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Transformation and reduction formulae for double q-Clausen series of type $\Phi^{1:2;\lambda}_{1:1;\mu}$

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Abstract

The Sears transformations are employed to establish several general series transformations for double q-Clausen hypergeometric series of type $\Phi^{1:2;\lambda}_{1:1;\mu}$. These transformations yield further a number of reduction and summation formulae on the double basic hypergeometric series.

Keywords: Basic hypergeometric series; q-Gauss summation formula; q-Chu–Vandermonde convolution; Sears transformations; Double q-Clausen series

1. Introduction

For two indeterminate x and q, the shifted factorial is defined by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = \prod_{k=0}^{n-1}(1-q^k x) \quad \text{with } n = 1, 2, \ldots.$$

When $|q| < 1$, we have the following well-defined infinite product expressions

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-q^k x) \quad \text{and} \quad (x;q)_n = \frac{(x;q)_\infty}{(q^n x;q)_\infty} \quad \text{for } n \in \mathbb{Z}.$$

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The factorial product is abbreviated to

$$[a, b, \dots, c; q]_n := (a; q)_n (b; q)_n \cdots (c; q)_n.$$

Following Gasper and Rahman [4], the basic hypergeometric series is defined by

$$ {}_{1+r}\phi_s \left[\begin{matrix} a_0, & a_1, & \dots, & a_r \\ & b_1, & \dots, & b_s \end{matrix} \middle| q; z \right] = \sum_{n=0}^{\infty} \left\{ (-1)^n q^{\binom{n}{2}} \right\}^{s-r} \frac{[a_0, a_1, \dots, a_r; q]_n}{[q, b_1, \dots, b_s; q]_n} z^n \tag{1.1}$$

where the base q will be restricted to $|q| < 1$ for non-terminating series.

As the q-analogue of Kampé de Fériet function, Srivastava and Karlsson [7, p. 349] defined the generalized bivariate basic hypergeometric function by

$$\Phi^{\lambda:r;s}_{\mu:u;v} \left[\begin{matrix} \alpha_1, \dots, \alpha_\lambda: & a_1, \dots, a_r; & c_1, \dots, c_s; & q: x, y \\ \beta_1, \dots, \beta_\mu: & b_1, \dots, b_u; & d_1, \dots, d_v; & i, j, k \end{matrix} \right] \tag{1.2a}$$

$$= \sum_{m,n=0}^{\infty} \frac{[\alpha_1, \dots, \alpha_\lambda; q]_{m+n}}{[\beta_1, \dots, \beta_\mu; q]_{m+n}} \frac{[a_1, \dots, a_r; q]_m [c_1, \dots, c_s; q]_n}{[b_1, \dots, b_u; q]_m [d_1, \dots, d_v; q]_n} \frac{x^m y^n q^{i\binom{m}{2} + j\binom{n}{2} + kmn}}{(q; q)_m (q; q)_n}. \tag{1.2b}$$

It is not hard to check that when $i, j, k \in \mathbb{N}_0$, the double series $\Phi^{\lambda:r;s}_{\mu:u;v}$ is convergent for $|x| < 1$, $|y| < 1$ and $|q| < 1$.

Following the definitions in [2], the series $\Phi^{\lambda:r;s}_{\mu:u;v}$ is said to be terminating (non-terminating) if it is terminating (non-terminating) with respect to both summation indices m and n. In the mixed case, we call $\Phi^{\lambda:r;s}_{\mu:u;v}$ semi-terminating if it is terminating with respect to one of summation indices m and n and non-terminating with respect to another summation index.

For double basic hypergeometric series, there are fewer literature on this work. See Chu and Srivastava [3], Chu and Jia [2], Jia and Wang [5], Singh [6] and Van der Jeugt [8] for references. Specially in [2,5], the authors gave several general transformations for $\Phi^{0:3;\lambda}_{1:1;\mu}$, $\Phi^{2:1;\lambda}_{0:2;\mu}$ and $\Phi^{2:1;\lambda}_{2:0;\mu}$ and also obtained a number of transformation, reduction and summation formulae on Φ^{122}_{022}, Φ^{123}_{023}, Φ^{033}_{111}, Φ^{034}_{112}, Φ^{212}_{201}, Φ^{213}_{202} and Φ^{214}_{203} as special cases.

In this paper, we shall investigate another type q-Clausen hypergeometric series $\Phi^{1:2;\lambda}_{1:1;\mu}$ again by employing the Sears transformations. Seven general transformations for non-terminating, semi-terminating and terminating series $\Phi^{1:2;\lambda}_{1:1;\mu}$ are established, some of which are closely related to other types of q-Clausen functions just mentioned. Furthermore, we derive several reduction and summation formulae for Φ^{122}_{111}, Φ^{123}_{112} and Φ^{124}_{113} as consequences.

2. Non-terminating double series $\Phi^{1:2;\lambda}_{1:1;\mu}$

Theorem 2.1 *(Transformation formula). For an arbitrary complex sequence $\{\Omega(j)\}$, there holds the following transformation*

$$\sum_{i,j=0}^{\infty} \frac{(a; q)_{i+j} (c; q)_i (e; q)_i}{(b; q)_{i+j} (d; q)_i (q; q)_i (q; q)_j} \left(\frac{bd}{ace} \right)^i \Omega(j) \tag{2.1a}$$

$$= \frac{[d/c, bd/ae; q]_\infty}{[d, bd/ace; q]_\infty} \sum_{i,j=0}^{\infty} \left(\frac{d}{c} \right)^i \frac{(b/e; q)_{i+j} [c, b/a; q]_i (a; q)_j}{(b; q)_{i+j} [q, bd/ae; q]_i [q, b/e; q]_j} \Omega(j) \tag{2.1b}$$

provided that two double series displayed above are absolutely convergent.

Proof. Recalling the q-analogue of the Kummer–Thomae–Whipple transformation [4, III-9]

$$
{}_3\phi_2\left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix}\,\middle|\, q; \frac{bd}{ace}\right] = \frac{[d/c, bd/ae; q]_\infty}{[d, bd/ace; q]_\infty}\,{}_3\phi_2\left[\begin{matrix} c, & b/a, & b/e \\ & b, & bd/ae \end{matrix}\,\middle|\, q; \frac{d}{c}\right] \tag{2.2}
$$

we can reformulate the double sum in (2.1a) as follows:

$$
\begin{aligned}
&\sum_{j=0}^{\infty} \frac{(a;q)_j}{(q;q)_j(b;q)_j} \Omega(j)\,{}_3\phi_2\left[\begin{matrix} q^j a, c, e \\ q^j b, d \end{matrix}\,\middle|\, q; \frac{bd}{ace}\right] \\
&\quad = \frac{[d/c, bd/ae; q]_\infty}{[d, bd/ace; q]_\infty} \sum_{j=0}^{\infty} \frac{(a;q)_j}{(q;q)_j(b;q)_j} \Omega(j)\,{}_3\phi_2\left[\begin{matrix} c, b/a, q^j b/e \\ q^j b,\ bd/ae \end{matrix}\,\middle|\, q; \frac{d}{c}\right].
\end{aligned}
$$

Writing the last double sum explicitly, we see that it coincides with (2.1b). □

When the Ω-sequence is specified by

$$
\Omega(j) = \frac{[u_1, u_2, \ldots, u_\lambda; q]_j}{[v_1, v_2, \ldots, v_\mu; q]_j} w^j \tag{2.3}
$$

the last theorem gives us a very general transformation between two non-terminating double series $\Phi^{1:2;\lambda}_{1:1;\mu}$ and $\Phi^{1:2;\lambda+1}_{1:1;\mu+1}$.

