(t) - p(t)Q(t), \quad t \neq \theta_i;\\ &\Delta z = \beta_i - \gamma_i, \quad t = \theta_i.\end{aligned} \tag{3.9}$$

By Lemma 3.1, the corresponding second-order impulsive equation

$$\begin{aligned}&y'' + \{p(t) - 2Q(t)\}y' + \{Q^2(t) - p(t)Q(t)\}y = 0, \quad t \neq \theta_i;\\ &\Delta y' + \{\beta_i - \gamma_i\}y = 0, \quad t = \theta_i\end{aligned} \tag{3.10}$$

is nonoscillatory. Let

$$m(t) = \exp \int_0^t \{p(s) - 2Q(s)\}\mathrm{d}s.$$

Then we may write (3.10) as

$$\begin{aligned}&(m(t)y')' + m(t)(Q(t) - p(t))Q(t)y = 0, \quad t \neq \theta_i;\\ &\Delta m(t)y' + m(\theta_i)\{\beta_i - \gamma_i\}y = 0, \quad t = \theta_i.\end{aligned} \tag{3.11}$$

On the other hand, since p and Q are ω-periodic with mean value zero, the function m becomes ω-periodic and hence there exists $m_1 > 0$ such that

$$\begin{aligned}\int_{n\omega}^{(n+1)\omega} \frac{1}{m(t)}\mathrm{d}t &= \int_{n\omega}^{(n+1)\omega} \left[\exp \int_0^t \{2Q(s) - p(s)\}\mathrm{d}s\right]\mathrm{d}t\\ &= \int_0^\omega \left[\exp \int_0^t \{2Q(s) - p(s)\}\mathrm{d}s\right]\mathrm{d}t = m_1 > 0, \quad n \in \mathbb{N}.\end{aligned} \tag{3.12}$$

Moreover, (2.7) results in

$$\int_{n\omega}^{(n+1)\omega} m(t)\{Q^2(t) - p(t)Q(t)\}\mathrm{d}t + \sum_{n\omega \le \theta_i < (n+1)\omega} m(\theta_i)\{\beta_i - \gamma_i\} =: m_2 > 0, \quad n \in \mathbb{N}. \tag{3.13}$$

It follows that from (3.12) and (3.13), respectively, that

$$\int_0^\infty \frac{1}{m(t)}\mathrm{d}t = \infty$$

and

$$\int_0^\infty m(t)\{Q^2(t) - p(t)Q(t)\}\mathrm{d}t + \sum_{0<\theta_i} m(\theta_i)\{\beta_i - \gamma_i\} = \infty.$$

Applying the Leighton–Wintner theorem for impulsive equations [11, Theorem 2.1], we conclude that (3.11) is oscillatory. This contradiction completes the proof of Theorem 2.2. □

Proof of Theorem 2.3. Proceeding as in the proof of Theorem 2.2 we obtain (3.12) and (3.13) with $m(t)$ replaced by $k(t)$. □

4. Examples

Example 4.1. Consider the impulsive Hill equation

$$\begin{aligned} &x'' + \lambda p(t)x' + q(t)x = 0, \quad t \ne \theta_i;\ \lambda \in \mathbb{R}, \\ &\Delta x' + \beta_i x = 0, \quad t = \theta_i, \end{aligned} \tag{4.1}$$

where

$$p(t) = \int_0^t q(s)\mathrm{d}s + \sigma \sum_{0 \le \theta_i < t} \beta_i, \quad t \in [0, \omega] \tag{4.2}$$

and $\beta_i > 0$ for all $i = 1, 2, \ldots, r$ and that $p(\omega) = 0$. Eq. (4.1) is nonoscillatory if $\lambda \ge 1$ and $\sigma \ge 1$ and oscillatory if $\lambda \le 1$ and $\sigma < 1$ (or $\lambda < 1$ and $\sigma \le 1$) by Theorems 2.1 and 2.2, respectively. Moreover, (4.1) is oscillatory if

$$(1-\lambda)\int_0^\omega p^2(t)e^{(\lambda-2)\int_0^t p(s)\mathrm{d}s}\mathrm{d}t + (1-\sigma)\sum_{i=1}^r \beta_i e^{(\lambda-2)\int_0^{\theta_i} p(s)\mathrm{d}s} > 0 \tag{4.3}$$

by Theorem 2.3.

Let us consider a special case. We take $q(t) = \sigma(2\pi)^{-1}$, $\beta_i = (-1)^i$, $\theta_i = i\pi/2$, and $\omega = 2\pi$. Then $r = 4$ and $p(t) = (t/\pi - 1 + (-1)^{i+1})\sigma/2$, $t \in ((i-1)\pi/2, i\pi/2]$, $i = 1, 2, 3, 4$.

