Transformation and reduction formulae for double $q$-Clausen series of type $\Phi_{1:1;\lambda}^{1:2;\mu}$

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Abstract

The Sears transformations are employed to establish several general series transformations for double $q$-Clausen hypergeometric series of type $\Phi_{1:1;\lambda}^{1:2;\mu}$. These transformations yield further a number of reduction and summation formulae on the double basic hypergeometric series.

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1. Introduction

For two indeterminate $x$ and $q$, the shifted factorial is defined by

$$(x; q)_0 = 1 \quad \text{and} \quad (x; q)_n = \prod_{k=0}^{n-1} (1 - q^k x) \quad \text{with} \ n = 1, 2, \ldots .$$

When $|q| < 1$, we have the following well-defined infinite product expressions

$$(x; q)_\infty = \prod_{k=0}^{\infty} (1 - q^k x) \quad \text{and} \quad (x; q)_n = \frac{(x; q)_\infty}{(q^n x; q)_\infty} \quad \text{for} \ n \in \mathbb{Z}. $$
The factorial product is abbreviated to
\[ (a, b, \ldots, c; q)_n := (a; q)_n (b; q)_n \cdots (c; q)_n. \]
Following Gasper and Rahman [4], the basic hypergeometric series is defined by
\[
1 + r \Phi_{\lambda;r:s} \left[ \begin{array}{c} a_0, a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \right| q; z \right] = \sum_{n=0}^{\infty} \left( \frac{(-1)^n q^n}{[q, b_1, \ldots, b_s; q]_n} \right) z^n \tag{1.1}
\]
where the base \( q \) will be restricted to \(|q| < 1 \) for non-terminating series.
As the \( q \)-analogue of Kampé de Fériet function, Srivastava and Karlsson [7, p. 349] defined the generalized bivariate basic hypergeometric function by
\[
\Phi_{\mu;w:v}^{\lambda;r:s} \left[ \begin{array}{c} \alpha_1, \ldots, \alpha_\lambda; \beta_1, \ldots, \beta_\mu; \\ c_1, \ldots, c_s; q; x, y \\
\end{array} \right] = \sum_{m,n=0}^{\infty} \frac{[\alpha_1, \ldots, \alpha_\lambda; q]_{m+n}}{[\beta_1, \ldots, \beta_\mu; q]_{m+n}} \frac{[c_1, \ldots, c_s; q]_n}{[b_1, \ldots, b_s; q]_n} x^m y^n q^{(m)\lambda+(n)\mu+k m n} \tag{1.2a}
\]
\[
\text{It is not hard to check that when } i, j, k \in \mathbb{N}_0, \text{ the double series } \Phi_{\mu;w:v}^{\lambda;r:s} \text{ is convergent for } |x| < 1, |y| < 1 \text{ and } |q| < 1.
\]
Following the definitions in [2], the series \( \Phi_{\mu;w:v}^{\lambda;r:s} \) is said to be terminating (non-terminating) if it is terminating (non-terminating) with respect to both summation indices \( m \) and \( n \). In the mixed case, we call \( \Phi_{\mu;w:v}^{\lambda;r:s} \) semi-terminating if it is terminating with respect to one of summation indices \( m \) and \( n \) and non-terminating with respect to another summation index.
For double basic hypergeometric series, there are fewer literature on this work. See Chu and Srivastava [3], Chu and Jia [2], Jia and Wang [5], Singh [6] and Van der Jeugt [8] for references. Specially in [2,5], the authors gave several general transformations for \( \Phi_{1:1;1}^{0:3;0} \), \( \Phi_{0:2;0}^{2:1;2} \) and \( \Phi_{2:0;2}^{2:1;0} \) and also obtained a number of transformation, reduction and summation formulae on \( \Phi_{122}^{022}, \Phi_{123}^{023}, \Phi_{111}^{033}, \Phi_{112}^{034}, \Phi_{212}^{210}, \Phi_{213}^{213} \) and \( \Phi_{203}^{214} \) as special cases.
In this paper, we shall investigate another type \( q \)-Clausen hypergeometric series \( \Phi_{1;1;1}^{1:2;\lambda} \) again by employing the Sears transformations. Seven general transformations for non-terminating, semi-terminating and terminating series \( \Phi_{1;1:1}^{1:2;\lambda} \) are established, some of which are closely related to other types of \( q \)-Clausen functions just mentioned. Furthermore, we derive several reduction and summation formulae for \( \Phi_{111}^{122}, \Phi_{112}^{123} \) and \( \Phi_{113}^{124} \) as consequences.

2. Non-terminating double series \( \Phi_{1;1;1}^{1:2;\lambda} \)

**Theorem 2.1** (Transformation formula). For an arbitrary complex sequence \( \{\Omega(j)\} \), there holds the following transformation
\[
\sum_{i,j=0}^{\infty} \frac{(a; q)_{i+j}(c; q)_i(e; q)_i}{(b; q)_i(d; q)_i(q; q)_i(q; q)_j} \left( \frac{bd}{ace} \right)^i \Omega(j) \tag{2.1a}
\]
\[
= \frac{[d/c, bd/ae; q]_\infty}{[d, bd/ace; q]_\infty} \sum_{i,j=0}^{\infty} \frac{(d/c)^i (b/e; q)_{i+j}[c, b/a; q]_i(a; q)_j}{(b; q)_{i+j}[q, bd/ae; q]_i[q, b/e; q]_j} \Omega(j) \tag{2.1b}
\]
provided that two double series displayed above are absolutely convergent.
Proof. Recalling the $q$-analogue of the Kummer–Thomae–Whipple transformation [4, III-9]

$$3\phi_2 \left[ \begin{array}{c|c} a, c, e \\ b, d \end{array} \right| q; \frac{bd}{ace} = \frac{[d/c, bd/ae; q]_{\infty}}{[d, bd/ace; q]_{\infty}} 3\phi_2 \left[ \begin{array}{c|c} c, b/a, b/e \\ b, bd/ae \end{array} \right| q; \frac{d}{c} \right]$$

we can reformulate the double sum in (2.1a) as follows:

$$\sum_{j=0}^{\infty} \frac{(a; q)_j}{(q; q)_j (b; q)_j} \Omega(j) 3\phi_2 \left[ \begin{array}{c|c} q^j a, c, e \\ q^j b, d \end{array} \right| q; \frac{bd}{ace} = \frac{[d/c, bd/ae; q]_{\infty}}{[d, bd/ace; q]_{\infty}} \sum_{j=0}^{\infty} \frac{(a; q)_j}{(q; q)_j (b; q)_j} \Omega(j) 3\phi_2 \left[ \begin{array}{c|c} c, b/a, q^j b/e \\ q^j b, bd/ae \end{array} \right| q; \frac{d}{c} \right].$$

Writing the last double sum explicitly, we see that it coincides with (2.1b).

