Defect numbers of singular integral operators with Carleman shift and almost periodic coefficients

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This paper is devoted to the study of properties of the kernel and the cokernel of singular integral operators with almost periodic coefficients and a Carleman shift. In particular, the dimensions of their kernels and cokernels are obtained. This is done by considering appropriate properties of the related almost periodic elements and, in special, the partial indices of some of their relevant factorizations.

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1. Introduction

The theory of singular integral operators with shifts exhibits strong consequences in a wide range of different areas. This is due to its high importance in the modelling of a large variety of applied problems. In this way, it is not surprising that their Fredholm theory – and, if possible, detailed information about the dimensions of their kernel and cokernel – is highly desirable for the understanding of the solvability of corresponding integral equations characterized by such kind of operators.

In view of this, it is significant to mention that although several characterizations for the Fredholm property are presently known for different classes of singular integral operators with shift (see, for instance, Karapetiants and Samko [17] and Litvinchuk [21]) not as much defect numbers characterizations are known for the same operators. Here, it is also relevant to notice that singular integral operators with shift are related with classes of boundary value problems in a natural way. For instance, the study of singular integral operators with conjugation may be considered to have begun more than half century ago with the investigation of boundary value problems for analytic functions with conjugation, namely by Markushevich (1946), Vekua (1952) and Boiarskii (1952). In particular, Vekua’s paper [25] is recognized as the first one in which singular integral equations with shift were considered. The classical monographs of F.D. Gakhov [16], N. Karapetiants and S. Samko [17], V.G. Kravchenko and G.S. Litvinchuk [20], G.S. Litvinchuk [21], S.G. Mikhlin and S. Prössdorf [22] describe the advances in this field of research in a very detailed way. For some more recent work within this scope we would like to refer to [2–15,18,19,23,24,26,27].

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In the present work, we are concerned with singular integral operators of the form
\[
T_{A, I} = AP + IQ : L^2(\mathbb{R}) \to L^2(\mathbb{R})
\] (1.1)
where the coefficient \( A \) is a functional operator of the type
\[
A = \phi_1 I + \phi_2 J,
\] (1.2)
with almost periodic elements \( \phi_1 \) and \( \phi_2 \), \( I \) denotes the identity operator and \( J \) is the reflection operator given by the rule
\[
Jf(x) = \tilde{f}(x) = f(-x), \quad x \in \mathbb{R}.
\]
Moreover, \( L^2(\mathbb{R}) \) stands for the usual Lebesgue space of measurable functions \( \phi \) on \( \mathbb{R} \) with norm
\[
\| \phi \|_{L^2(\mathbb{R})} = \left( \int_{\mathbb{R}} |\phi(x)|^2 \, dx \right)^{1/2}.
\]
On \( L^2(\mathbb{R}) \), we consider the Cauchy singular integral operator \( S = S_{\mathbb{R}} \) defined by
\[
(S\varphi)(x) = \frac{1}{\pi i} \int_{\mathbb{R}} \varphi(\tau) \frac{\tau}{\tau - x} \, d\tau, \quad x \in \mathbb{R},
\]
where the integral is understood in the sense of principal value. Since \( S^2 = I \), the operator \( S \) generates two complementary projections denominated by Riesz projections and defined by
\[
P = \frac{1}{2} (I + S) \quad \text{and} \quad Q = \frac{1}{2} (I - S),
\] (1.3)
which are being used in (1.1).

The Hardy space may be now quickly defined by \( H^2(\mathbb{R}) := P(L^2(\mathbb{R})). \)

In a parallel way to (1.2), we will be also using the auxiliary operator
\[
\tilde{A} = \phi_1 I - \phi_2 J.
\]
Additionally, let \( M = M(J) : [L^2(\mathbb{R})]^2 \to [L^2(\mathbb{R})]^2 \) be the operator given in the form of a \( 2 \times 2 \) matrix by
\[
M = \frac{1}{\sqrt{2}} \begin{pmatrix} I & 0 \\ J & -I \end{pmatrix}.
\]
Then,
\[
S_{A,B} = M \begin{pmatrix} T_{A,I} & 0 \\ 0 & T_{\tilde{A},I} \end{pmatrix} M^{-1}
\] (1.4)
is a (pure) singular integral operator. Namely
\[
S_{A,B} = AP + BQ,
\]
with \( 2 \times 2 \) matrix coefficients given by
\[
A = \begin{pmatrix} \phi_1 & 0 \\ \phi_2 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & \phi_2 \\ 0 & \phi_1 \end{pmatrix},
\] (1.5)
Setting
\[
w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\] (1.6)
then, it is relevant to observe that \( B = w\tilde{A}w \).

The identity (1.4) allows us to define the map
\[
\Phi : T_{A,I} \mapsto S_{A,B},
\] (1.7)
for operators (1.1), where the functional operator \( A \) is given by (1.2). The relation between \( A \) and \( \mathcal{A} \) is established in (1.4) and (1.5). The values of \( \Phi \) are (two by two matrix) singular integral operators without shift and with almost periodic coefficients.

The knowledge about the properties of singular integral operators without shift is much more developed than the one about corresponding operators with shift. Thus, using the map (1.7) (with the operator relation (1.4)), we would like to obtain a factorization of singular integral operator with shift \( T_{A,I} \) and then, to derive some of its properties, in particular,
the dimensions of its kernel and cokernel. In the present work, this will be done considering \( \phi_1 \) and \( \phi_2 \) in the class of almost periodic functions \( AP \) by using some of their specific properties (and, consequently, \( A, B \in \mathbb{R}^{2 \times 2} \)).

In the next section we will present all the necessary background. After that, we will obtain several relations between different special factorizations which will be helpful later on. Section 4 is devoted to the construction of an operator equivalence relation between the initial operator and a new operator which will be simpler to analyse. Moreover, the structure and the "representatives" of this last operator will be described. In the last section, we use the previous material to deduce our final main result where a defect numbers description of the operator in study is finally obtained.

2. The Besicovitch space, almost periodic matrix functions and their generalized factorization

The smallest closed subalgebra of \( L^\infty(\mathbb{R}) \) that contains all the functions \( e_\lambda (\lambda \in \mathbb{R}) \), where \( e_\lambda (x) = e^{i\lambda x}, x \in \mathbb{R} \), is denoted by \( AP \) and called the algebra of almost periodic functions: \( AP := \text{alg}_{L^\infty(\mathbb{R})} \{ e_\lambda : \lambda \in \mathbb{R} \} \).