After some tedious calculations we see that if $\lambda = 2$ and $\sigma < 0$, then (4.3) is satisfied, and so

$$\begin{aligned} &x'' + \{t/\pi - 1 + (-1)^{i+1}\}\sigma x' + \sigma(2\pi)^{-1}x = 0, \quad t \ne i\pi/2; \\ &\Delta x' + (-1)^i x = 0, \quad t = i\pi/2 \end{aligned} \tag{4.4}$$

is oscillatory.

If we choose $\sigma(\lambda - 2) = -2$, $\sigma < 1, 39419$, and $\lambda < 0, 56548$, then in view of (4.3), (4.4) is oscillatory.

Example 4.2. Consider the impulsive Hill equation

$$\begin{aligned} &x'' + \sin(2t)x' + \cos(2t)x = 0, \quad t \ne i\pi/4; \\ &\Delta x' + (\beta + \cos(i\pi/2))x = 0, \quad t = i\pi/4, \end{aligned} \tag{4.5}$$

where $\beta \ge 1$. It can be seen that conditions (i)–(iii) are satisfied with $\omega = \pi$ and $r = 4$. Let

$$Q(t) = \int_0^t \cos(2s)\mathrm{d}s + \sum_{0 \le \theta_i < t} \sin(i\pi/2), \quad t \in [0, \pi] \tag{4.6}$$

where $J_{imp} = \{\pi/4, \pi/2, 3\pi/4, \pi\}$. Note that

$$Q(\pi) = \int_0^\pi \cos(2s)\mathrm{d}s + \sum_{i=1}^3 \sin(i\pi/2) = 0$$

and

$$Q(t)=\frac{1}{2}\sin(2t)+\frac{1}{2}\begin{cases}-1, & t\in[0,\pi/4)\\ 1, & t\in(\pi/4,\pi/2)\\ 1, & t\in(\pi/2,3\pi/4)\\ -1, & t\in(3\pi/4,\pi).\end{cases} \tag{4.7}$$

It can be easily seen that the function Q is π-periodic with mean value zero, and that

$$[p(t)-Q(t)]Q(t)=-\frac{1}{4}\cos^2 2t\le 0,\quad t\in[0,\pi]\setminus J_{imp}. \tag{4.8}$$

Thus, we conclude from Theorem 2.2 that (4.5) is oscillatory. We remark that if the impulses are dropped, then the corresponding Hill equation

$$x''+\sin(2t)x'+\cos(2t)x=0 \tag{4.9}$$

is nonoscillatory by Theorem 1.1.

Example 4.3. Consider the equation

$$\begin{aligned}&x''+ax'+bx=0,\quad t\ne i\sigma;\\ &\Delta x'-b\sigma x=0,\quad t=i\sigma\end{aligned} \tag{4.10}$$

where a, b, c and σ are real constants with $b>0$, $\sigma>0$ and $a\ge 2b\sigma$. It can be seen that conditions (i)–(iii) are satisfied with $\omega=r\sigma$. Let

$$Q(t)=\int_0^t b\mathrm{d}s+\sum_{0\le i\sigma<t}(-b\sigma),\quad t\in[0,r\sigma],$$

then a simple calculation gives $Q(t)=bt-b\sigma(i-1),\ t\in((i-1)\sigma,i\sigma]$.

We see that $Q(\omega)=\int_0^\omega b\mathrm{d}s+\sum_{0\le\theta_i<\omega}(-b\sigma)=0$. Define

$$\mathcal{H}(t):=\{a-bt+b\sigma(i-1)\}\{bt-b\sigma(i-1)\},\quad t\in((i-1)\sigma,i\sigma].$$

Clearly,

$$\mathcal{H}'(t)=2b^2(i\sigma-t-\sigma)+ab\ge b(a-2b\sigma)\ge 0,\quad t\in((i-1)\sigma,i\sigma].$$

So the function $\mathcal{H}(t)$ is an increasing function on $((i-1)\sigma,i\sigma]$. Since $\inf_{((i-1)\sigma,i\sigma]}\mathcal{H}(t)=0$, $\mathcal{H}((i\sigma)^+)=\{a-bt+b\sigma i\}$ $x\{bt-b\sigma i\}|_{t=i\sigma}=0$, the condition (2.3) of Theorem 2.1 is satisfied. It follows that (4.10) is nonoscillatory. Note that

$$x''+ax'+bx=0 \tag{4.11}$$

is oscillatory if $a^2<4b$.

Acknowledgments

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