When the $\Omega$-sequence is specified by

$$\Omega(j) = \left[ \frac{u_1, u_2, \ldots, u_\lambda; q}{v_1, v_2, \ldots, v_\mu; q} \right]_{w_j}$$

the last theorem gives us a very general transformation between two non-terminating double series $\Phi^{1;2;\lambda}_{1;1;\mu}$ and $\Phi^{1;2;\lambda+1}_{1;1;\mu+1}$.

In the proof of the last theorem, if we apply, instead of (2.2), the Hall transformation [4, III-10]

$$3\phi_2 \left[ \begin{array}{c|c} a, c, e \\ b, d \end{array} \right| q; \frac{bd}{ace} = \frac{[c, bd/ace, bd/ce; q]_{\infty}}{[b, d, bd/ace; q]_{\infty}} 3\phi_2 \left[ \begin{array}{c|c} b/c, d/c, bd/ace \\ bd/ac, bd/ce \end{array} \right| q; c \right]$$

then we can establish another transformation formula.

Theorem 2.2 (Transformation formula). For an arbitrary complex sequence $\{\Omega(j)\}$, there holds the following transformation

$$\sum_{i,j=0}^{\infty} \frac{(a; q)_{i+j}(c; q)_i(e; q)_i}{(b; q)_{i+j}(d; q)_i(q; q)_i(q; q)_j} \left( \frac{bd}{ace} \right)^i \Omega(j) = \frac{[c, bd/ace, bd/ce; q]_{\infty}}{[b, d, bd/ace; q]_{\infty}} \sum_{i,j=0}^{\infty} \frac{(b/c; q)_{i+j}[d/c, bd/ace; q]_i(a; q)_j}{(bd/ce; q)_{i+j}[q, bd/ace; q]_i[q, b/c; q]_j} \Omega(j)$$

provided that two double series displayed above are absolutely convergent.

Under specification (2.3), this theorem also yields a transformation between two non-terminating double series $\Phi^{1;2;\lambda}_{1;1;\mu}$ and $\Phi^{1;2;\lambda+1}_{1;1;\mu+1}$.

2.1.

Specifying in Theorem 2.1 with

$$\Omega(j) = \left[ \frac{[b/e, \beta; q]_j}{(a; q)_j} \left( \frac{e}{\beta} \right)^j \right]$$
and then evaluating the sum with respect to \( j \) displayed in (2.1b) by means of the \( q \)-Gauss summation theorem [4, II-8]

\[
2\phi_1\left[ \begin{array}{c} a, b \\ c \\
\frac{c}{ab} \end{array} \right | q; \frac{c/a, c/b; q}_\infty = \frac{[c/a, c/b; q]}{[c, c/ab; q]}_\infty,
\]

we find after some trivial simplification the following reduction formula.

**Proposition 2.3** *(Reduction formula).*

\[
\Phi_{1:2;2}^1\left[ \begin{array}{c} a, c, e, b/e, \beta \\ d; \gamma, q \end{array} \right | q; q \]
\[
\frac{[b/c, \beta; q]}{[d/c, b/d; a/e; q]}_\infty
\]

\[
\frac{[c, b/a, b/e; q]}{[d/b, b/d; q]}_\infty
\]

\[
\frac{[c, c/b, c/ab; q]}{[c, c/ab; q]}_\infty
\]

we find after some trivial simplification the following reduction formula.

**Proposition 2.4** *(Reduction formula).*

\[
\Phi_{1:2;3}^1\left[ \begin{array}{c} a, c, e; q^n, b/e, \gamma, q \end{array} \right | q; q \]
\[
\frac{[d/c, \beta; q]}{[d/c, b/d; a/e; q]}_\infty
\]

\[
\frac{[c, c/b, c/ab; q]}{[c, c/ab; q]}_\infty
\]

For \( a = q^{-n} \) in particular, we get another reduction formula from the last proposition.

**Corollary 2.5** *(Reduction formula).*

\[
\Phi_{1:2;2}^1\left[ \begin{array}{c} q^n: a, c, e; b/e, \gamma, q \end{array} \right | q; q \]
\[
\frac{[d/c, \beta; q]}{[d/c, b/d; a/e; q]}_\infty
\]

\[
\frac{[c, c/b, c/ab; q]}{[c, c/ab; q]}_\infty
\]

we find the following reduction formula.

2.2.

Letting in Theorem 2.1

\[
\Omega(j) = \frac{[q^{-n}, b/e, \gamma, q]}{[a, q^{1-n} \gamma / e; q]}_\infty
\]

and then reformulating the corresponding (2.1b) by using the \( q \)-Pfaff–Saalschütz formula [4, II-12]

\[
3\phi_2\left[ \begin{array}{c} q^{-n}, \alpha, \beta \\ \gamma, \alpha \gamma, \alpha \beta \end{array} \right | q; q \] = \frac{[\gamma / \alpha, \gamma / \beta; q]}{[\gamma, \alpha \beta; q]}
\]

we find the following reduction formula.

**Proposition 2.4** *(Reduction formula).*

\[
\Phi_{1:2;3}^1\left[ \begin{array}{c} a, c, e; q^n, b/e, \gamma, q \end{array} \right | q; q \]
\[
\frac{[d/c, \beta; q]}{[d/c, b/d; a/e; q]}_\infty
\]

\[
\frac{[c, c/b, c/ab; q]}{[c, c/ab; q]}_\infty
\]

2.3.

Setting in Theorem 2.2

\[
\Omega(j) = \frac{[b/c, \beta; q]}{[a, q]}_\infty \left( \frac{d}{e^\beta} \right)_j
\]
and then rewriting the corresponding (2.1b) by (2.6) again, we find the following reduction formula.