An almost periodic polynomial is a function \( f : \mathbb{R} \to \mathbb{C} \) which can be represented as a finite sum of the form \( f := \sum \lambda f_\lambda e_\lambda \), with \( f_\lambda \in \mathbb{C} \) and \( \lambda \in \mathbb{R} \). The set of all almost periodic polynomials will be denoted by \( AP \). The Besicovitch space \( B^2 \) is defined as the completion of \( AP \) with respect to the norm:

\[
\|f\|_{B^2} := \left( \sum_{\lambda} |f_\lambda|^2 \right)^{\frac{1}{2}}.
\]

Let \( \mathbb{R}_B \) denote the Bohr compactification of \( \mathbb{R} \) and \( d\mu \) the normalized Haar measure on \( \mathbb{R}_B \). It is known that \( AP \) may be identified with \( C(\mathbb{R}_B) \) and \( B^2 \) may be identified with \( L^2(\mathbb{R}_B, d\mu) \). Thus \( B^2 := L^2(\mathbb{R}_B, d\mu) \) is a (non-separable) Hilbert space with the inner product \( (f, g) := \int_{\mathbb{R}_B} f(\xi)\overline{g(\xi)}d\mu(\xi) \). For \( f, g \in AP \) we also have \( (f, g) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x)\overline{g(x)}dx \). Since \( \mu(\mathbb{R}_B) = 1 \), it follows that \( AP \subset \mathbb{B}^2 \), and moreover, \( AP \) is a dense subset of \( \mathbb{B}^2 \).

The mean value

\[
M(f) = \int_{\mathbb{R}_B} f(\xi) d\mu(\xi)
\]

exists and is finite for every \( f \in \mathbb{B}^2 \). The set \( \Omega(f) = \{ \lambda \in \mathbb{R} : M(f e_{-\lambda}) \neq 0 \} \) is called the Bohr–Fourier spectrum of \( f \in \mathbb{B}^2 \), which can be shown to be at most countable.

Analogously to the Besicovitch space \( B^2 \), one can define Besicovitch spaces \( B^p \) (\( 1 \leq p \leq \infty \)). For \( 1 \leq p < \infty \), \( B^p \) may be viewed as the completion of the set of all almost periodic polynomials with respect to the norm

\[
\|f\|_{B^p} = \left( \sum_{\lambda} |f_\lambda|^p \right)^{\frac{1}{p}} = \left( M(|f|^p) \right)^{\frac{1}{p}} = \left( \int_{\mathbb{R}_B} |f(\lambda)|^p d\mu(\lambda) \right)^{\frac{1}{p}}.
\]

\( B^\infty \) is the C*-algebra of all essentially bounded and \( \mu \)-measurable functions on \( \mathbb{R}_B \) and its norm is given by

\[
\|f\|_{B^\infty} = \text{ess sup}_{x \in \mathbb{R}_B} |f(\lambda)|.
\]

For \( p \in [1, 2, \infty) \), put \( B^p_\Delta := \{ \varphi \in B^p : \Omega(\varphi) \subset \mathbb{R}_B \} \) and \( B^p_\circ := \{ \varphi \in B^p : \Omega(\varphi) \subset \mathbb{R}_B \} \). Let \( \ell^2(\mathbb{R}) \) denote the collection of all functions \( f : \mathbb{R} \to \mathbb{C} \) for which the set \( \{ \lambda \in \mathbb{C} : f(\lambda) \neq 0 \} \) is at most countable and \( \|f\|_{\ell^2(\mathbb{R})} := \sum |f(\lambda)|^2 < \infty \). Additionally, \( \ell^\infty(\mathbb{R}) \) denote the set of all functions \( f : \mathbb{R} \to \mathbb{C} \) such that

\[
\|f\|_{\ell^\infty(\mathbb{R})} := \sup_{\lambda \in \mathbb{R}} |f(\lambda)| < \infty.
\]

The Bohr–Fourier transform \( F_B : \ell^2(\mathbb{R}) \to \mathbb{B}^2 \), which sends a function \( f \in \ell^2(\mathbb{R}) \) with a finite support to the function

\[
(F_B f)(\lambda) = \sum_{x \in \mathbb{R}} f(x) e^{i\lambda x}, \quad x \in \mathbb{R},
\]

can be extended by continuity to all \( \ell^2(\mathbb{R}) \). The operator \( F_B \) is an isometric isomorphism and its inverse acts by the rule

\[
(F_B^{-1} f)(\lambda) = M(f e_{-\lambda}), \quad \lambda \in \mathbb{R}.
\]

If \( \phi \in \ell^\infty(\mathbb{R}) \), then the operator \( \psi(\phi) = F_B \phi F_B^{-1} \) is bounded.

The Fredholm characteristics of a singular integral operator with bounded measurable coefficients in \( [L^2(\mathbb{R})]^{N \times N} \) can be obtained through certain factorizations of the involved symbols, in particular, the so-called generalized right AP factorization.
Definition 1. (See [3].) A generalized right AP factorization of a matrix function $\Psi$ in $\mathcal{GAP}^{N \times N}$ is a representation

$$
\Psi = \Psi_{-}A\Psi_{+}
$$

where $A = \text{diag}(e^{i\lambda_1 x}, \ldots, e^{i\lambda_N x})$, with $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$ and

$$
\Psi_{-} \in \mathcal{G}[B^{2}_{-}]^{N \times N}, \quad \Psi_{+} \in \mathcal{G}[B^{2}_{+}]^{N \times N}, \quad \Psi_{-} \tilde{P} \Psi_{-}^{-1} I \in \mathcal{L}(B^{2})^{N}.
$$

Here, $\tilde{P}$ is the projection $\tilde{P} := P_{\mathbb{R}} \chi_{+} P_{\mathbb{R}}^{-1} \in \mathcal{L}(B^{2})^{N}$ (with $\chi_{+}$ being the characteristic function of $\mathbb{R}_{+}$).

3. Special matrix AP factorizations

Let us denote by $\mathcal{AP}^{2 \times 2}_{w}$ the subset of $\mathcal{AP}^{2 \times 2}$ consisting of all matrix functions $\Psi$ for which there exist $A \in \mathcal{AP}^{2 \times 2}$ such that $\Psi = w\tilde{A}^{-1}w A$. We refer to matrix $A$ as a representative of the element $\Psi \in \mathcal{AP}^{2 \times 2}_{w}$. Note that the representatives are not unique. We cite a property of elements of $\mathcal{AP}^{2 \times 2}_{w}$ that will be used below: if $\Psi \in \mathcal{AP}^{2 \times 2}_{w}$, then $\Psi_{-} = w\tilde{\Psi} w$.