In the proof of the last theorem, if we apply, instead of (2.2), the Hall transformation [4, III-10]

$$
{}_3\phi_2\left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix}\,\middle|\, q; \frac{bd}{ace}\right] = \frac{[c, bd/ac, bd/ce; q]_\infty}{[b, d, bd/ace; q]_\infty}\,{}_3\phi_2\left[\begin{matrix} b/c, d/c, bd/ace \\ bd/ac, bd/ce \end{matrix}\,\middle|\, q; c\right] \tag{2.4}
$$

then we can establish another transformation formula.

Theorem 2.2 *(Transformation formula). For an arbitrary complex sequence $\{\Omega(j)\}$, there holds the following transformation*

$$
\sum_{i,j=0}^{\infty} \frac{(a;q)_{i+j}(c;q)_i(e;q)_i}{(b;q)_{i+j}(d;q)_i(q;q)_i(q;q)_j} \left(\frac{bd}{ace}\right)^i \Omega(j) \tag{2.5a}
$$

$$
= \frac{[c, bd/ac, bd/ce; q]_\infty}{[b, d, bd/ace; q]_\infty} \sum_{i,j=0}^{\infty} c^i \frac{(b/c;q)_{i+j}[d/c, bd/ace; q]_i (a;q)_j}{(bd/ce;q)_{i+j}[q, bd/ac; q]_i [q, b/c; q]_j} \Omega(j) \tag{2.5b}
$$

provided that two double series displayed above are absolutely convergent.

Under specification (2.3), this theorem also yields a transformation between two non-terminating double series $\Phi^{1:2;\lambda}_{1:1;\mu}$ and $\Phi^{1:2;\lambda+1}_{1:1;\mu+1}$.

2.1.

Specifying in Theorem 2.1 with

$$
\Omega(j) = \frac{[b/e, \beta; q]_j}{(a;q)_j} \left(\frac{e}{\beta}\right)^j
$$

and then evaluating the sum with respect to j displayed in (2.1b) by means of the q-Gauss summation theorem [4, II-8]

$$ {}_2\phi_1\left[\begin{matrix} a,\ b \\ c \end{matrix}\,\Big|\, q;\ \frac{c}{ab}\right] = \frac{[c/a, c/b; q]_\infty}{[c, c/ab; q]_\infty}, \tag{2.6} $$

we find after some trivial simplification the following reduction formula.

Proposition 2.3 *(Reduction formula).*

$$ \Phi^{1:2;2}_{1:1;1}\left[\begin{matrix} a\colon & c,\ e; & b/e,\ \beta; & q\colon & bd/ace, & e/\beta \\ b\colon & d; & a; & 0, & 0, & 0 \end{matrix}\right] = \frac{[e, b/\beta, d/c, bd/ae; q]_\infty}{[b, d, e/\beta, bd/ace; q]_\infty}\, {}_3\phi_2\left[\begin{matrix} c, b/a, b/e \\ b/\beta, bd/ae \end{matrix}\,\Big|\, q;\ \frac{d}{c}\right]. $$

2.2.

Letting in Theorem 2.1

$$ \Omega(j) = \frac{[q^{-n}, b/e, \gamma; q]_j}{[a, q^{1-n}\gamma/e; q]_j} q^j $$

and then reformulating the corresponding (2.1b) by using the q-Pfaff–Saalschütz formula [4, II-12]

$$ {}_3\phi_2\left[\begin{matrix} q^{-n},\ \alpha, & \beta \\ \gamma, & q^{1-n}\alpha\beta/\gamma \end{matrix}\,\Big|\, q; q\right] = \frac{[\gamma/\alpha, \gamma/\beta; q]_n}{[\gamma, \gamma/\alpha\beta; q]_n}, \tag{2.7} $$

we find the following reduction formula.

Proposition 2.4 *(Reduction formula).*

$$ \Phi^{1:2;3}_{1:1;2}\left[\begin{matrix} a\colon & c,\ e; & q^{-n},\ b/e, & \gamma; & q\colon & bd/ace, & q \\ b\colon & d; & a, & q^{1-n}\gamma/e; & 0, & 0, & 0 \end{matrix}\right] = \frac{[e, b/\gamma; q]_n}{[b, e/\gamma; q]_n} \frac{[d/c, bd/ae; q]_\infty}{[d, bd/ace; q]_\infty}\, {}_4\phi_3\left[\begin{matrix} c, b/a, b/e, q^n b/\gamma \\ b/\gamma, bd/ae, q^n b \end{matrix}\,\Big|\, q;\ \frac{d}{c}\right]. $$

For $a = q^{-n}$ in particular, we get another reduction formula from the last proposition.

Corollary 2.5 *(Reduction formula).*

$$ \Phi^{1:2;2}_{1:1;1}\left[\begin{matrix} q^{-n}\colon & c,\ e; & b/e, & \gamma; & q\colon & q^n bd/ce, & q \\ b\colon & d; & & q^{1-n}\gamma/e; & 0, & 0, & 0 \end{matrix}\right] = \frac{[e, b/\gamma; q]_n}{[b, e/\gamma; q]_n} \frac{[d/c, q^n bd/e; q]_\infty}{[d, q^n bd/ce; q]_\infty}\, {}_3\phi_2\left[\begin{matrix} c, b/e, q^n b/\gamma \\ b/\gamma, q^n bd/e \end{matrix}\,\Big|\, q; d/c\right]. $$

2.3.

Setting in Theorem 2.2

$$ \Omega(j) = \frac{[b/c, \beta; q]_j}{(a; q)_j}\left(\frac{d}{e\beta}\right)^j $$

and then rewriting the corresponding (2.1b) by (2.6) again, we find the following reduction formula.

Proposition 2.6 *(Reduction formula).*

$$\Phi^{1:2;2}_{1:1;1}\left[\begin{matrix} a: & c, & e; & b/c, & \beta; & q: & bd/ace, & d/e\beta \\ b: & & d; & & a; & 0, & 0, & 0 \end{matrix}\right]$$
$$= \frac{[c, bd/ac, bd/ce\beta, d/e; q]_\infty}{[b, d, bd/ace, d/e\beta; q]_\infty} {}_3\phi_2\left[\begin{matrix} b/c, d/c, bd/ace \\ bd/ac, bd/ce\beta \end{matrix} \middle| q; c\right].$$

2.4.

Taking in Theorem 2.2

$$\Omega(j) = \frac{[q^{-n}, b/c, \beta; q]_j}{[a, q^{1-n}e\beta/d; q]_j} q^j$$

and then reformulating the corresponding j sum in (2.1b) by (2.7), we establish the following reduction formula.