**Proposition 2.6 (Reduction formula).**

\[
\Phi_{1; 2; 2}^{1; 1; 1} \left[ \begin{array}{l}
\alpha: \ c, \ e; \ b/c, \ \beta; \ q: \ bd/ace, \ d/e\beta \\
\beta: \ d; \ a; \ 0, \ 0, \ 0
\end{array} \right] = [c, bd/ace, bd/ce\beta, d/e; q]_\infty 3\phi_2 \left[ \begin{array}{l}
b/c, d/c, bd/ace \\
b/d/ac, bd/ce\beta
\end{array} \right] q;c.
\]

2.4.

Taking in Theorem 2.2

\[
\Omega(j) = \frac{[q^{-n}, b/c, \beta; q]_j}{[a, q^{1-n}\beta/d, q]_j} q^j
\]

and then reformulating the corresponding \( j \) sum in (2.1b) by (2.7), we establish the following reduction formula.

**Proposition 2.7 (Reduction formula).**

\[
\Phi_{1; 2; 3}^{1; 1; 2} \left[ \begin{array}{l}
\alpha: \ c, \ e; \ q^{-n}, \ b/c, \ b/c, \ q^{1-n}\beta/d; \ a, \ 0, \ 0, \ 0
\end{array} \right] = \frac{[d/e, bd/ce\beta; q]_n}{[bd/ce, d/e\beta; q]_n} [c, bd/ace, bd/ce; q]_\infty 4\phi_3 \left[ \begin{array}{l}
b/c, d/c, bd/ace, q^{nbd/ce\beta} \\
b/d/ac, q^{nb/d/ce, bd/ce\beta}
\end{array} \right] q;c.
\]

3. Semi-terminating double series \( \Phi_{1; 1; \mu}^{1; 2; \lambda} \)

**Theorem 3.1 (Transformation formula).** For an arbitrary complex sequence \( \{\Omega(j)\} \), there holds the following transformation

\[
\frac{[b, d; q]_n}{[c, bd/ace; q]_n} \sum_{i, j=0}^{\infty} \frac{(a; q)_i+j(q^{-n}; q)_i(c; q)_i}{(b; q)_{i+j}(d; q)_i(q; q)_i(q; q)_j} q^i \Omega(j) = a^n \sum_{i, j=0}^{\infty} q^{i(n-i)} \frac{(b/c; q)_i+j(q^{-n}; q)_i(d/c; q)_i(a; q)_j}{[q, q^{1-n}/c, bd/ace; q]_i[q, q^n b, c/b; q]_j} \left( \frac{q}{a} \right)^i \Omega(j)
\]

provided that two semi-terminating double series displayed above are absolutely convergent.

**Proof.** By means of the Sears transformation (cf. [1, p. 79]):

\[
3\phi_2 \left[ \begin{array}{l}
q^{-n}, \ a, \ c; q \\
b, \ d
\end{array} \right] = \frac{[c, bd/ace; q]_n}{[b, d; q]_n} a^n 3\phi_2 \left[ \begin{array}{l}
q^{-n}, \ b/c, \ d/c; q \\
q^{1-n}/c, bd/ace
\end{array} \right] q; \left( \frac{q}{a} \right)
\]

we can proceed as follows:
Eq. (3.1a) \[
\frac{(b; q)_n (d; q)_n}{(c; q)_n (bd/ac; q)_n} \sum_{j=0}^{\infty} \frac{(a; q)_j (q; q)_j (b; q)_j}{(q; q)_j (b; q)_j} \Omega(j) \phi_2 \left[ \begin{array}{c} q^{-n}, q^i a, c \\ q^i b, d \\ \end{array} \right] \left[ q; q \right]
\]
\[= a^n \sum_{j=0}^{\infty} q^{ijn} \frac{(a; q)_j (q; q)_j (b; q)_j}{(q; q)_j (q^n b; q)_j} \Omega(j) \phi_2 \left[ \begin{array}{c} q^{-n}, q^i b/c, d/c \\ q^{-n}/c, bd/ac \\ \end{array} \right] \left[ q; q^{1-j}/a \right] \],

which leads us to (3.1b) when writing as a double sum. \(\square\)

Under specification (2.3), this theorem gives a transformation between two semi-terminating double series \(\Phi^{1,2}_1; \lambda_1, \mu_1\) and \(\Phi^{1,2}_1; \lambda_1+1, \mu_1+1\).

In the proof of the last theorem, if we apply instead of (3.2) another Sears transformation (cf. [1, p. 79]):

\[
3 \phi_2 \left[ \begin{array}{c} q^{-n}, a, c \\ b, d \\ \end{array} \right] = \frac{(d/a; q)_n a^n_3 \phi_2}{(d; q)_n} \left[ \begin{array}{c} q^{-n}, a, b/c \\ b, q^{1-n} a/d \\ \end{array} \right] \left[ q; q c/d \right],
\]

we would obtain the following general transformation formulae.

**Theorem 3.2 (Transformation formula).** For an arbitrary complex sequence \(\{\Omega(j)\}\), there holds the following transformation

\[
\frac{(d; q)_n}{(d/a; q)_n} \sum_{i,j=0}^{\infty} \frac{(a; q)_i (q^{-n}; q)_i (c; q)_i}{(b; q)_i+j (q^{-n}; q)_i (q; q)_j} q^i \Omega(j)
\]
\[= a^n \sum_{i,j=0}^{\infty} \left( \frac{q c}{d} \right)^i \left[ a, b/c, a; q \right]_{i+j} (q^{-n}; q)_i (qa/d; q)_j \Omega(j)
\]

provided that both semi-terminating double series are absolutely convergent.

Under specification (2.3), this theorem reduces to a transformation between two semi-terminating double series \(\Phi^{1,2}_1; \lambda_1, \mu_1\) and \(\Phi^{2,1}_2; \lambda_1+1, \mu_1+1\).

It should be pointed out that the special case of the last theorem has first been established in [3] by a different method.