Let us introduce the subalgebra $\mathcal{M}^{4 \times 4}_{w}$ of $[L^\infty(\mathbb{R})]^{4 \times 4}$ consisting of all matrix functions $M$ satisfying the relation

$$
M = \begin{pmatrix}
0 & w & w \\
0 & w & 0 \\
0 & w & 0
\end{pmatrix}
$$

Note that $\mathcal{M}^{4 \times 4}_{w}$ contains the subalgebra of block diagonal matrices of the form

$$
M = \begin{pmatrix}
A & 0 \\
0 & w\tilde{A}w
\end{pmatrix}
$$

Definition 2. A special AP factorization of a matrix $M$ in the algebra $\mathcal{M}^{4 \times 4}_{w}$ is a block diagonal factorization of $M$ in the form:

$$
M = \begin{pmatrix}
\mathcal{X} & 0 \\
0 & \mathcal{X}^\prime
\end{pmatrix}
\begin{pmatrix}
R & 0 \\
0 & wRw
\end{pmatrix}
\begin{pmatrix}
\mathcal{Y}_{+} & 0 \\
0 & \mathcal{Y}_{-}
\end{pmatrix}
$$

where $\mathcal{X} = w\tilde{\mathcal{X}} w$, $R \in \mathcal{GAP}^{2 \times 2}$, $\mathcal{Y}_{+} = w\tilde{\mathcal{Y}}_{+} w \in \mathcal{G}[B^{2}_{+}]^{N \times N}$.

Proposition 3. Let $A \in \mathcal{AP}^{2 \times 2}_{w}$ be given. The matrix $M$ admits a special factorization in the algebra $\mathcal{AP}^{4 \times 4} \cap \mathcal{M}^{4 \times 4}_{w}$ if and only if the matrix function $\Psi := B^{-1} A$, where $B = w\tilde{A}w$ admits the representation

$$
\Psi = \mathcal{X}_{-} \mathcal{U} \mathcal{X}_{+}
$$

where $\mathcal{X}_{-} = w\tilde{\mathcal{X}}_{+}^{-1} w \in \mathcal{G}[B^{2}_{-}]^{2 \times 2}$ and $\mathcal{U} \in \mathcal{AP}^{2 \times 2}_{w}$. The relation between the special AP factorization of $M$ and the factorization of $\Psi$ is given by:

$$
\mathcal{X}_{-} = \mathcal{Y}_{-}^{-1}, \quad \mathcal{X}_{+} = \mathcal{Y}_{+} = R^{-1} \mathcal{X}_{-} A, \quad \text{and} \quad \mathcal{U} = w\tilde{R}^{-1} wR.
$$

Proof. Suppose that the matrix function $M$ (3.1) admits a special factorization in $\mathcal{AP}^{4 \times 4} \cap \mathcal{M}^{4 \times 4}_{w}$, and set

$$
\mathcal{X}_{-} = \mathcal{Y}_{-}^{-1}, \quad \mathcal{X}_{+} = \mathcal{Y}_{+}, \quad \text{and} \quad \mathcal{U} = w\tilde{R}^{-1} wR.
$$

It follows that $\mathcal{X}_{-} \in \mathcal{G}[B^{2}_{-}]^{4 \times 4}$, $\mathcal{X}_{+} \in \mathcal{G}[B^{2}_{+}]^{4 \times 4}$ and $\mathcal{U} \in \mathcal{AP}^{4 \times 4}_{w}$. Additionally, $\mathcal{X}_{-}^{-1} = \mathcal{Y}_{-} = w\tilde{\mathcal{Y}}_{+} w = w\tilde{\mathcal{X}}_{+} w$ and

$$
A = \mathcal{X} R \mathcal{X}_{+}
$$

$$
B = w\tilde{A}w = w\tilde{\mathcal{X}} \tilde{R} \tilde{\mathcal{X}}_{+} w = \mathcal{X} w\tilde{R} w \mathcal{X}_{-}^{-1} = \mathcal{X} R U^{-1} \mathcal{X}_{-}^{-1},
$$

where the last equality follows from the definition of $\mathcal{U}$, i.e., $R^{-1} w\tilde{R} w = U^{-1} \Leftrightarrow w\tilde{R} w = R U^{-1}$.

Thus, we obtain

$$
B^{-1} A = (\mathcal{X} R U^{-1} \mathcal{X}_{-}^{-1})^{-1} (\mathcal{X} R \mathcal{X}_{+}) = \mathcal{X} U R^{-1} \mathcal{X}_{-}^{-1} \mathcal{X} R \mathcal{X}_{+} = \mathcal{X} U R \mathcal{X}_{+}.
$$

Suppose now that the matrix function $\Psi$ admits the representation (3.3), with $\mathcal{X}_{+} = R^{-1} \mathcal{X}_{-} A$, where $R$ is such that $\mathcal{U} = w R^{-1} w R$. It follows that $\mathcal{X} = A \mathcal{X}_{+} R^{-1}$ and
\[ w \tilde{\chi} w = w \tilde{\chi} w^{-1} \tilde{R}^{-1} w = B w \tilde{\chi}^{-1} R^{-1} w = B \tilde{\chi} w R^{-1} w = A \psi^{-1} \chi w R^{-1} w \]

\[ = A \psi^{-1} \chi \Lambda R^{-1} = A \psi^{-1} \chi \Lambda^{-1} = A \chi^{-1} R^{-1} \]

\[ = \chi. \]

Thus, the matrix functions \( A \) and \( B \) satisfy the equalities (3.4). Consequently, the block-diagonal matrix function \( M \) can be factorized in the form (3.2), with \( \tilde{Y}_+ \) given by \( \tilde{Y}_- = \chi^{-1} \) and \( \tilde{Y}_+ = \chi_+ \), and we have just concluded that \( M \) admits a special factorization in \( AP_{w^2} \cap \Lambda R \). \( \square \)

### 4. Operator equivalence relation between \( T_{A,\lambda} \) and a new simpler operator

In this section we will start by analyzing several factorization properties which at the end of the first subsection will help us to find an equivalence relation [1] between \( T_{A,\lambda} \) and a new operator – in a sense simpler than the first one. Then, in the second subsection we will look for simple forms of the representatives associated to the last mentioned operator.