Proposition 2.7 *(Reduction formula).*

$$\Phi^{1:2;3}_{1:1;2}\left[\begin{matrix} a: & c, & e; & q^{-n}, & b/c, & \beta; & q: & bd/ace, & q \\ b: & & d; & & a, & q^{1-n}e\beta/d; & 0, & 0, & 0 \end{matrix}\right]$$
$$= \frac{[d/e, bd/ce\beta; q]_n}{[bd/ce, d/e\beta; q]_n} \frac{[c, bd/ac, bd/ce; q]_\infty}{[b, d, bd/ace; q]_\infty} {}_4\phi_3\left[\begin{matrix} b/c, d/c, bd/ace, q^n bd/ce\beta \\ bd/ac, q^n bd/ce, bd/ce\beta \end{matrix} \middle| q; c\right].$$

3. Semi-terminating double series $\Phi^{1:2;\lambda}_{1:1;\mu}$

Theorem 3.1 *(Transformation formula). For an arbitrary complex sequence $\{\Omega(j)\}$, there holds the following transformation*

$$\frac{[b, d; q]_n}{[c, bd/ac; q]_n} \sum_{i,j=0}^{\infty} \frac{(a; q)_{i+j}(q^{-n}; q)_i (c; q)_i}{(b; q)_{i+j}(d; q)_i (q; q)_i (q; q)_j} q^i \Omega(j) \tag{3.1a}$$
$$= a^n \sum_{i,j=0}^{\infty} q^{j(n-i)} \frac{(b/c; q)_{i+j}(q^{-n}; q)_i (d/c; q)_i (a; q)_j}{[q, q^{1-n}/c, bd/ac; q]_i [q, q^n b, b/c; q]_j} \left(\frac{q}{a}\right)^i \Omega(j) \tag{3.1b}$$

provided that two semi-terminating double series displayed above are absolutely convergent.

Proof. By means of the Sears transformation (cf. [1, p. 79]):

$${}_3\phi_2\left[\begin{matrix} q^{-n}, & a, & c \\ & b, & d \end{matrix} \middle| q; q\right] = \frac{[c, bd/ac; q]_n}{[b, d; q]_n} a^n {}_3\phi_2\left[\begin{matrix} q^{-n}, & b/c, & d/c \\ & q^{1-n}/c, & bd/ac \end{matrix} \middle| q; \frac{q}{a}\right] \tag{3.2}$$

we can proceed as follows:

$$\text{Eq. (3.1a)} = \frac{(b;q)_n(d;q)_n}{(c;q)_n(bd/ac;q)_n}\sum_{j=0}^{\infty}\frac{(a;q)_j}{(q;q)_j(b;q)_j}\Omega(j)\,{}_3\phi_2\left[\begin{matrix} q^{-n},\ q^j a,\ c \\ q^j b,\ d\end{matrix}\,\middle|\, q;q\right]$$
$$= a^n\sum_{j=0}^{\infty} q^{jn}\frac{(a;q)_j}{(q;q)_j(q^n b;q)_j}\Omega(j)\,{}_3\phi_2\left[\begin{matrix} q^{-n}, q^j b/c, d/c \\ q^{1-n}/c, bd/ac\end{matrix}\,\middle|\, q;q^{1-j}/a\right],$$

which leads us to (3.1b) when writing as a double sum. □

Under specification (2.3), this theorem gives a transformation between two semi-terminating double series $\Phi^{1:2;\lambda}_{1:1;\mu}$ and $\Phi^{1:2;\lambda+1}_{0:2;\mu+2}$.

In the proof of the last theorem, if we apply instead of (3.2) another Sears transformation (cf. [1, p. 79]):

$$ {}_3\phi_2\left[\begin{matrix} q^{-n},\ a,\ c \\ b,\ d\end{matrix}\,\middle|\, q;q\right] = \frac{(d/a;q)_n}{(d;q)_n}a^n\,{}_3\phi_2\left[\begin{matrix} q^{-n},\ a,\ \ b/c \\ b,\ q^{1-n}a/d\end{matrix}\,\middle|\, q;qc/d\right], \tag{3.3}$$

we would obtain the following general transformation formulae.

Theorem 3.2 *(Transformation formula). For an arbitrary complex sequence* $\{\Omega(j)\}$*, there holds the following transformation*

$$\frac{(d;q)_n}{(d/a;q)_n}\sum_{i,j=0}^{\infty}\frac{(a;q)_{i+j}(q^{-n};q)_i(c;q)_i}{(b;q)_{i+j}(d;q)_i(q;q)_i(q;q)_j}q^i\Omega(j) \tag{3.4a}$$
$$= a^n\sum_{i,j=0}^{\infty}\left(\frac{qc}{d}\right)^i\frac{[a,b/c;q]_{i+j}(q^{-n};q)_i(qa/d;q)_j}{[b,q^{1-n}a/d;q]_{i+j}(q;q)_i[q,b/c;q]_j}\Omega(j) \tag{3.4b}$$

provided that both semi-terminating double series are absolutely convergent.

Under specification (2.3), this theorem reduces to a transformation between two semi-terminating double series $\Phi^{1:2;\lambda}_{1:1;\mu}$ and $\Phi^{2:1;\lambda+1}_{2:0;\mu+1}$.

It should be pointed out that the special case of the last theorem has first been established in [3] by a different method.

Theorem 3.3 *(Transformation formula). For an arbitrary complex sequence* $\{\Omega(j)\}$*, there holds the following transformation*

$$\frac{(d;q)_n}{(d/c;q)_n}\sum_{i,j=0}^{\infty}\frac{(a;q)_{i+j}(q^{-n};q)_i(c;q)_i}{(b;q)_{i+j}(d;q)_i(q;q)_i(q;q)_j}q^i\Omega(j) \tag{3.5a}$$
$$= c^n\sum_{i,j=0}^{\infty}q^{ij}\frac{[q^{-n},c,b/a;q]_i}{(b;q)_{i+j}[q,q^{1-n}c/d;q]_i}\frac{(a;q)_j}{(q;q)_j}\left(\frac{qa}{d}\right)^i\Omega(j) \tag{3.5b}$$

provided that two semi-terminating double series displayed above are absolutely convergent.

Under specification (2.3), this theorem becomes a transformation between two semi-terminating double series $\Phi^{1:2;\lambda}_{1:1;\mu}$ and $\Phi^{0:3;\lambda+1}_{1:1;\mu+1}$.

3.1.

Letting in Theorem 3.1

$$\Omega(j)=\frac{[q^{-m},q^{-n}e,q^nb,b/c;q]_j}{[a,q^{-m}bde/ac,qa/d;q]_j}q^j$$

and then evaluating the corresponding (3.1b) by [2, Proposition 3.3]

$$\Phi^{1:2;2}_{0:2;2}\left[\begin{matrix} e: & q^{-n}, & a; & q^{-m}, & q^{-n}b; & q: & q^ncd/ae, & q^{1+n} \\ -: & c, & d; & q^{-m}bd, & qe/d; & 0, & 0, & -1 \end{matrix}\right] \tag{3.6a}$$

$$=\frac{(d/e;q)_n[qe/bd,q^{1-n}/d;q]_m}{(d;q)_n[q/bd,q^{1-n}e/d;q]_m}\times {}_4\phi_3\left[\begin{matrix} q^{-n},c/a,e,q^{1+m}e/bd \\ c,qe/bd,q^{1+m-n}e/d \end{matrix}\,\middle|\,q;q\right], \tag{3.6b}$$

we get the following reduction formula.