**Theorem 3.3 (Transformation formula).** For an arbitrary complex sequence \(\{\Omega(j)\}\), there holds the following transformation

\[
\frac{(d; q)_n}{(d/c; q)_n} \sum_{i,j=0}^{\infty} \frac{(a; q)_i (q^{-n}; q)_i (c; q)_i}{(b; q)_i+j (q^{-n}; q)_i (q; q)_j} q^i \Omega(j)
\]
\[= c^n \sum_{i,j=0}^{\infty} q^{-ij} \left[ a, b/a, c; q \right]_{i+j} (q^{-n} c/d; q)_i (qa/d; q)_j \left( \frac{qa}{d} \right)^i \Omega(j)
\]

provided that two semi-terminating double series displayed above are absolutely convergent.

Under specification (2.3), this theorem becomes a transformation between two semi-terminating double series \(\Phi^{1,2}_1; \lambda_1, \mu_1\) and \(\Phi^{0,3}_2; \lambda_1+1, \mu_1+1\).
3.1.

Letting in Theorem 3.1

\[ \Omega(j) = \frac{[q^{-m}, q^{-n}e, q^n b, b/c; q]_j q^j}{[a, q^{-m}bd/ac, qa/d; q]_j} \]

and then evaluating the corresponding (3.1b) by [2, Proposition 3.3]

\[ \Phi_{1:2:2}^{1:2:4} \left[ e: q^{-n}, a; q^{-m}, q^{-n}b; q: q^n cd/ae, q^{1+n} \right] = \frac{(d/e; q)_n[qe/bd, q^{1-n}/d; q]_m}{(d; q)_n[q/bd, q^{1-n}e/d; q]_m} \times 4 \Phi_3 \left[ q^{-n}, c/a, e, q^{1+m}e/bd | q; q \right] \times \left[ q^{-n}, c/a, e, q^{1+m}e/bd | q; q \right], \]

we get the following reduction formula.

**Proposition 3.4 (Reduction formula).**

\[ \Phi_{1:2:4}^{1:2:4} \left[ a: q^{-n}, c; q^{-m}, q^{-n}e, q^n b, b/c; q: q, q \right] = \frac{a^n}{[b, d; q)_n \left( qa/de, q^{1-n}ac/bd; q]_m \right.} \frac{q^{-n}, q^{1-n}/d, b/c, q^{1+m}a/de}{q^{1-n}/c, qa/de, q^{1-n}+m a/d | q; q} \times \left. 4 \Phi_3 \left[ q^{-n}, c/a, e, q^{1+m}e/bd | q; q \right] \right]. \]

Alternatively, letting in Theorem 3.1

\[ \Omega(j) = \frac{[q^{-m}, q^{-n}e, b/c; q]_j q^j}{[a, q^{1-n}+m e/c; q]_j} \]

and then simplifying the corresponding (3.1b) by (3.6a)–(3.6b) again, we establish another reduction formula.

**Proposition 3.5 (Reduction formula).**

\[ \Phi_{1:2:3}^{1:2:3} \left[ a: q^{-n}, c; q^{-m}, q^{-n}e, b/c; q: q, q \right] = \left( \frac{ac}{b} \right)^n \frac{(bd/ac; q)_n}{(d; q)_n \left( qa/e, q^{n+e}b/c; q]_m \right.} \frac{q^{-n}, b/a, b/c, q^{n+m}b/e}{q^{n+b/e, q^{n+m}b, bd/ac | q; q} \left. 4 \Phi_3 \left[ q^{-n}, b/a, b/c, q^{n+m}b/e | q; q \right] \right]. \]

3.2.

Specializing in Theorem 3.1 with

\[ \Omega(j) = \left\{ \frac{[\alpha, \beta; q]_j \left( \frac{b}{a} \right)^j}{(a; q)_j} \right\}, \]

and then transforming the corresponding (3.1b) by [2, Proposition 3.4]
\[ \Phi_{1;2}^{1;2_2} \left[ \begin{array}{c} e: a, q^{-n}; \alpha, \beta; q: q^n a, q e / c; \alpha, \beta; q: q^n a, q e / c \beta; q: q_n \end{array} \right] = \left( d / a; q \right)_n \left[ q e / c, q e / c \beta; q: q \right] \left[ q e / c, q e / c \beta; q: q \right]_\infty 4 \phi_3 \left[ \begin{array}{c} q^{-n}, a, c e / a, c \beta / e \end{array} \right], \]

we derive the following two reduction formulae.

**Proposition 3.6 (Reduction formula).**

\[ \Phi_{1;2}^{1;1_1} \left[ \begin{array}{c} a: q^{-n}, c; \alpha, \beta; q: q, b / \alpha \beta \end{array} \right] = \left( d / c; q \right)_n \left[ q a / d, q a / d \beta; q: q \right] \left[ q a / d, q a / d \beta; q: q \right]_\infty 4 \phi_3 \left[ \begin{array}{c} q^{-n}, c, b / a, b / \alpha \beta \end{array} \right], \]

**Proposition 3.7 (Reduction formula).**

Letting further \( \beta \to 1 / c \) and \( \gamma \to a \), the last \( 4 \phi_3 \)-series reduces to a \( 2 \phi_1 \)-series. Evaluating it by the \( q \)-Chu–Vandermonde convolution formula [4, II-6]

\[ 2 \phi_1 \left[ \begin{array}{c} q^{-n}, a \end{array} \right] = \left( c / a; q \right)_n a^n, \]

we obtain the following closed formula.

**Corollary 3.8 (Summation formula).**

\[ \Phi_{1;2}^{1;1_1} \left[ \begin{array}{c} a: q^{-n}, c; q^n b, 1 / c; q: q, q^{-n} c / d \end{array} \right] = \left[ q / d, q a c / d; q \right]_\infty, \]

3.3.