#### 4.1. Operator equivalence relation

**Proposition 4.** Let \( \Psi \in G A P_{w^2}^{2 \times 2} \) and suppose that \( \Psi = \psi - A \psi_+ \) is a generalized right AP factorization of \( \psi \) with \( \Lambda = \text{diag}(\varepsilon, 1) \) (\( \lambda_1 \geq \lambda_2 \)). Then, the outer factors of the factorization of \( \Psi \) satisfy:

\[ \tilde{\Psi} = w \tilde{\psi}^{-1} \tilde{H} \]

where \( \tilde{H} = (\tilde{H})_{jk}, \) \( j, k = 1 \) is a matrix function in \( G(B_1^{1 \times 1}) \) such that

\[ \tilde{H} = \Lambda^{-1} \tilde{H}^{-1} \Lambda \]

with \( \det \tilde{H} = -1 \). Moreover,

(i) if \( \lambda_1 = \lambda_2 \), then \( \tilde{H} \) is a constant matrix with \( \tilde{H} = \tilde{H}^{-1} \);

(ii) if \( \lambda_1 > \lambda_2 \), then \( \tilde{H} \) is a lower triangular matrix of the form

\[ \tilde{H} = \begin{pmatrix} \varepsilon & 0 \\ p & -\varepsilon \end{pmatrix} \]

\[ \text{where } \varepsilon \in [-1, 1] \text{ and } p \text{ is an entire function such that } \]

\[ \tilde{p}(x) = e^{(\lambda_1 - \lambda_2)x} p(x). \]

**Proof.** Suppose that \( \Psi = \psi - A \psi_+ \) is a generalized right AP factorization of \( \Psi \in AP_{w^2}^{2 \times 2} \). Any \( \Psi \in AP_{w^2}^{2 \times 2} \) fulfills the relation

\[ \psi = w \psi^{-1} w. \]

Therefore, we also have that

\[ \tilde{\Psi} = w \tilde{\psi}^{-1} \Lambda^{-1} \psi^{-1} \tilde{W} = w \tilde{\psi}^{-1} \Lambda \psi^{-1} \tilde{W} \]

since \( \tilde{\Lambda} = \Lambda^{-1} \). Additionally, we have that \( \tilde{\psi}^{-1} \in G(B_1^{2 \times 2}) \) and \( \tilde{\psi}^{-1} \in G(B_1^{2 \times 2}) \). Consequently, \( \Psi = \psi \Lambda \psi_+ \), with \( \psi_+ = w \psi_+ \) and \( \psi_+ = w \psi^{-1} \) is another factorization of \( \Psi \in AP_{w^2}^{2 \times 2} \).

The equality \( \psi_+ \Lambda = \Lambda \psi_+ \psi_+^{-1} \) implies that

\[ (\psi_+^{-1})^{-1} \psi_+ \Lambda = \Lambda \psi_+ \psi_+^{-1}. \]

Setting \( \tilde{H}_- = (\psi_0^{-1})^{-1} \tilde{H}_- \) and \( \tilde{H}_+ = \psi_0 \psi_+^{-1} \), from (4.5) we obtain the following identity

\[ (\tilde{H})_{jk}(x) = (\tilde{H})_{jk}(x) e^{(\lambda_1 - \lambda_2)x} \]

where \( (\tilde{H})_{jk} \) is the \( j, k \) entry of \( \tilde{H}_\pm \).

If \( \lambda_j \geq \lambda_k \), then the element in the left-hand side of (4.6) is in the class \( B_1 \) and the function in the right-hand side belongs to \( B_1 \), which implies that there exist constants \( c_{jk} \) such that

\[ (\tilde{H}_{-})_{jk}(x) = c_{jk} = (\tilde{H}_{+})_{jk}(x) e^{(\lambda_1 - \lambda_2)x}. \]

Therefore, \( c_{jk} = c_{jk} e^{(\lambda_1 - \lambda_2)x} \). Thus, if \( \lambda_j > \lambda_k \), we obtain that \( c_{jk} = 0 \) and in the case \( \lambda_j = \lambda_k \), we conclude that \( c_{jk} \) are constants. We have proved that
\[(\mathcal{H}_-)_{jk}(x) = \begin{cases} 0, & \text{if } \lambda_j > \lambda_k, \\ c_{jk} = \text{const}, & \text{if } \lambda_j = \lambda_k. \end{cases} \quad (4.7)\]

Assume now that \(\lambda_j < \lambda_k\). We will rewrite equality (4.6) in the following way

\[(\mathcal{H}_-)_{jk}(x)e^{i(\lambda_k - \lambda_j)x} = (\mathcal{H}_+)_{jk}(x). \quad (4.8)\]

In (4.8), the right-hand side belongs to \(B^1_+\) and in the left-hand side we have the product of a \(B^1\) function and a (Wiener) almost periodic element with Bohr–Fourier spectrum equal to \(\lambda_k - \lambda_j > 0\). Therefore, to have the equality in (4.8) we must guaranty that \((\mathcal{H}_-)_{jk}\) has Bohr–Fourier spectrum contained in \([\lambda_j - \lambda_k, 0]\). Thus, the left-hand side of (4.8) extends to an analytic function in \(C^-\) and the right-hand side extends to an analytic function in \(C^+\), and consequently, \((\mathcal{H}_-)_{jk}(x)\) is the restriction to the real line of an entire function.

Letting

\[\mathcal{H} = \mathcal{H}_-, \quad (4.9)\]

we conclude that \(\mathcal{H}\) is an entire function which satisfies (4.7). The triangular structure of \(\mathcal{H}\) implies that \(\det \mathcal{H}(x)\) is a constant and since \(\mathcal{H} = \tilde{\Psi} w \Psi_-\), this constant cannot be zero. Thus, in addition to (4.7), we have that

\[(\mathcal{H})_{jk}(x) = \begin{cases} 0, & \text{if } \lambda_j > \lambda_k, \\ c_{jk} = \text{const} \neq 0, & \text{if } \lambda_j = \lambda_k. \end{cases} \]

Additionally, from (4.9), we obtain that

\[\mathcal{H} = (\Psi_0^-)^{-1} \Psi_- \iff \Psi_0^- \mathcal{H} = \Psi_- \iff w\tilde{\Psi}_-^{-1} \mathcal{H} = \Psi_-.

Thus, (4.1) holds true.

Since \(\mathcal{H} = (\Psi_0^-)^{-1} \Psi_-\), recalling (4.5), we have \(\mathcal{H} = \Lambda \Psi_0^+ \Psi_-^{-1} \Lambda^{-1}\). Thus, we obtain successively:

\[
\begin{align*}
\mathcal{H} &= \Lambda \tilde{\Psi}_0^+ \Psi_-^{-1} \Lambda^{-1} = \Lambda^{-1} \Psi_0^+ \Psi_-^{-1} \Lambda = \Lambda^{-1} \left[ \tilde{\Psi}_0^+ (\Psi_0^-)^{-1} \right]^{-1} \Lambda \\
&= \Lambda^{-1} \left[ \tilde{\Psi}_0^+ w \Psi_-^{-1} \right] \Lambda = \Lambda^{-1} \left[ (\Psi_0^-)^{-1} w w \Psi_-^{-1} \right] \Lambda = \Lambda^{-1} \left[ (\Psi_0^-)^{-1} \Psi_-^{-1} \right] \Lambda \\
&= \Lambda^{-1} \mathcal{H}^{-1} \Lambda,
\end{align*}
\]

and (4.2) is verified.