Proposition 3.4 *(Reduction formula).*

$$\Phi^{1:2;4}_{1:1;3}\left[\begin{matrix} a: & q^{-n}, & c; & q^{-m}, & q^{-n}e, & q^nb, & b/c; & q: & q, & q \\ b: & & d; & & a, & q^{-m}bde/ac, & qa/d; & 0, & 0, & 0 \end{matrix}\right]$$
$$=a^n\frac{[c,d/a;q]_n}{[b,d;q]_n}\frac{[qa/de,q^{1-n}ac/bd;q]_m}{[qac/bde,q^{1-n}a/d;q]_m}{}_4\phi_3\left[\begin{matrix} q^{-n},q^{1-n}/d,b/c,q^{1+m}a/de \\ q^{1-n}/c,qa/de,q^{1-n+m}a/d \end{matrix}\,\middle|\,q;q\right].$$

Alternatively, letting in Theorem 3.1

$$\Omega(j)=\frac{[q^{-m},q^{-n}e,b/c;q]_j}{[a,q^{1-n-m}e/c;q]_j}q^j$$

and then simplifying the corresponding (3.1b) by (3.6a)–(3.6b) again, we establish another reduction formula.

Proposition 3.5 *(Reduction formula).*

$$\Phi^{1:2;3}_{1:1;2}\left[\begin{matrix} a: & q^{-n}, & c; & q^{-m}, & q^{-n}e, & b/c; & q: & q, & q \\ b: & & d; & & a, & q^{1-m-n}e/c; & 0, & 0, & 0 \end{matrix}\right]$$
$$=\left(\frac{ac}{b}\right)^n\frac{(bd/ac;q)_n}{(d;q)_n}\frac{[c,q^nb/e;q]_m}{[b,q^nc/e;q]_m}{}_4\phi_3\left[\begin{matrix} q^{-n},b/a,b/c,q^{n+m}b/e \\ q^nb/e,q^mb,bd/ac \end{matrix}\,\middle|\,q;q\right].$$

3.2.

Specializing in Theorem 3.1 with

$$\Omega(j)=\begin{cases} \dfrac{[\alpha,\beta;q]_j}{(a;q)_j}\left(\dfrac{b}{\alpha\beta}\right)^j, \\ \dfrac{[q^nb,\beta,\gamma;q]_j}{[a,qa/d;q]_j}\left(\dfrac{q^{1-n}a}{d\beta\gamma}\right)^j, \end{cases}$$

and then transforming the corresponding (3.1b) by [2, Proposition 3.4]

$$\Phi^{1:2;2}_{0:2;2}\left[\begin{matrix} e: & a, & q^{-n}; & \alpha, & \beta; & q: & q^n cd/ae, & qe/c\alpha\beta \\ -: & c, & d; & qe/c, & e; & 0, & 0, & -1 \end{matrix}\right]$$
$$= \frac{(d/a;q)_n}{(d;q)_n}\frac{[qe/c\alpha, qe/c\beta;q]_\infty}{[qe/c, qe/c\alpha\beta;q]_\infty}\,{}_4\phi_3\left[\begin{matrix} q^{-n}, a, c\alpha/e, c\beta/e \\ c, q^{1-n}a/d, c\alpha\beta/e \end{matrix}\,\Big|\, q;q\right],$$

we derive the following two reduction formulae.

Proposition 3.6 *(Reduction formula).*

$$\Phi^{1:2;2}_{1:1;1}\left[\begin{matrix} a: & q^{-n}, & c; & \alpha, & \beta; & q: & q, & b/\alpha\beta \\ b: & & d; & & a; & 0, & 0, & 0 \end{matrix}\right]$$
$$= c^n\frac{(d/c;q)_n}{(d;q)_n}\frac{[b/\alpha, b/\beta;q]_\infty}{[b, b/\alpha\beta;q]_\infty}\,{}_4\phi_3\left[\begin{matrix} q^{-n}, c, b/a, b/\alpha\beta \\ q^{1-n}c/d, b/\alpha, b/\beta \end{matrix}\,\Big|\, q;\frac{qa}{d}\right].$$

Proposition 3.7 *(Reduction formula).*

$$\Phi^{1:2;3}_{1:1;2}\left[\begin{matrix} a: & q^{-n}, & c; & q^n b, & \beta, & \gamma; & q: & q, & q^{1-n}a/d\beta\gamma \\ b: & & d; & & a, & qa/d; & 0, & 0, & 0 \end{matrix}\right]$$
$$= \left(\frac{ac}{d}\right)^n\frac{(bd/ac;q)_n}{(b;q)_n}\frac{[qa/d\beta, qa/d\gamma;q]_\infty}{[qa/d, qa/d\beta\gamma;q]_\infty}\,{}_4\phi_3\left[\begin{matrix} q^{-n}, d/c, d\beta/a, d\gamma/a \\ d, bd/ac, d\beta\gamma/a \end{matrix}\,\Big|\, q;q\right].$$

Letting further $\beta \to 1/c$ and $\gamma \to a$, the last ${}_4\phi_3$-series reduces to a ${}_2\phi_1$-series. Evaluating it by the q-Chu–Vandermonde convolution formula [4, II-6]

$$ {}_2\phi_1\left[\begin{matrix} q^{-n}, & a \\ & c \end{matrix}\,\Big|\, q;q\right] = \frac{(c/a;q)_n}{(c;q)_n}a^n \tag{3.7}$$

we obtain the following closed formula.

Corollary 3.8 *(Summation formula).*

$$\Phi^{1:2;2}_{1:1;1}\left[\begin{matrix} a: & q^{-n}, & c; & q^n b, & 1/c; & q: & q, & q^{1-n}c/d \\ b: & & d; & & qa/d; & 0, & 0, & 0 \end{matrix}\right] = \frac{[q/d, qac/d;q]_\infty}{[qa/d, qc/d;q]_\infty}.$$

3.3.

Taking in Theorem 3.1

$$\Omega(j) = \frac{[q^n b, \alpha;q]_j}{(\gamma;q)_j}\left(\frac{q^{-n}\gamma}{a\alpha}\right)^j$$

and then evaluating the corresponding (3.1b) by [2, Proposition 3.9]

$$\Phi^{1:2;2}_{0:2;2}\left[\begin{matrix} e: & a, & q^{-n}; & \alpha, & q^{1-n}ae/cd; & q: & q^n cd/ae, & q^{n-1}cd\gamma/ae\alpha \\ -: & c, & d; & \gamma, & e; & 0, & 0, & -1 \end{matrix}\right]$$
$$= \frac{[a, cd/ae;q]_n}{[c,d;q]_n}\frac{[\gamma/\alpha, cd\gamma/qae;q]_\infty}{[\gamma, cd\gamma/qae\alpha;q]_\infty}\,{}_4\phi_3\left[\begin{matrix} q^{-n}, c/a, d/a, cd\gamma/qae\alpha \\ q^{1-n}/a, cd/ae, cd\gamma/qae \end{matrix}\,\Big|\, q;q\right],$$

we derive the following reduction formula.

Proposition 3.9 *(Reduction formula).*

$$\Phi^{1:2;2}_{1:1;1}\left[\begin{matrix} a: & q^{-n}, & c; & q^n b, & \alpha; & q: & q, & q^{-n}\gamma/a\alpha \\ b: & & d; & & \gamma; & 0, & 0, & 0 \end{matrix}\right] = a^n \frac{(b/a;q)_n}{(b;q)_n}\frac{[\gamma/a,\gamma/\alpha;q]_\infty}{[\gamma,\gamma/a\alpha;q]_\infty}{}_4\phi_3\left[\begin{matrix} q^{-n},a,d/c,qa/\gamma \\ d,qa\alpha/\gamma,q^{1-n}a/b\end{matrix}\,\Big|\, q;\frac{qc\alpha}{b}\right].$$

3.4.