Taking in Theorem 3.1

\[ \Omega(j) = \frac{[q^n b, \alpha; q]_j}{(\gamma; q)_j} \left( \frac{q^{-n} \gamma}{a \alpha} \right)^j \]

and then evaluating the corresponding (3.1b) by [2, Proposition 3.9]

\[ \Phi_{0;2}^{1;2_2} \left[ \begin{array}{c} e: a, q^{-n}; \alpha, \beta; q: q^n c d / a e, q^n - 1 c d \gamma / a e \alpha \end{array} \right] = \left[ a, c d / a e; q \right]_n \left[ \gamma / \alpha, c d \gamma / q a e; q \right]_\infty 4 \phi_3 \left[ \begin{array}{c} q^{-n}, c / a, d / a, c d \gamma / q a e \end{array} \right], \]

we derive the following reduction formula.
Proposition 3.9 \((\text{Reduction formula})\).

\[
\Phi_{1:2;2}^{a: q^{-n}, c; q^n b, \alpha; q: q, q^{-n} \gamma / \alpha \alpha} \\
\Phi_{1:1;1}^{b: d; \gamma; 0, 0, 0} \\
= a^n \frac{(b/a; q)_n [\gamma / a, \gamma / \alpha; q]_\infty}{(b; q)_n [\gamma, \gamma / a \alpha; q]_\infty} 4\phi_3 \left[ q^{-n}, a, d/c, qa / \gamma \right. \\
\left. d, qa / \gamma, q^{1-n} a/b \middle| q; \frac{qc \alpha}{b} \right].
\]

3.4. Setting in Theorem 3.1 with

\[
\Omega(j) = \begin{cases} \\
\left[b/c, \alpha, \beta; q \right]_j \left( q^{1-n} c \right)^j \\
\left[a, q^{1-n} \alpha \beta / d; q \right]_j \left( q d \right), \quad \left[q^n b, b/c, \alpha, \beta; q \right]_j \left( q^{1-n} c \right)^j \\
\left[a, qa / d, b \alpha \beta / a; q \right]_j \left( q d \right),
\end{cases}
\]

and then transforming the corresponding (3.1b) by \([2, \text{Proposition 3.10}]\)

\[
\Phi_{0:2;2}^{e: a, q^{-n}; \alpha, \beta; q: q^n cd / ae, q / a} \\
\Phi_{1:1;1}^{c: d; \gamma; 0, 0, -1} \\
= \frac{(d/e; q)_n [q/c, qe / a; q]_\infty}{(d; q)_n [q/a, qe / c; q]_\infty} 4\phi_3 \left[ e, qe / d, c \alpha / a, c \beta / a \right. \\
\left. qe / a, q^{1-n} e / d, c \alpha \beta / a \middle| q; q^{1-n} / c \right],
\]

we have two further reduction formulae.

Proposition 3.10 \((\text{Reduction formula})\).

\[
\Phi_{1:2;3}^{a: q^{-n}, c; q^n b, \alpha, \beta; q: q, q^{1-n} c/d} \\
\Phi_{1:1;2}^{b: d; a, q^{1-n} \alpha \beta / d; 0, 0, 0} \\
= a^n [d/a; q)_n \left[c, q \alpha / d, q \beta / d; q \right]_\infty \left[b/c, qa / d, q^{1-n} a / d, q^{1-n} \beta / d \right. \\
\left. qb / d, q^{1-n} a / d, q^{1-n} \alpha \beta / d \middle| q; c \right].
\]

Proposition 3.11 \((\text{Reduction formula})\).

\[
\Phi_{1:2;4}^{a: q^{-n}, c; q^n b, \alpha, \beta; q: q, q^{1-n} c/d} \\
\Phi_{1:1;3}^{b: d; a, qa / d, b \alpha \beta / a; 0, 0, 0} \\
= \left(\frac{a c}{b} \right)^n \frac{(bd / ac; q)_n [qac / bd, qb / d; q]_\infty}{(d; q)_n [qa / d, qc / d; q]_\infty} 4\phi_3 \left[ q^n b, b / c, b \alpha / a, b \beta / a \right. \\
\left. b, qb / d, b \alpha \beta / a \middle| q; q^{1-n} ac / bd \right].
\]

3.5. Specializing in Theorem 3.1

\[
c = q^m b \quad \text{and} \quad \Omega(j) = \begin{cases} \\
[q^{-m}, \alpha, \beta; q]_j q^j, \quad [a, q^{1-m} \alpha \beta / b; q]_j q^j, \quad [q^{-m}, q^n b, \alpha, \beta; q]_j q^j, \quad [a, qa / d, q^{n-m} \alpha \beta / a; q]_j q^j,
\end{cases}
\]

and then rewriting the corresponding (3.1b) through \([2, \text{Proposition 3.11}]\)
Proposition 3.12 (Reduction formula).

\[ \Phi_{1:2;3}^{0:2;2} \left[ q^{-m}, q^{-n}, a; \alpha, \beta; q: q^{n+m} cd/a, q^{1+n} \right] \]

\[ = \frac{q^n c \alpha, q^n c \beta; q m}{[c, q^n c \alpha; q] m} \times 4 \Phi_3 \left[ q^{-m}, q^{-n}, d/a, q^{1-n-m} / c \alpha \beta \right] \]

we deduce the following reduction formulae respectively.

Proposition 3.13 (Reduction formula).

\[ \Phi_{1:1;2}^{1:1} \left[ a: q^{-n}, q^{-m} b; q^{-m}, \alpha, \beta; q: q, q \right] \]

\[ = q^m d^n (q^{-m} d/a; q) n \left[ b, d; \alpha, q^{1-n} a \beta / b; q, q \right] \]

\[ \times 4 \Phi_3 \left[ q^{-m}, q^{-n}, b/a, q^{n+m} a / \alpha \beta \right] \]

Proposition 3.14 (Reduction formula).

\[ \Phi_{1:1;3}^{1:2;4} \left[ a: q^{-n}, q^{-m} b; q^{-m}, q^n b, \alpha, \beta; q: q, q \right] \]

\[ = q^m d^n (q^{-m} d/a; q) n \left[ q^{-1-n} a / d \alpha, q^{1-n} a / d \beta; q \right] m \]

\[ \times 4 \Phi_3 \left[ q^{-m}, q^{-n}, q^{1-n} / d, q^{1-n} a / d \alpha \beta \right] \]

3.6.