From the above relation, we conclude that if \(\lambda_1 = \lambda_2\) then \(\mathcal{H}\) is a constant non-singular matrix such that \(\mathcal{H} = \mathcal{H}^{-1}\) (recall that the constants \(c_{jk}\) obviously satisfy \(\tilde{c}_{jk} = c_{jk}\)). In the case \(\lambda_1 > \lambda_2\), we have that

\[
\begin{pmatrix}
c_1 \\
p(x)
\end{pmatrix} = \begin{pmatrix}
c_1^{-1} \\
-c_1^{-1} c_2^{-1} e^{i(\lambda_1 - \lambda_2)x} p(x)
\end{pmatrix}
\]

\[
(4.10)
\]

from which one concludes that the constants \(c_j, j = 1, 2, \) in the main diagonal of \(\mathcal{H}\) are such that \(c_j^2 = 1\).

To draw the other conclusions in the proposition we now use the fact that the shift operator \(J\) has one fixed point \((x = 0)\). As \(\Psi \in AP^{2 \times 2}\), \(\Psi = w A^{-1} w A\) for some \(A \in AP^{2 \times 2}\), we have

\[
\det \Psi(0) = 1 = \det \Psi_-(0) \det \Lambda(0) \det \Psi_+(0) = \det \Psi_-(0) \det \Psi_+(0).
\]

From (4.1), we have \(\mathcal{H} = \tilde{\Psi}_+ w \Psi_-\) and it follows that

\[
\det \mathcal{H} = \det \mathcal{H}(0) = \det \tilde{\Psi}_+(0) \det w \det \Psi_-(0) = - \det \Psi_+(0) \det \Psi_-(0) = -1.
\]

For the case \(\lambda_1 > \lambda_2\), this implies that the constants \(c_j (j = 1, 2)\) in the main diagonal of \(\mathcal{H}\) are such that \(c_1 = -c_2 = \varepsilon \in \{-1, 1\}\) and from (4.10), we also conclude that \(\tilde{p} = e^{i(\lambda_1 - \lambda_2)x} p\). This completes the proof. \(\square\)

**Theorem 5.** Let \(\Psi \in AP^{2 \times 2}_w\) and suppose that \(\Psi = \tilde{\Psi}_- A \Psi_+\) is a generalized right AP factorization of \(\Psi\) in \(AP^{2 \times 2}_w\) with \(\Lambda(x) = \text{diag}(e^{i\lambda_1 x}, e^{i\lambda_2 x})\). Considering the matrix function \(\mathcal{H}\) described in Proposition 4, we have that \(\Psi\) admits the representation

\[
\Psi = \Xi^- D_w \Xi_+,
\]

where:
1. \( X_\sim = \overline{w X_+^{-1}} \in \mathcal{G}\mathbb{B}^2 \times^2 \) is given by \( X_\sim = \Psi_- F^{-1} \), and with \( F \) chosen as follows:
   (i) If \( \lambda_1 = \lambda_2 \), then \( F \) is any matrix such that \( \mathcal{H} F = w F \), with \( \mathcal{H} \) as defined in Proposition 4;
   (ii) If \( \lambda_1 > \lambda_2 \), then \( F = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \).

2. \( D_w \) is any anti-diagonal matrix function, \( D_w = D_w \), with \( D \) depending on the matrix \( \mathcal{H} \) as follows:

\[
D = \text{diag}\{ e^{i\lambda_1 x}, -e^{i\lambda_2 x} \}
\]

except for the case \( \lambda_1 = \lambda_2 \) where we simply have \( D_w = e^{i\lambda_1 x} I \).

**Proof.** Starting with the given factorization of \( \Psi \) in \( \mathcal{A}_w^2 \times^2 \), taking into account the results (4.1) and (4.2) of Proposition 4, we have

\[
\Psi = \Psi_- A \Psi_+ = \Psi_- A \mathcal{H} \Psi_-^* w = \Psi_- A A^{-1} \mathcal{H}^{-1} A \mathcal{H}^{-1} w = \Psi_- \mathcal{H}^{-1} A \mathcal{H}^{-1} w
\]  

(4.11)

where the matrix \( \mathcal{H} \) is defined in Proposition 4.

To proceed, we consider two cases, depending on whether the partial indices of \( \Psi \) are equal or not.

(i) If \( \lambda_1 = \lambda_2 \), then according to Proposition 4, \( \mathcal{H} \) is a constant matrix such that \( \mathcal{H} = \mathcal{H}^{-1} \). Thus, we have

\[
\mathcal{H}^{-1} A = \mathcal{H}^{-1} e^{i\lambda_1 x} = \mathcal{H} e^{i\lambda_1 x}.
\]

Since \( \det \mathcal{H} = -1 \), the matrices \( w \) and \( \mathcal{H} \) are similar. Therefore, there exists a constant non-singular matrix \( F \) such that \( \mathcal{H} F = w F \). Setting \( D_w = D_w = e^{i\lambda_1 x} I \), we have

\[
\mathcal{H} e^{i\lambda_1 x} = F^{-1} w F e^{i\lambda_1 x} = F^{-1} e^{i\lambda_1 x} w F = F^{-1} D_w F.
\]

(ii) If \( \lambda_1 > \lambda_2 \), then from Proposition 4, we know that \( \mathcal{H} \) is a lower triangular matrix function of the form (4.3). Introducing the matrix functions

\[
F = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} \varepsilon p & 1 \end{pmatrix}, \quad D = \text{diag}\{ e^{i\lambda_1 x}, -e^{i\lambda_2 x} \}, \quad \varepsilon \in \{-1, 1\},
\]

and using the property (4.4) of the function \( p \), we obtain that

\[
\mathcal{H}^{-1} A = F^{-1} D^F = F^{-1} D_w F.
\]

For both cases (i) and (ii), substituting the representation obtained for \( \mathcal{H}^{-1} A \) in (4.11), we conclude that

\[
\Psi = \mathcal{X}_- D_w \mathcal{X}_+,
\]

where \( D_w = D_w \) and \( \mathcal{X}_- = \Psi_- F^{-1}, \mathcal{X}_+ = w \mathcal{X}_- w^{-1} \).

\[\square\]

Theorem 5 shows that if we consider a matrix function \( \Psi \in \mathcal{A}_w^2 \times^2 \) which admits a generalized right \( \mathcal{A}_w \) factorization, then we can obtain a representation such that the outer factors are connected in the way we need in (3.3) (Proposition 3), and also the central factor \( D_w \) in that representation belongs to \( \mathcal{A}_w^2 \times^2 \) (as in (3.3)). We have proved in Proposition 3 that obtaining a factorization of this type for a matrix function \( \Psi = B^{-1} A \in \mathcal{A}_w^2 \times^2 \) is equivalent to obtain a special \( \mathcal{A}_w \) factorization of the block-diagonal matrix function \( M = \text{diag}\{ A, B \} \) in \( \mathcal{L}^{2 \times^2} \). The special factorization of \( M \) is just what we need to factorize the operator \( S_{A, B} \), and, consequently, to factorize the operator \( T_{A, I} \). Using these results, we can state the following result.