Setting in Theorem 3.1 with

$$\Omega(j)=\begin{cases}\dfrac{[b/c,\alpha,\beta;q]_j}{[a,q^{1-n}\alpha\beta/d;q]_j}\left(\dfrac{q^{1-n}c}{d}\right)^j, \\ \dfrac{[q^n b,b/c,\alpha,\beta;q]_j}{[a,qa/d,b\alpha\beta/a;q]_j}\left(\dfrac{q^{1-n}c}{d}\right)^j,\end{cases}$$

and then transforming the corresponding (3.1b) by [2, Proposition 3.10]

$$\Phi^{1:2;2}_{0:2;2}\left[\begin{matrix} e: & a, & q^{-n}; & \alpha, & \beta; & q: & q^n cd/ae, & q/a \\ -: & c, & d; & qe/c, & c\alpha\beta/a; & 0, & 0, & -1 \end{matrix}\right] = \frac{(d/e;q)_n}{(d;q)_n}\frac{[q/c,qe/a;q]_\infty}{[q/a,qe/c;q]_\infty}{}_4\phi_3\left[\begin{matrix} e,qe/d,c\alpha/a,c\beta/a \\ qe/a,q^{1-n}e/d,c\alpha\beta/a\end{matrix}\,\Big|\, q;q^{1-n}/c\right],$$

we have two further reduction formulae.

Proposition 3.10 *(Reduction formula).*

$$\Phi^{1:2;3}_{1:1;2}\left[\begin{matrix} a: & q^{-n}, & c; & b/c, & \alpha, & \beta; & q: & q, & q^{1-n}c/d \\ b: & & d; & & a, & q^{1-n}\alpha\beta/d; & 0, & 0, & 0 \end{matrix}\right] = a^n \frac{(d/a;q)_n}{(d;q)_n}\frac{[c,qb/d;q]_\infty}{[b,qc/d;q]_\infty}{}_4\phi_3\left[\begin{matrix} b/c,qa/d,q^{1-n}\alpha/d,q^{1-n}\beta/d \\ qb/d,q^{1-n}a/d,q^{1-n}\alpha\beta/d\end{matrix}\,\Big|\, q;c\right].$$

Proposition 3.11 *(Reduction formula).*

$$\Phi^{1:2;4}_{1:1;3}\left[\begin{matrix} a: & q^{-n}, & c; & q^n b, & b/c, & \alpha, & \beta; & q: & q, & q^{1-n}c/d \\ b: & & d; & & a, & qa/d, & b\alpha\beta/a; & 0, & 0, & 0 \end{matrix}\right] = \left(\frac{ac}{b}\right)^n \frac{(bd/ac;q)_n}{(d;q)_n}\frac{[qac/bd,qb/d;q]_\infty}{[qa/d,qc/d;q]_\infty}{}_4\phi_3\left[\begin{matrix} q^n b,b/c,b\alpha/a,b\beta/a \\ b,qb/d,b\alpha\beta/a\end{matrix}\,\Big|\, q;\frac{q^{1-n}ac}{bd}\right].$$

3.5.

Specializing in Theorem 3.1

$$c=q^m b \quad\text{and}\quad \Omega(j)=\begin{cases}\dfrac{[q^{-m},\alpha,\beta;q]_j}{[a,q^{1-m}\alpha\beta/b;q]_j}q^j, \\ \dfrac{[q^{-m},q^n b,\alpha,\beta;q]_j}{[a,qa/d,q^{n-m}d\alpha\beta/a;q]_j}q^j,\end{cases}$$

and then rewriting the corresponding (3.1b) through [2, Proposition 3.11]

$$\Phi^{1:2;2}_{0:2;2}\left[\begin{matrix} q^{-m}: & q^{-n}, & a; & \alpha, & \beta; & q: & q^{n+m}cd/a, & q^{1+n} \\ -: & c, & d; & q^{1-m}/c, & q^n c\alpha\beta; & 0, & 0, & -1 \end{matrix}\right]$$
$$= \frac{[q^n c\alpha, q^n c\beta; q]_m}{[c, q^n c\alpha\beta; q]_m} \times {}_4\phi_3\left[\begin{matrix} q^{-m}, q^{-n}, d/a, q^{1-n-m}/c\alpha\beta \\ d, q^{1-n-m}/c\alpha, q^{1-n-m}/c\beta \end{matrix}\,\middle|\, q; q\right],$$

we deduce the following reduction formulae respectively.

Proposition 3.12 *(Reduction formula).*

$$\Phi^{1:2;3}_{1:1;2}\left[\begin{matrix} a: & q^{-n}, & q^m b; & q^{-m}, & \alpha, & \beta; & q: & q, & q \\ b: & & d; & & a, & q^{1-m}\alpha\beta/b; & 0, & 0, & 0 \end{matrix}\right]$$
$$= q^{mn}a^n \frac{(q^{-m}d/a; q)_n}{(d; q)_n} \frac{[b/\alpha, b/\beta; q]_m}{[b, b/\alpha\beta; q]_m} {}_4\phi_3\left[\begin{matrix} q^{-m}, q^{-n}, b/a, b/\alpha\beta \\ q^{-m}d/a, b/\alpha, b/\beta \end{matrix}\,\middle|\, q; q\right].$$

Proposition 3.13 *(Reduction formula).*

$$\Phi^{1:2;4}_{1:1;3}\left[\begin{matrix} a: & q^{-n}, & q^m b; & q^{-m}, & q^n b, & \alpha & \beta; & q: & q, & q \\ b: & & d; & & q^{n-m}d\alpha\beta/a, & qa/d, & a; & 0, & 0, & 0 \end{matrix}\right]$$
$$= q^{mn}a^n \frac{[q^m b, q^{-m}d/a; q]_n}{[b,\ d; q]_n} \frac{[q^{1-n}a/d\alpha, q^{1-n}a/d\beta; q]_m}{[qa/d, q^{1-n}a/d\alpha\beta; q]_m}$$
$$\times {}_4\phi_3\left[\begin{matrix} q^{-m}, q^{-n}, q^{1-n}/d, q^{1-n}a/d\alpha\beta \\ q^{1-n-m}/b, q^{1-n}a/d\alpha, q^{1-n}a/d\beta \end{matrix}\,\middle|\, q; q\right].$$

3.6.

Putting in Theorem 3.1 with

$$c = q^m b \quad \text{and} \quad \Omega(j) = \begin{cases} \dfrac{[q^{-m}, \beta; q]_j}{(\gamma; q)_j}\left(\dfrac{q^m b\gamma}{a\beta}\right)^j, \\ \dfrac{[q^{-m}, q^n b, \beta; q]_j}{[qa/d, \gamma; q]_j}\left(\dfrac{q^{1+m-n}\gamma}{d\beta}\right)^j, \end{cases}$$

and then reformulating the corresponding (3.1b) through [2, Proposition 3.12]

$$\Phi^{1:2;2}_{0:2;2}\left[\begin{matrix} q^{-m}: & q^{-n}, a; & q^{1-n-m}a/cd, \beta; & q: & q^n cd/ae, & q^{n+m}d\gamma/a\beta \\ -: & c, d; & q^{1-m}/c, \gamma; & & 0, \quad 0, & -1 \end{matrix}\right]$$
$$= \frac{[q^n cd/a, \gamma/\beta; q]_m}{[c, \gamma; q]_m} \times {}_4\phi_3\left[\begin{matrix} q^{-m}, d/a,\ q^n d,\ q^{1-m}/\gamma \\ d, q^n cd/a, q^{1-m}\beta/\gamma \end{matrix}\,\middle|\, q; q^m c\beta\right],$$

we find the following reduction formulae respectively.