Putting in Theorem 3.1 with

\[ c = q^m b \quad \text{and} \quad \Omega (j) = \begin{bmatrix} [q^{-m}, \beta; q j] (q^{m} b \gamma) j \\ (\gamma; q) j \\ [q^{-m}, q^n b, \beta; q j] (q^{1+m-n} \gamma) j \\ [q a / d, \gamma; q j] \end{bmatrix} \]

and then reformulating the corresponding (3.1b) through [2, Proposition 3.12]

\[ \Phi_{0:2;2}^{1:2;2} \left[ q^{-m}, q^{-n}, a; q^{1-n-m} a / c d, \beta; q: q^{n+m} a \gamma / a \beta \right] \]

\[ = \frac{q^n c d / a, \gamma / \beta; q m}{[c, \gamma; q] m} \times 4 \Phi_3 \left[ q^{-m}, d / a, q^n d, q^{1-m} / \gamma \right] \]

we find the following reduction formulae respectively.

Proposition 3.14 (Reduction formula).

\[ \Phi_{1:1;1}^{1:2;2} \left[ a: q^{-n}, q^{-m} b; q^{-m}, \beta; q: q, q^{m} b \gamma / a \beta \right] \]

\[ = a^n (d / a; q) n (b / a; q) m \left[ q^{-m}, a, q a / d, \gamma / \beta \right] \]

\[ q^{1-m} a / b, q^{1-n} a / d, \gamma \quad q, q . \]
Proposition 3.15 (Reduction formula).

\[
\Phi_{1:2;3}^{1:2:3} \left[ \begin{array}{l}
\alpha: q^{-n}, q^n b, q^{-n}, \beta; q, q, q^{1+n-m} \gamma / d \beta \\
\beta: d; q a / d, \gamma; 0, 0, 0 \\
\gamma: \end{array} \right] = q^{m n} a^n \left( \frac{q^{-n} d / a; q}_n (q^{1-n} / d; q)_m (q a / d; q)_m \right) 4 \phi_3 \left[ \begin{array}{l}
q^{-n}, a, q^n b, \gamma / \beta \\
q^{n-m} d, b, \gamma \\
\end{array} \right].
\]

3.7.

Putting in Theorem 3.2

\[
\Omega(j) = \left[ \frac{[\alpha, \beta; q]_j (q b / d \alpha \beta)}{(q a / d; q)_j} \right]^j
\]

and then rewriting the corresponding (3.4b) through [5, Proposition 2.4]

\[
\Phi_{2:1;2}^{2:1:2} \left[ \begin{array}{l}
a, c; e; \beta, \gamma; q; b d / a c e, b d / a e \beta \gamma \\
b, d; -; c; 0, 0, 0 \\
\end{array} \right] = \left[ \frac{[a, b d / a c e, b d / a e \beta \gamma; q]_\infty}{[b, d, b d / a c e, b d / a e \beta \gamma; q]_\infty} \right] 4 \phi_3 \left[ \begin{array}{l}
b / a, d / a, b d / a c e, b d / a e \beta \gamma \\
q / a \\
\end{array} \right].
\]

we get the following reduction formula.

Proposition 3.16 (Reduction formula).

\[
\Phi_{1:2;2}^{1:2:2} \left[ \begin{array}{l}
a: q^{-n}, c; \alpha, \beta; q, q b / d \alpha \beta \\
b: d; q a / d; 0, 0, 0 \\
\end{array} \right] = c^n \left( \frac{d / c; q}_n (q b / d \alpha \beta; q)_\infty \right) 4 \phi_3 \left[ \begin{array}{l}
b / a, q c / d, q^{1-n} / d, q b / d \alpha \beta \\
q^{1-n} c / d, q b / d \alpha, q b / d \beta \\
\end{array} \right].
\]

3.8.

Similarly, letting in Theorem 3.2

\[
\Omega(j) = \left[ \frac{[b / c, \beta, \gamma; q]_j (q a c / d \beta \gamma)}{[a, q a / d; q]_j} \right]^j
\]

and then transforming the corresponding (3.4b) through (3.8a)–(3.8b) again, we establish another reduction formula.

Proposition 3.17 (Reduction formula).

\[
\Phi_{1:2;3}^{1:2:3} \left[ \begin{array}{l}
a: q^{-n}, c; b / c, \beta, \gamma; q, q a c / d \beta \gamma \\
b: d; a, q a / d; 0, 0, 0 \\
\end{array} \right] = c^n \left( \frac{d / c; q}_n [b / c, q a c / d \beta, q a c / d \gamma; q]_\infty \right) 4 \phi_3 \left[ \begin{array}{l}
c, q c / d, q a c / d \beta \gamma, q^{1-n} a c / b d \\
q a c / d \beta, q a c / d \gamma, q^{1-n} c / d \\
\end{array} \right].
\]
3.9.

Specializing in Theorem 3.3 with
\[
\Omega(j) = \begin{cases}
[b/c, \alpha, \beta; q]_j & \left(\frac{q^{1-n}c}{d}\right)^j, \\
[qa/d, q^{-n}\alpha\beta; q]_j & \frac{q}{d}, \\
[q^n b, \alpha, \beta; q]_j & \left(\frac{q^{1-n}c}{d}\right)^j,
\end{cases}
\]
and then reformulating the corresponding (3.5b) by [2, Proposition 2.5]
\[
\Phi_{0;3,4}^{1;1;2} \left[ -; a, b, c; d/a, d/b, \alpha, \beta; q: de/abc, e \\
\right. \\
\left. d: e; de/abc, c\alpha\beta; 0, 0, 1 \right] \\
= [e/c, de/ab; q]_\infty \times 4\phi_3 \left[ d/a, d/b, c\alpha, c\beta; q; e/c \right],
\]
we deduce the following two reduction formulae.

**Proposition 3.18** (Reduction formula).
\[
\Phi_{1;1;2}^{1;2;3} \left[ a; q^{-n}, c; b/c, \alpha, \beta; q: q^{1-n}c/d \\
b; d; qa/d, q^{-n}\alpha\beta; 0, 0, 0 \right] \\
= a^n \left(\frac{d/a}{d}; q\right)_n \times 4\phi_3 \left[ a, b/c, q^{-n}\alpha, q^{-n}\beta; q; q^{1-n}c/d \right].
\]

**Proposition 3.19** (Reduction formula).
\[
\Phi_{1;1;2}^{1;2;3} \left[ a; q^{-n}, c; q^n b, \alpha, \beta; q: q, q^{1-n}c/d \\
b; d; qa/d, c\alpha\beta; 0, 0, 0 \right] \\
= \left(\frac{q}{d}; q\right)_n \times 4\phi_3 \left[ a, q^n b, c\alpha, c\beta; q; q^{1-n}c/d \right].
\]

3.10.