**Proposition 6.** Let \( A \in \mathcal{G}\mathcal{A}_w^2 \times^2 \) and \( B = w \tilde{A} w \). If the matrix function \( \Psi = B^{-1} A \) admits a generalized right \( \mathcal{A}_w \) factorization, then the matrix function \( M = \text{diag}\{ A, B \} \) admits a special \( \mathcal{A}_w \) factorization in the algebra \( \mathcal{L}^{2 \times^2} \).

We have just collected results which permit us to achieve the desired equivalence operator relation for the singular integral operator \( T_{A, I} \).

**Theorem 7.** Let \( T_{A, I} = \Phi_1 P + \Phi_2 J P + Q \) with coefficients \( \Phi_1, \Phi_2 \in \mathcal{A}_w \) and suppose that the matrix functions \( A \) and \( B \) given by (1.5) are such that \( B \) is invertible in \( \mathcal{A}_w^2 \times^2 \) and \( \Psi = B^{-1} A \) admits a generalized right \( \mathcal{A}_w \) factorization. Then \( T_{A, I} \) is equivalent to \( T_{A_0, B_0} \) in the way that

\[
T_{A, I} = T_{A_1, B_1} T_{A_2, B_2} T_{A_3, B_3}
\]

(4.12)

for invertible operators \( T_{A_j, B_j}, j = 1, 2 \), on the right-hand side of (4.12). Moreover, the new equivalent operator \( T_{A_0, B_0} \) has simple coefficients in the sense that \( \Phi(T_{A_0, B_0}) = S_{R_0, w R_0} \) for \( R \) a representative of the anti-diagonal matrix function \( D_w \in \mathcal{A}_w^2 \times^2 \) given in Theorem 5.
**Proof.** Let $S_{A,B} = \Phi(T_{A,I})$ with $B = w\tilde{A}w$ and set $M = \text{diag}(A, B)$. Suppose that the matrix function $\Psi$ admits a generalized right AP factorization. Then, from Proposition 6, $M$ admits a special factorization

$$M = \text{diag}(Y, X) \text{diag}(R, w\tilde{R}w) \text{diag}(Y_+, Y_-).$$

Since the product of a singular integral operator with equal coefficients by an arbitrary singular integral operator is again a singular integral operator and the product of an arbitrary singular integral operator by a singular integral operator of the form $X_+P + X_-Q$, with $X_{\pm} \in [B_0^2]^{1 \times 2}$ is again a singular integral operator, it follows that $S_{A,B}$ admits the factorization

$$S_{A,B} = S_X \cdot \Phi(S_{R,W} \tilde{w} S_{Y,\tilde{Y}}),$$

where each operator on the right-hand side belongs to the image of the map $\Phi$, and moreover, the outer operators are invertible. Consequently, setting

$$T_{A_1,B_1} = \Phi^{-1}(S_X), \quad T_{A_2,B_2} = \Phi^{-1}(S_{Y,\tilde{Y}}), \quad T_{A_0,B_0} = \Phi^{-1}(S_{R,W} \tilde{w} S_{Y,\tilde{Y}}),$$

we have (4.12) where the outer operators on the right-hand side are invertible. \[\square\]

4.2. On the structure of the representatives

We know already from Theorem 7 that if the matrix $\Psi = B^{-1}A$ admits a generalized right $AP$ factorization then $T_{A,I}$ is equivalent to an operator $T_{A_0,B_0}$ such that

$$\Phi(T_{A_0,B_0}) = w\tilde{R}w(D_w P + Q)$$

for $R$ a representative of $D_w$, with $D_w$ given in Theorem 5. In view of this, our present goal is to obtain a simpler representative which depends on the parameters $\epsilon, \lambda_1$ and $\lambda_2$.

**Lemma 8.** Let $D$ be a diagonal matrix function such that $D_w = D_w \in AP^{2 \times 2}$ with representative $\tilde{R}$. Then

$$\tilde{R}^0 = \tilde{R} \text{diag}\{e^{-i\frac{1}{2}\lambda}x, e^{-i\frac{1}{2}\lambda}x\}$$

is a representative of $D_w^0 = D^0 w$ with $D^0 = \text{diag}(e^{-i\lambda_1 x}, e^{-i\lambda_2 x})D$, for $\lambda_1, \lambda_2 \in \mathbb{R}$.

**Proof.** If $\tilde{R}$ is a representative of $D_w$, then $D_w = D_w = w\tilde{R}^{-1}w\tilde{R}$ and so $wDw = \tilde{R}^{-1}w\tilde{R}$. Let us prove that $D_w^0 = w(\tilde{R}^0)^{-1}w\tilde{R}^0$. In fact, one has

$$w(\tilde{R}^0)^{-1}w\tilde{R}^0 = w \text{ diag}\{e^{-i\frac{1}{2}\lambda}x, e^{-i\frac{1}{2}\lambda}x\} \tilde{R}^{-1}w\tilde{R} \text{ diag}\{e^{-i\frac{1}{2}\lambda}x, e^{-i\frac{1}{2}\lambda}x\} = w \text{ diag}\{e^{-i\frac{1}{2}\lambda}x, e^{-i\frac{1}{2}\lambda}x\} D_w \text{ diag}\{e^{-i\frac{1}{2}\lambda}x, e^{-i\frac{1}{2}\lambda}x\} = w \text{ diag}\{e^{-i\lambda_1 x}, e^{-i\lambda_2 x}\} D_w = D^0 w = D_w^0.$$ 

Thus, $\tilde{R}^0$ is a representative of $D_w^0$, with $D^0 = \text{diag}(e^{-i\lambda_1 x}, e^{-i\lambda_2 x})D$. \[\square\]

**Theorem 9.** Let $D_w \in AP^{2 \times 2}$ be the matrix function given in Theorem 5. Then, $\tilde{R} = \text{diag}(1, e^{i\lambda_1 x})$ is a representative of $D_w$ if $\lambda_1 = \lambda_2$. If $\lambda_1 > \lambda_2$, a representative of $D_w$ is given by

$$\tilde{R} = \left(\begin{array}{cc}
e^{i\frac{3}{2}\lambda_2}x & -\epsilon \ne^{i\frac{1}{2}\lambda_2}x \\
\epsilon \ne^{i\frac{3}{2}\lambda_2}x & \ne^{i\frac{3}{2}\lambda_2}x \end{array}\right) = \left(\begin{array}{cc}
f & -\epsilon \ne^{i\lambda_1 x}g \\
\epsilon \ne^{i\lambda_2 x}f & g \end{array}\right)$$