Proposition 3.14 *(Reduction formula).*

$$\Phi^{1:2;2}_{1:1;1}\left[\begin{matrix} a: & q^{-n}, & q^m b; & q^{-m}, & \beta; & q: & q, & q^m b\gamma/a\beta \\ b: & & d; & & \gamma; & 0, & 0, & 0 \end{matrix}\right]$$
$$= a^n \frac{(d/a; q)_n}{(d; q)_n} \frac{(b/a; q)_m}{(b; q)_m} {}_4\phi_3\left[\begin{matrix} q^{-m}, a, qa/d, \gamma/\beta \\ q^{1-m}a/b, q^{1-n}a/d, \gamma \end{matrix}\,\middle|\, q; q\right].$$

Proposition 3.15 *(Reduction formula).*

$$\Phi^{1:2;3}_{1:1;2}\left[\begin{matrix} a: & q^{-n}, \; q^m b; & q^{-m}, \; q^n b, \; \beta; & q: \; q, \; q^{1+m-n}\gamma/d\beta \\ b: & d; & qa/d, \; \gamma; & 0, \; 0, \; 0 \end{matrix}\right]$$
$$= q^{mn}a^n \frac{(q^{-m}d/a;q)_n}{(d;q)_n}\frac{(q^{1-n}/d;q)_m}{(qa/d;q)_m}{}_4\phi_3\left[\begin{matrix} q^{-m}, a, q^n b, \gamma/\beta \\ q^{n-m}d, b, \gamma \end{matrix} \,\Big|\, q; q\right].$$

3.7.

Putting in Theorem 3.2

$$\Omega(j) = \frac{[\alpha,\beta;q]_j}{(qa/d;q)_j}\left(\frac{qb}{d\alpha\beta}\right)^j$$

and then rewriting the corresponding (3.4b) through [5, Proposition 2.4]

$$\Phi^{2:1;2}_{2:0;1}\left[\begin{matrix} a, \; c: & e; & \beta, \; \gamma; & q: \; bd/ace, \; bd/ae\beta\gamma \\ b, \; d: & -; & c; & 0, \; 0, \; 0 \end{matrix}\right] \tag{3.8a}$$
$$= \frac{[a, bd/ac, bd/ae\beta, bd/ae\gamma;q]_\infty}{[b, d, bd/ace, bd/ae\beta\gamma;q]_\infty}{}_4\phi_3\left[\begin{matrix} b/a, d/a, bd/ace, bd/ae\beta\gamma \\ bd/ac, bd/ae\beta, bd/ae\gamma \end{matrix} \,\Big|\, q; a\right], \tag{3.8b}$$

we get the following reduction formula.

Proposition 3.16 *(Reduction formula).*

$$\Phi^{1:2;2}_{1:1;1}\left[\begin{matrix} a: & q^{-n}, \; c; & \alpha, \; \beta; & q: \; q, \; qb/d\alpha\beta \\ b: & d; & qa/d; & 0, \; 0, \; 0 \end{matrix}\right]$$
$$= c^n \frac{(d/c;q)_n}{(d;q)_n}\frac{[a, qb/d\alpha, qb/d\beta;q]_\infty}{[b, qa/d, qb/d\alpha\beta;q]_\infty}{}_4\phi_3\left[\begin{matrix} b/a, qc/d, q^{1-n}/d, qb/d\alpha\beta \\ q^{1-n}c/d, qb/d\alpha, qb/d\beta \end{matrix} \,\Big|\, q; a\right].$$

3.8.

Similarly, letting in Theorem 3.2

$$\Omega(j) = \frac{[b/c,\beta,\gamma;q]_j}{[a, qa/d;q]_j}\left(\frac{qac}{d\beta\gamma}\right)^j$$

and then transforming the corresponding (3.4b) through (3.8a)–(3.8b) again, we establish another reduction formula.

Proposition 3.17 *(Reduction formula).*

$$\Phi^{1:2;3}_{1:1;2}\left[\begin{matrix} a: & q^{-n}, \; c; & b/c, \; \beta, \; \gamma; & q: \; q, \; qac/d\beta\gamma \\ b: & d; & a, \; qa/d; & 0, \; 0, \; 0 \end{matrix}\right]$$
$$= c^n \frac{(d/c;q)_n}{(d;q)_n}\frac{[b/c, qac/d\beta, qac/d\gamma;q]_\infty}{[b, qa/d, qac/d\beta\gamma;q]_\infty}$$
$$\times {}_4\phi_3\left[\begin{matrix} c, qc/d, qac/d\beta\gamma, q^{1-n}ac/bd \\ qac/d\beta, qac/d\gamma, q^{1-n}c/d \end{matrix} \,\Big|\, q; \frac{b}{c}\right].$$

3.9.

Specializing in Theorem 3.3 with

$$\Omega(j)=\begin{cases}\dfrac{[b/c,\alpha,\beta;q]_j}{[qa/d,q^{-n}\alpha\beta;q]_j}\left(\dfrac{q^{1-n}c}{d}\right)^j,\\ \dfrac{[q^nb,\alpha,\beta;q]_j}{[qa/d,c\alpha\beta;q]_j}\left(\dfrac{q^{1-n}c}{d}\right)^j,\end{cases}$$

and then reformulating the corresponding (3.5b) by [2, Proposition 2.5]

$$\Phi^{0:3;4}_{1:1;2}\left[\begin{matrix}-: & a,b,c; & d/a,d/b,\alpha,\beta; & q: & de/abc,e\\ d: & e; & de/abc,c\alpha\beta; & & 0,0,1\end{matrix}\right]$$
$$=\frac{[e/c,de/ab;q]_\infty}{[e,de/abc;q]_\infty}{}_4\phi_3\left[\begin{matrix}d/a,\ d/b,c\alpha,c\beta\\ d,\ de/ab,c\alpha\beta\end{matrix}\,\middle|\,q;e/c\right],$$

we deduce the following two reduction formulae.

Proposition 3.18 *(Reduction formula).*

$$\Phi^{1:2;3}_{1:1;2}\left[\begin{matrix}a: & q^{-n}, & c; & b/c,\alpha,\beta; & q: & q, & q^{1-n}c/d\\ b: & & d; & qa/d,q^{-n}\alpha\beta; & 0, & 0, & 0\end{matrix}\right]$$
$$=a^n\frac{(d/a;q)_n}{(d;q)_n}\times{}_4\phi_3\left[\begin{matrix}a,b/c,q^{-n}\alpha,q^{-n}\beta\\ b,q^{1-n}a/d,q^{-n}\alpha\beta\end{matrix}\,\middle|\,q;qc/d\right].$$

Proposition 3.19 *(Reduction formula).*

$$\Phi^{1:2;3}_{1:1;2}\left[\begin{matrix}a: & q^{-n}, & c; & q^nb, & \alpha, & \beta,; & q: & q, & q^{1-n}c/d\\ b: & & d; & & qa/d, & c\alpha\beta; & 0, & 0, & 0\end{matrix}\right]$$
$$=\frac{[q/d,qac/d;q]_\infty}{[qa/d,qc/d;q]_\infty}\times{}_4\phi_3\left[\begin{matrix}a,q^nb,c\alpha,c\beta\\ b,qac/d,c\alpha\beta\end{matrix}\,\middle|\,q;q^{1-n}/d\right].$$

3.10.