Taking in Theorem 3.3
\[
\Omega(j) = \begin{cases}
[b/c, q^n b, \beta; q]_j & \left(\frac{q^{-n}c\gamma}{b\beta}\right)^j, \\
[a, \gamma; q]_j & \end{cases}
\]
and then reformulating the corresponding (3.5b) through [2, Proposition 2.10]
\[
\Phi_{0;3,3}^{1;1;1} \left[ -; q^n d, b, c; q^{-n}, d/b, \beta; q: q^{-n}e/bc, q^n b\gamma/\beta \right. \\
\left. d: e; \gamma; 0, 0, 1 \right] \\
= e^n \left(\frac{q}{d}; q\right)_n \times 4\phi_3 \left[ q^{-n}, d/b, \gamma/\beta, q^{-n}e/bc; q; q \right],
\]
we get the following reduction formula.
Proposition 3.20 (Reduction formula).

\[
\Phi_{1:1;2}^{\lambda:3} \left[ a: q^{-n}, c; b/c, q^n b, \beta; q: q, q^{-n} c \gamma/\beta \right] = a^n \left[ c, bd/ac; q \right]_n \left[ q \right]_\infty \Phi_{1:1;3}^{\lambda} \left[ q^{-n}, b/c, d/c, qb/c \gamma \right]_\infty \left[ q; q \beta/a \right].
\]

4. Terminating double series \( \Phi_{1:1;\mu}^{1:2;\lambda} \)

Theorem 4.1 (Transformation formula). For an arbitrary complex sequence \( \{ \Omega(j) \} \), there holds the following transformation

\[
\frac{(d; q)_n}{(d/a; q)_n} \sum_{i,j=0}^n \frac{(q^{-n}; q)_i (a; q)_i (c; q)_i}{(b; q)_i (d; q)_i (q; q)_i (q; q)_j} q^i \Omega(j) = a^n \sum_{i,j=0}^n \frac{[q^{-n}, b/c; q]_i (a; q)_i (q^{1-n}/d; q)_j}{[b, q^{1-n}a/d; q]_i (q, b/c; q)_j} \left( q^c \right)^i \Omega(j).
\]

Under specification (2.3), this theorem gives a transformation between two terminating double series \( \Phi_{1:1;\mu}^{1:2;\lambda} \) and \( \Phi_{2:0;\mu+1}^{2:1;\lambda+1} \).

Proof. Recalling transformation (3.3), we may manipulate the double sum in (4.1a) as follows:

\[
\sum_{j=0}^n \frac{(q^{-n}; q)_j}{(q; q)_j (b; q)_j} \Omega(j) \sum_{j=0}^n \frac{(q^{-n}; q)_j}{(q; q)_j (b; q)_j} \Omega(j) = a^n \sum_{i,j=0}^n \frac{[q^{-n}, q^{1-n}/d; q]_j}{[q, b, q^{1-n}a/d; q]_i (q, b/c; q)_j} \left( q^c \right)^i \Omega(j).
\]

Writing the last expression as a double sum, we confirm the theorem stated in Theorem 4.1.

In the proof of the last theorem, if we apply (3.2) instead of (3.3), we would find the following transformation formula.

Theorem 4.2 (Transformation formula). For an arbitrary complex sequence \( \{ \Omega(j) \} \), there holds the following transformation:

\[
\frac{[b, d; q]_n}{[c, bd/ac; q]_n} \sum_{i,j=0}^n \frac{(q^{-n}; q)_i (a; q)_i (c; q)_i}{(b; q)_i (d; q)_i (q; q)_i (q; q)_j} q^i \Omega(j) = a^n \sum_{i,j=0}^n \frac{(q^{-n}/c, bd/ac; q)_i (q, b/c; q)_j}{[q^{1-n}/c, bd/ac; q]_i (q, b/c; q)_j} \left( q^{1-n}/d; q \right)_j \left( d/ ac \right)^j \Omega(j).
\]

Under specification (2.3), this theorem yields a transformation between two terminating double series \( \Phi_{1:1;\mu}^{1:2;\lambda} \) and \( \Phi_{2:0;\mu+1}^{2:1;\lambda+1} \).
4.1.

Taking in Theorem 4.1 by
\[ \Omega(j) = \frac{[b/c, \alpha, \beta; q]_j}{[q^{1-n}/d, d\alpha\beta/c; q]_j} q^j \]
and then evaluating the corresponding \( j \)-sum in (4.1b) by [5, Proposition 2.8]
\[ \Phi_2^{2:1:2} q^{-n}, c: e; \alpha, \beta; \quad q: q^{nbd/ce}, q \]
\[ b, d: -; \quad q^{1-n}ce\alpha\beta/bd; 0, 0, 0 \]
\[ = \frac{[c, bd/ce\alpha, bd/ce\beta; q]_n}{[b, d, bd/ce\alpha\beta; q]_n} 4\Phi_3 \left[ \begin{array}{c} q^{-n}, b/c, d/c, bd/ce\alpha\beta \\ q^{1-n}/c, bd/ce\alpha, bd/ce\beta \end{array} \right | q \right] , \]
we can establish the following reduction formula.

**Proposition 4.3 (Reduction formula).**

\[ \Phi_1^{1:2:3} q^{-n}, a, c; b/c, \alpha, \beta; \quad q: q, q \]
\[ b, d: -; \quad q^{1-n}ac\alpha/c, d\beta/c \]
\[ = \frac{[c, bd/ac; q]_n}{[b, d, ac/b; q]_n} a^n 4\Phi_3 \left[ \begin{array}{c} q^{-n}, b/c, d\alpha/c, d\beta/c \\ q^{1-n}/c, bd/ac, d\alpha\beta/c \end{array} \right | q/a \right] . \]

For \( \alpha \to b/a \) and \( \beta \to q^{1-n}/d \) in particular, the last \( 4\Phi_3 \)-series reduces to a \( 2\Phi_1 \)-series. Evaluating it by (2.6), we get the following identity.