(4.13)

where

$$g = e^{i\frac{1}{2}\lambda_2}x, \quad f = e^{i\frac{3}{2}\lambda_2}x.$$ 

(4.14)

**Proof.** If $\lambda_1 = \lambda_2$ then $D_w = e^{i\lambda_1 x}I$ (see Theorem 5). Observing that

$$e^{i\lambda_1 x}I = w \text{ diag}\{1, e^{i\lambda_1 x}\} w \text{ diag}\{1, e^{i\lambda_1 x}\},$$

we obtain that $\tilde{R} = \text{diag}(1, e^{i\lambda_1 x})$ is a representative of $D_w$. 

\[\square\]
Now, we consider the case $\lambda_1 > \lambda_2$ for which, according to Theorem 5, $D_w$ is an anti-diagonal matrix function belonging to $A^{D_{w}^{2\times 2}}$.

Using Lemma 8, we obtain that a representative of the matrix function $D_w$ is given by

$$\mathcal{R} = \mathcal{R}^0 \text{diag}[f \cdot g],$$

(4.15)

where $\mathcal{R}^0$ is a representative of $D_w^0 = D^0 w$ with $D^0 = \text{diag}[\epsilon, \epsilon] = \epsilon I$. A representative $\mathcal{R}^0$ of $D_w^0$ is any diagonalizing matrix of $w$, for instance

$$\mathcal{R}^0 = \left( \begin{array}{cc} 1 & -\epsilon \\ \epsilon & 1 \end{array} \right).$$

(4.16)

Substituting the representative of $\mathcal{R}^0$ of $D_w^0$ into (4.15), we obtain the formula for the representative $\mathcal{R}$ of $D_w$ given in the statement of the theorem. □

5. The defect numbers of $T_{A, I}$

In this last section, we are now in a position to obtain the defect numbers of $T_{A, I}$ upon the above considered factorizations and operator relations. We recall, from Theorem 7, that the operator $T_{A, I}$ is equivalent to the simpler operator $T_{A_0, B_0}$ such that

$$\Phi(T_{A_0, B_0}) = \mathcal{R} P + w\mathcal{R} w Q = w\mathcal{R} w(D_w P + Q),$$

for $\mathcal{R}$ a representative of the anti-diagonal matrix function $D_w$ introduced in Theorem 5.

If $\lambda_1 = \lambda_2$, we have that $T_{A, I}$ is simply equivalent to the singular integral operator $T_{A_0, B_0} = e^{i\lambda_1 x} P + Q$. This operator is obviously invertible in case of $\lambda_1 = 0$; left-invertible and properly $n$-normal in case $\lambda_1 > 0$; right-invertible and properly $d$-normal if $\lambda_1 < 0$.

Now, consider the case $\lambda_1 > \lambda_2$. In this situation, a representative $\mathcal{R}$ of $D_w$ is of the form (4.13) with $f$ and $g$ given by (4.14). Thus, $T_{A_0, B_0} = A_0 P + B_0 Q$, with

$$A_0 = f(I + \epsilon e^{-i\lambda_2 x} f), \quad B_0 = \widetilde{g}(I - \epsilon e^{i\lambda_1 x} f).$$

Since $f^{-1} \widetilde{f} = e^{-i\lambda_2 x}$, $g^{-1} \widetilde{g} = e^{-i\lambda_1 x}$, we have

$$A_0 = (I + \epsilon f) f I, \quad B_0 = (I - \epsilon f) \widetilde{g} I.$$  

Let us fix some notation. For $\sigma \in \{-1, 1\}$, introduce the operator

$$P_\sigma = \frac{1}{2}(I + \sigma f),$$

which is a projection operator in $L^2(\mathbb{R})$, having $P_{-\sigma}$ as complementary projection. With this notation, we have

$$A_0 = 2P_\sigma f I, \quad B_0 = 2P_{-\sigma} \widetilde{g} I.$$  

This implies that the operator $T_{A_0, B_0}$ is a direct sum of two corresponding operators:

$$T_{A_0, B_0} = 2P_\sigma f I_+ \oplus 2P_{-\sigma} \widetilde{g} I_-$$

where $I_\pm$ are the identity operators on $\text{im} P$ and $\text{im} Q$, respectively. Additionally, note that $J B_0 f = P_{-\sigma} g$. Hence,

$$T_{A_0, B_0} = 2P_\sigma f I_+ \oplus 2P_{-\sigma} g J I_-.$$  

(5.1)

Let us display this result in a more convenient form. Consider the parameter $\sigma \in \{-1, 1\}$ and a function $h \in \mathcal{G}A\mathcal{P}$ such that

$$h^{-1} \widetilde{h} = e^{-i\lambda x}$$

(5.2)

(for some $\lambda \in \mathbb{R}$) and define the operator

$$V_{\sigma, h} = 2P_\sigma h I_+ : \text{im} P \to \text{im} P_\sigma.$$  

(5.3)

Notice that (by the Bohr representation theorem) for any $h \in \mathcal{G}A\mathcal{P}$ there exists $k(h)$ and $a \in \mathcal{A}\mathcal{P}$ such that $h(x) = e^{i(k(h)x)} e^{a(x)}$, and therefore the existence of $h$ under condition (5.2) is guaranteed. Then

$$T_{A_0, B_0} = V_{\sigma, h} I_+ \oplus J V_{-\sigma, g} J I_-.$$  

(5.4)

Thus, to characterize the operator $T_{A_0, B_0}$ we only need to characterize the operator $V_{\sigma, h}$. 

Proposition 10. Let \( V_{\sigma,h} \) be the operator defined by (5.3) with \( h \) satisfying (5.2): \( h^{-1}h = e^{-i\lambda x} \). Then,

\[
\dim \ker V_{\sigma,h} = \begin{cases} 
0, & \text{if } \lambda > 0, \\
\infty, & \text{if } \lambda < 0,
\end{cases} \quad \codim \im V_{\sigma,h} = \begin{cases} 
0, & \text{if } \lambda \leq 0, \\
\infty, & \text{if } \lambda > 0.
\end{cases}
\]

Proof. Let us start by determining the properties associated with the kernel of the operator \( V_{\sigma,h} \). Suppose that \( \phi^+ \in \AP^+ \) belongs to \( \ker V_{\sigma,h} \), that is \( P_{\sigma} h \phi^+ = 0 \) and set \( \phi^- = J \phi^+ (\phi^- \in \tilde{B}_2^2) \). Thus \( \phi^+ = -\sigma h^{-1}h \phi^- \). Since \( h^{-1}h = e^{-i\lambda x} \), it follows that \( \phi^+ \) and \( \phi^- \) are solutions of the problem

\[
\phi^+ = -\sigma e^{-i\lambda x} \phi^-.
\]

(i) If \( \lambda \geq 0 \), then Eq. (5.5) has only the trivial solution and thus, in this case, \( \ker V_{\sigma,h} = \{0\} \).