Taking in Theorem 3.3

$$\Omega(j)=\frac{[b/c,q^nb,\beta;q]_j}{[a,\gamma;q]_j}\left(\frac{q^{-n}c\gamma}{b\beta}\right)^j$$

and then reformulating the corresponding (3.5b) through [2, Proposition 2.10]

$$\Phi^{0:3;3}_{1:1;1}\left[\begin{matrix}-: & q^nd,b,c; & q^{-n},d/b,\beta; & q: & q^{-n}e/bc,q^nb\gamma/\beta\\ d: & e; & \gamma; & & 0,\quad 0,\quad 1\end{matrix}\right]$$
$$=c^n\frac{[b,qb/e;q]_n}{[d,qbc/e;q]_n}\frac{[e/b,e/c;q]_\infty}{[e,e/bc;q]_\infty}{}_4\phi_3\left[\begin{matrix}q^{-n},d/b,\gamma/\beta,q^{-n}e/bc\\ \gamma,q^{1-n}/b,q^{-n}e/b\end{matrix}\,\middle|\,q;q\right],$$

we get the following reduction formula.

Proposition 3.20 *(Reduction formula).*

$$\Phi^{1:2;3}_{1:1;2}\left[\begin{matrix} a: & q^{-n}, & c; & b/c, & q^n b, & \beta; & q: & q, & q^{-n}c\gamma/\beta \\ b: & & d; & & a, & \gamma; & & 0, & 0, \quad 0 \end{matrix}\right]$$
$$= a^n \frac{[c, bd/ac; q]_n}{[b, d; q]_n} \frac{[c\gamma/b, \gamma/\beta; q]_\infty}{[\gamma, c\gamma/b\beta; q]_\infty} {}_4\phi_3\left[\begin{matrix} q^{-n}, b/c, d/c, qb/c\gamma \\ q^{1-n}/c, bd/ac, qb\beta/c\gamma \end{matrix}\,\Big|\, q; q\beta/a\right].$$

4. Terminating double series $\Phi^{1:2;\lambda}_{1:1;\mu}$

Theorem 4.1 *(Transformation formula). For an arbitrary complex sequence $\{\Omega(j)\}$, there holds the following transformation*

$$\frac{(d;q)_n}{(d/a;q)_n} \sum_{i,j=0}^{n} \frac{(q^{-n};q)_{i+j}(a;q)_i(c;q)_i}{(b;q)_{i+j}(d;q)_i(q;q)_i(q;q)_j} q^i \Omega(j) \tag{4.1a}$$

$$= a^n \sum_{i,j=0}^{n} \frac{[q^{-n}, b/c; q]_{i+j}(a;q)_i(q^{1-n}/d;q)_j}{[b, q^{1-n}a/d; q]_{i+j}(q;q)_i[q, b/c; q]_j} \left(\frac{qc}{d}\right)^i \Omega(j). \tag{4.1b}$$

Under specification (2.3), this theorem gives a transformation between two terminating double series $\Phi^{1:2;\lambda}_{1:1;\mu}$ and $\Phi^{2:1;\lambda+1}_{2:0;\mu+1}$.

Proof. Recalling transformation (3.3), we may manipulate the double sum in (4.1a) as follows:

$$\sum_{j=0}^{n} \frac{(q^{-n};q)_j}{(q;q)_j(b;q)_j} \Omega(j) {}_3\phi_2\left[\begin{matrix} q^{-n+j}, & a, & c \\ & q^j b, & d \end{matrix}\,\Big|\, q; q\right]$$
$$= \sum_{j=0}^{n} \frac{(q^{-n};q)_j}{(q;q)_j(b;q)_j} \Omega(j) \frac{(d/a;q)_{n-j}}{(d;q)_{n-j}} a^{n-j} {}_3\phi_2\left[\begin{matrix} q^{-n+j}, a, q^j b/c \\ q^j b, q^{1-n+j}a/d \end{matrix}\,\Big|\, q; qc/d\right]$$
$$= a^n \frac{(d/a;q)_n}{(d;q)_n} \sum_{j=0}^{n} \frac{[q^{-n}, q^{1-n}/d; q]_j}{[q, b, q^{1-n}a/d; q]_j} \Omega(j) {}_3\phi_2\left[\begin{matrix} q^{-n+j}, a, q^j b/c \\ q^j b, q^{1-n+j}a/d \end{matrix}\,\Big|\, q; qc/d\right].$$

Writing the last expression as a double sum, we confirm the theorem stated in Theorem 4.1. □

In the proof of the last theorem, if we apply (3.2) instead of (3.3), we would find the following transformation formula.

Theorem 4.2 *(Transformation formula). For an arbitrary complex sequence $\{\Omega(j)\}$, there holds the following transformation*:

$$\frac{[b, d; q]_n}{[c, bd/ac; q]_n} \sum_{i,j=0}^{n} \frac{(q^{-n};q)_{i+j}(a;q)_i(c;q)_i}{(b;q)_{i+j}(d;q)_i(q;q)_i(q;q)_j} q^i \Omega(j) \tag{4.2a}$$

$$= a^n \sum_{i,j=0}^{n} \left(\frac{q}{a}\right)^i \frac{[q^{-n}, b/c; q]_{i+j}(d/c;q)_i(q^{1-n}/d;q)_j}{[q^{1-n}/c, bd/ac; q]_{i+j}(q;q)_i[q, b/c; q]_j} \left(\frac{d}{ac}\right)^j \Omega(j). \tag{4.2b}$$

Under specification (2.3), this theorem yields a transformation between two terminating double series $\Phi^{1:2;\lambda}_{1:1;\mu}$ and $\Phi^{2:1;\lambda+1}_{2:0;\mu+1}$.

4.1.

Taking in Theorem 4.1 by

$$\Omega(j)=\frac{[b/c,\alpha,\beta;q]_j}{[q^{1-n}/d,d\alpha\beta/c;q]_j}q^j$$

and then evaluating the corresponding j-sum in (4.1b) by [5, Proposition 2.8]

$$\Phi^{2:1;2}_{2:0;1}\left[\begin{matrix} q^{-n},\ c: & e; & \alpha, & \beta; & q: & q^nbd/ce, & q \\ b,\ \ d: & -; & q^{1-n}ce\alpha\beta/bd; & & 0, & 0, & 0\end{matrix}\right]$$
$$=\frac{[c,bd/ce\alpha,bd/ce\beta;q]_n}{[b,d,bd/ce\alpha\beta;q]_n}{}_4\phi_3\left[\begin{matrix} q^{-n},b/c,d/c,bd/ce\alpha\beta \\ q^{1-n}/c,bd/ce\alpha,bd/ce\beta\end{matrix}\,\middle|\, q;q\right],$$

we can establish the following reduction formula.