**Corollary 4.4 (Summation formula).**

\[ \Phi_1^{1:2:2} q^{-n}, a, c; b/a, b/c; q: q, q \]
\[ b, d: -; \quad q^{1-n}d/ac; 0, 0, 0 \]
\[ = \frac{[a, c, bd/ac; q]_n}{[b, d, ac/b; q]_n} \left( \frac{ac}{b} \right)^n \]

4.2.

Putting in Theorem 4.1
\[ \Omega(j) = \frac{[b/a, \beta; q]_j}{(\gamma; q)_j} \left( \frac{q^na\gamma}{\beta} \right)^j \]
and then reformulating the corresponding (4.1b) by [5, Proposition 2.10]
\[ \Phi_2^{2:1:3} q^{-n}, c: e; b/e, d/e, \beta; \quad q: q^nbd/ce, q^n\gamma/\beta \]
\[ b, d: -; \quad c, \gamma; \quad 0, 0, 0 \]
\[ = \frac{[e, bd/ce; q]_n}{[b, d; q]_n} 4\Phi_3 \left[ \begin{array}{c} q^{-n}, b/e, d/e, \gamma/\beta \\ q^{1-n}/e, bd/ce, \gamma \end{array} \right | q \right] , \]
we obtain the following reduction formula.
Proposition 4.5 (Reduction formula).

\[
\Phi_{1; 2: 2}^{1: 1; 1} \left[ q^{-n}; a, c; b/a, \beta; q; q, q^n a \gamma/b \beta \right] = c_n^{a, d/c; q}_n \quad \frac{q^{-n}, q^{1-n}/d, b/a, \gamma/b}{[b, d; q]_n} 4 \Phi_3 \left[ q^{-n}, q^{1-n}/d, b/a, \gamma/b \left| q; q \right. \right]
\]

4.3.

Setting in Theorem 4.1

\[
\Omega(j) = \frac{[b/a, b/c, \beta; q]_j}{[q^{1-n}/d, \gamma; q]_j} \left( \frac{q ac \gamma}{bd \beta} \right)^j,
\]

and then rewriting the corresponding (4.1b) by [5, Proposition 2.11]

\[
\Phi_{2; 1: 2}^{1: 2; 0} \left[ q^{-n}, c; b/e, \beta; q; q^n b d/c e, q^n d \gamma/c \beta \right] = \frac{(d/c; q)_n}{(d; q)_n} 4 \Phi_3 \left[ q^{-n}, c, b/e, \gamma/b \left| q; q \right. \right] \times 4 \Phi_3 \left[ q^{-n}, b/a, b/c, \gamma/b \left| q; q \right. \right],
\]

we derive the following reduction formula.

Proposition 4.6 (Reduction formula).

\[
\Phi_{1; 2: 3}^{1: 1; 2} \left[ q^{-n}; a, c; b/a, b/c, \beta; q; q, q ac \gamma/bd \beta \right] = \left( \frac{ac}{b} \right)^n (bd/ac; q)_n \quad \frac{q^{1-n}/d, \gamma; q}{[d; q]_n} 4 \Phi_3 \left[ q^{-n}, b/a, b/c, \gamma/b \left| q; q \right. \right] \times 4 \Phi_3 \left[ q^{-n}, c, b/e, \gamma/b \left| q; q \right. \right],
\]

4.4.

Specializing in Theorem 4.1 with

\[
\Omega(j) = \left\{ \begin{array}{l}
\frac{[\alpha, \beta; q]_j}{(q^{1-n} ac \alpha \beta/bd; q)_j} \left( \frac{q ac}{d} \right)^j, \\
\frac{[b/a, \alpha, \beta; q]_j}{[q^{1-n}/d, \alpha \beta; q]_j} \left( \frac{q ac}{d} \right)^j, \\
\end{array} \right.
\]

and then reformulating the corresponding (4.1b) through [5, Proposition 2.12]

\[
\Phi_{2; 1: 3}^{2; 1: 0} \left[ a, c; b/e, \alpha, \beta; q; bd/ace, bd/ae \right] = \frac{[d/a, bd/ce; q]_{\infty}}{[d, bd/ace; q]_{\infty}} 4 \Phi_3 \left[ a, b/e, b \alpha/c, b \beta/c \left| q; d/a \right. \right] \times 4 \Phi_3 \left[ a, b/e, b \alpha/c, b \beta/c \left| q; d/a \right. \right] \times 4 \Phi_3 \left[ a, b/e, b \alpha/c, b \beta/c \left| q; d/a \right. \right] \times 4 \Phi_3 \left[ a, b/e, b \alpha/c, b \beta/c \left| q; d/a \right. \right],
\]

we have the following reduction formulae, respectively.
Proposition 4.7 (Reduction formula).
\[
\Phi_{1;2}^{1;1:1}
\begin{bmatrix}
q^{-n}; & a, c; & \alpha, \beta; \\
b; & d; & q^{1-n}ac\alpha\beta / bd;
\end{bmatrix}
\begin{bmatrix}
q; & q, qac/d^n
\end{bmatrix}
\]
\[
= \left( \frac{ac}{d} \right)^n \frac{(bd/ac\alpha, bd/ac\beta; q)_n}{[b, bd/ac\alpha; q]_n} 4\phi_3
\begin{bmatrix}
q^{-n}, d/a, d/c, bd/ac\alpha\beta \\
d, bd/ac\alpha, bd/ac\beta
\end{bmatrix}
\begin{bmatrix}
q; & q
\end{bmatrix}.
\]

Proposition 4.8 (Reduction formula).
\[
\Phi_{1;2}^{1;1:2}
\begin{bmatrix}
q^{-n}; & a, c; & b/a, \alpha, \beta; \\
b; & d; & q^{1-n}/d, c\alpha\beta;
\end{bmatrix}
\begin{bmatrix}
q; & q, qac/d^n
\end{bmatrix}
\]
\[
= \frac{(d/c; q)_n}{(d; q)_n} c^n \times 4\phi_3
\begin{bmatrix}
q^{-n}, b/a, c\alpha, c\beta \\
b, q^{1-n}c/d, c\alpha\beta
\end{bmatrix}
\begin{bmatrix}
q; & qa/d
\end{bmatrix}.
\]

References