(ii) Assume now \( \lambda < 0 \). In the right-hand side of Eq. (5.5) we have the product of \( B_2^2 \) and \( \tilde{B}_2^2 \) functions. To have the equality we must guarantee that the right-hand side belongs to \( B_1^1 \) and therefore, \( \phi^- \) must have Bohr–Fourier spectrum contained in \([\lambda, 0]\). From (5.5) we have that

\[
\phi^+ \in e^{i\lambda x} B_2^2 \cap B_2^2 = \{ \phi \in B^2 : \Omega(\phi) \subset [\lambda, 0] \}.
\]

Since \( \phi^+ = J \phi^- \), we conclude that, in this case, \( \dim \ker V_{\sigma,h} = \infty \).

Now we turn to the analysis of the properties associated with the image of \( V_{\sigma,h} \). For this purpose, consider the adjoint operator \( V^*_{\sigma,h} \), which can be viewed as an operator from \( \im P_{\sigma} \) into \( L^2(\mathbb{R}) \).

Since \( P^{*} = P \), \( J^* = J \) and \( P_{\sigma}^* = P_{\sigma} \), it follows that

\[
V^*_{\sigma,h} = 2P \tilde{h}P_{\sigma} : \im P_{\sigma} \rightarrow \im P.
\]

Suppose that \( \psi \in \ker V^*_{\sigma,h} \), i.e., \( \psi \in \im P_{\sigma} \cap \AP \) and \( P \tilde{h} \psi = 0 \). Then the function \( \psi^- = \tilde{h} \psi \in \im Q_+ \), in particular \( \psi^- \in \AP^- \).

Since \( \psi \in \im P_{\sigma} \), we have that \( \sigma \tilde{h} \psi = \psi \). Taking into account (5.2), it follows that \( \tilde{h} = e^{-i\lambda x} \tilde{h} \psi \). Therefore, we obtain

\[
\tilde{h} \psi = \sigma e^{-i\lambda x} \tilde{h} \psi.
\]

Defining \( \psi^+ = J \psi^- \), we have

\[
\psi^- = \sigma e^{-i\lambda x} \psi^+.
\]

This problem is analogous to the one considered in the first part of the proof.

(j) If \( \lambda \leq 0 \), then the equation has only the trivial solution and thus \( \ker V^*_{\sigma,h} = \{0\} \).

(ji) Suppose now \( \lambda \geq 0 \). The left-hand side of (5.6) belongs to \( \tilde{B}_2^2 \) and in the right-hand side we have the product of \( B_2^2 \) and \( \tilde{B}_2^2 \) functions. Thus, \( \psi^- \) must have Bohr–Fourier spectrum contained in \([0, \lambda]\). From (5.6) we have that

\[
\psi^+ \in e^{i\lambda x} B_2^2 \cap B_2^2 = \{ \psi \in B^2 : \Omega(\psi) \subset [0, \lambda] \}.
\]

Thus, in this case, \( \dim \ker V^*_{\sigma,h} = \infty \).

Since \( \codim \im V_{\sigma,h} = \dim \ker V^*_{\sigma,h} \), this completes the proof. \( \square \)

Theorem 11. Let \( T_{A,B} \) be the singular integral operator defined in (1.1) whose coefficients belong to the \( \AP \) algebra and suppose that the matrix functions \( A \) and \( B \) given by (1.5) are such that \( \tilde{B} \) is invertible and \( \Psi = B^{-1} A \in \AP_{w^2} \) admits a generalized right \( \AP \) factorization in \( \AP_{w^2} \) with partial indices \( \lambda_1 \) and \( \lambda_2 \) (\( \lambda_1 \geq \lambda_2 \)). Then

\[
\dim \ker T_{A,I} = \begin{cases} 
0, & \text{if } \lambda_i \geq 0, \\
\infty, & \text{otherwise},
\end{cases} \quad \codim \im T_{A,I} = \begin{cases} 
0, & \text{if } \lambda_i \leq 0, \\
\infty, & \text{otherwise.}
\end{cases}
\]

\( i = 1, 2 \).

Proof. From Theorem 7 we know already that \( T_{A,I} \) is equivalent to the operator \( T_{A_0,B_0} \) such that \( \Phi(T_{A_0,B_0}) = w \tilde{R}w \times (D_w P + Q) \) and where \( \tilde{R} \) is a representative of \( D_w \).

If \( \lambda_1 = \lambda_2 =: \lambda \), then \( T_{A_0,B_0} \) is the pure singular integral operator \( T_{A_0,B_0} = e^{i\lambda x} P + Q \). Therefore, see e.g. [3, Theorem 2.28], it follows that

\[
\dim \ker T_{A,I} = \begin{cases} 
0, & \text{if } \lambda \geq 0, \\
\infty, & \text{if } \lambda < 0,
\end{cases} \quad \codim \im T_{A,I} = \begin{cases} 
0, & \text{if } \lambda \leq 0, \\
\infty, & \text{if } \lambda > 0.
\end{cases}
\]

(5.7)
In the other case, taking into account the direct sum decompositions (5.4), we have
\begin{align}
\dim \ker T_{A_0, b_0} &= \dim \ker V_{\epsilon, f} + \dim \ker V_{-\epsilon, g}, \\
\text{codim} \im T_{A_0, b_0} &= \text{codim} \im V_{\epsilon, f} + \text{codim} \im V_{-\epsilon, g},
\end{align}

(5.8) (5.9)

where $\epsilon$ is the parameter arising in the characterization obtained in Proposition 4 and $f$ and $g$ are given in (4.14). Additionally, applying Proposition 10, it follows:
\begin{align}
\dim \ker V_j &= \begin{cases} 
0, & \text{if } \lambda_j > 0, \\
\infty, & \text{otherwise},
\end{cases} \\
\text{codim} \im V_j &= \begin{cases} 
0, & \text{if } \lambda_j \leq 0, \\
\infty, & \text{otherwise},
\end{cases}
\end{align}

(5.10)

$j = 1, 2$. Joining (5.8)–(5.9) and (5.10), we obtain the result stated in the theorem.

References