Proposition 4.3 *(Reduction formula).*

$$\Phi^{1:2;3}_{1:1;2}\left[\begin{matrix} q^{-n}: & a,\ c; & b/c, & \alpha, & \beta; & q: & q, & q \\ b: & d; & & q^{1-n}/d, & d\alpha\beta/c; & 0, & 0, & 0\end{matrix}\right]$$
$$=\frac{[c,bd/ac;q]_n}{[b,d;q]_n}a^n{}_4\phi_3\left[\begin{matrix} q^{-n},b/c,d\alpha/c,d\beta/c \\ q^{1-n}/c,bd/ac,d\alpha\beta/c\end{matrix}\,\middle|\, q;q/a\right].$$

For $\alpha\to b/a$ and $\beta\to q^{1-n}/d$ in particular, the last ${}_4\phi_3$-series reduces to a ${}_2\phi_1$-series. Evaluating it by (2.6), we get the following identity.

Corollary 4.4 *(Summation formula).*

$$\Phi^{1:2;2}_{1:1;1}\left[\begin{matrix} q^{-n}: & a,c; & b/a,b/c; & q: & q,q \\ b: & d; & q^{1-n}d/ac; & & 0,0,0\end{matrix}\right]=\frac{[a,c,bd/ac;q]_n}{[b,d,ac/b;q]_n}\left(\frac{ac}{b}\right)^n.$$

4.2.

Putting in Theorem 4.1

$$\Omega(j)=\frac{[b/a,\beta;q]_j}{(\gamma;q)_j}\left(\frac{q^na\gamma}{\beta}\right)^j$$

and then reformulating the corresponding (4.1b) by [5, Proposition 2.10]

$$\Phi^{2:1;3}_{2:0;2}\left[\begin{matrix} q^{-n},\ c: & e; & b/e,d/e,\ \beta; & q: & q^nbd/ce,q^ne\gamma/\beta \\ b,\ d: & -; & c,\ \gamma; & & 0,\ \ 0,\ \ 0\end{matrix}\right]$$
$$=\frac{[e,bd/ce;q]_n}{[b,d;q]_n}{}_4\phi_3\left[\begin{matrix} q^{-n}, & b/e, & d/e, & \gamma/\beta \\ & q^{1-n}/e, & bd/ce, & \gamma\end{matrix}\,\middle|\, q;q\right],$$

we obtain the following reduction formula.

Proposition 4.5 *(Reduction formula).*

$$\Phi^{1:2;2}_{1:1;1}\left[\begin{matrix} q^{-n}: & a,\ c; & b/a,\ \beta; & q:\ q,\ q^n a\gamma/\beta \\ b: & d; & \gamma; & 0,\ 0,\ 0 \end{matrix}\right]$$
$$= c^n \frac{[a, d/c; q]_n}{[b, d; q]_n} {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1-n}/d, b/a, \gamma/\beta \\ q^{1-n}/a, q^{1-n}c/d, \gamma \end{matrix}\,\Big|\, q; q\right].$$

4.3.

Setting in Theorem 4.1

$$\Omega(j) = \frac{[b/a, b/c, \beta; q]_j}{[q^{1-n}/d, \gamma; q]_j}\left(\frac{qac\gamma}{bd\beta}\right)^j$$

and then rewriting the corresponding (4.1b) by [5, Proposition 2.11]

$$\Phi^{2:1;2}_{2:0;1}\left[\begin{matrix} q^{-n}, c: & e; & b/e, \beta; & q:\ q^n bd/ce, q^n d\gamma/c\beta \\ b, d: & -; & \gamma; & 0,\ 0,\ 0 \end{matrix}\right]$$
$$= \frac{(d/c; q)_n}{(d; q)_n} {}_4\phi_3\left[\begin{matrix} q^{-n}, & c, & b/e, & \gamma/\beta \\ & q^{1-n}c/d, & b, & \gamma \end{matrix}\,\Big|\, q; q\right],$$

we derive the following reduction formula.

Proposition 4.6 *(Reduction formula).*

$$\Phi^{1:2;3}_{1:1;2}\left[\begin{matrix} q^{-n}: & a,\ c; & b/a,\ b/c,\ \beta; & q:\ q,\ qac\gamma/bd\beta \\ b: & d; & q^{1-n}/d,\ \gamma; & 0,\ 0,\ 0 \end{matrix}\right]$$
$$= \left(\frac{ac}{b}\right)^n \frac{(bd/ac; q)_n}{(d; q)_n} \times {}_4\phi_3\left[\begin{matrix} q^{-n},\ b/a,\ b/c,\ \gamma/\beta \\ b,\ bd/ac,\ \gamma \end{matrix}\,\Big|\, q; q\right].$$

4.4.

Specializing in Theorem 4.1 with

$$\Omega(j) = \begin{cases} \dfrac{[\alpha, \beta; q]_j}{(q^{1-n}ac\alpha\beta/bd; q)_j}\left(\dfrac{qac}{d}\right)^j, \\ \dfrac{[b/a, \alpha, \beta; q]_j}{[q^{1-n}/d, c\alpha\beta; q]_j}\left(\dfrac{qac}{d}\right)^j, \end{cases}$$

and then reformulating the corresponding (4.1b) through [5, Proposition 2.12]

$$\Phi^{2:1;3}_{2:0;2}\left[\begin{matrix} a,\ c: & e; & b/e,\ \alpha,\ \beta; & q:\ bd/ace,\ bd/ac \\ b,\ d: & -; & c,\ b\alpha\beta/c; & 0,\ 0,\ 0 \end{matrix}\right]$$
$$= \frac{[d/a, bd/ce; q]_\infty}{[d, bd/ace; q]_\infty} {}_4\phi_3\left[\begin{matrix} a, b/e, b\alpha/c, b\beta/c \\ b, bd/ce, b\alpha\beta/c \end{matrix}\,\Big|\, q; d/a\right],$$

we have the following reduction formulae, respectively.

Proposition 4.7 *(Reduction formula).*

$$\Phi^{1:2;2}_{1:1;1}\left[\begin{matrix} q^{-n}: & a,\ c; & \alpha, & \beta; & q: & q,\ qac/d \\ b: & d; & q^{1-n}ac\alpha\beta/bd; & & 0,\ 0, & 0 \end{matrix}\right]$$
$$= \left(\frac{ac}{d}\right)^n \frac{[bd/ac\alpha, bd/ac\beta; q]_n}{[b, bd/ac\alpha\beta; q]_n} {}_4\phi_3\left[\begin{matrix} q^{-n}, d/a, d/c, bd/ac\alpha\beta \\ d, bd/ac\alpha, bd/ac\beta \end{matrix}\,\middle|\, q; q\right].$$

Proposition 4.8 *(Reduction formula).*

$$\Phi^{1:2;3}_{1:1;2}\left[\begin{matrix} q^{-n}: & a, c; & b/a, \alpha, \beta; & q: q, qac/d \\ b: & d; & q^{1-n}/d, c\alpha\beta; & 0, \quad 0, \quad 0 \end{matrix}\right]$$
$$= \frac{(d/c; q)_n}{(d; q)_n} c^n \times {}_4\phi_3\left[\begin{matrix} q^{-n}, b/a, c\alpha, c\beta \\ b,\ q^{1-n}c/d,\ c\alpha\beta \end{matrix}\,\middle|\, q; qa/d\right].$$

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