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Subharmonic Solutions of Second Order Subquadratic Hamiltonian Systems with Potential Changing Sign

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With the aid of some symplectic transformations, the existence of subharmonic solutions of the second order Hamiltonian system $-\ddot{x} = V_x(t, x)$ is considered, where the potential V is subquadratic in x as $|x|$ goes to infinity and can change sign with respect to t. © 2000 Academic Press

1. INTRODUCTION AND RESULTS

The existence of periodic and subharmonic solutions of the Hamiltonian system

$$-\ddot{x} = V_x(t, x), \tag{1.1}$$

where $x \in \mathbf{R}^n$, has been extensively studied in the past two decades. When the potential V is positive at infinity, there are many existence results of such solutions for both subquadratic and superquadratic potential, see [4, 18] and the references therein. The case that V changes sign in t and is superquadratic in x, some existence results for periodic solutions of (1.1) has also been obtained by several authors (see [1–3, 5–8, 12, 13]).

In this paper, we consider the case that the potential V is subquadratic which can change sign in t. Some existence results of periodic and subharmonic solutions for (1.1) will be given. In the case of $n = 1$, we will further give some results for the existence of Aubry–Mather sets, which correspond to quasiperiodic solutions in the generalized sense, by the Aubry–Mather theory.

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Throughout this paper, we will assume that the potential V consists of two parts, $V(t,x) = V_1(t,x) + V_2(t,x)$, where both V_1 and V_2 are 1-periodic in t, V_1 is positive at infinity, and V_2 changes sign when t varies. Furthermore, we suppose that V_2 satisfies $\int_0^1 V_{2,x}(t,x)\,dt = 0$. It turns out that with this condition, the problem caused by the changing sign of the potential can be easily overcome by introducing a time dependent symplectic transformation, which is 1-periodic with respect to time. Combining with other symplectic transformations enables us to deduce our results from some known existence results.

The main results of this paper are the following theorems.

THEOREM 1.1. *Let $V(t,x) = V_1(t,x) + V_2(t,x)$. Suppose that V_1 and V_2 satisfy following conditions*:

(V1) $V_1 \in C^1$, *there are positive constants a_1, a_2, and α_1 with $1 > \alpha_1 > 0$ such that*

$$\left|V_{1,x}(t,x)\right| \le a_1|x|^{\alpha_1} + a_2, \forall (t,x),$$

$$\lim_{|x|\to\infty} \frac{V_1(t,x)}{|x|^{(3\alpha_1+1)/2}} = +\infty \text{ uniformly in } t.$$

(V2) $V_2 \in C^2$, $\int_0^1 V_{2,x}(t,x)\,dt = 0$, *and there are positive constants a_3 and α_2 with $\alpha_2 \le (1+\alpha_1)/2$ such that*

$$\left|V_{2,xx}(t,x)\right| \le a_3|x|^{\alpha_2 - 1}, \qquad |x| \gg 1,$$

$$\lim_{|x|\to\infty} \frac{\left|V_{2,x}(s,x)\right|^2}{V_1(t,x)} = 0 \text{ uniformly in } s, t.$$

Then for any positive integer k, (1.1) *has a k-periodic solution x_k such that* $\max_t(|x_k(t)| + |\dot{x}_k(t)|) \to +\infty$ *as* $k \to +\infty$.

THEOREM 1.2. *Let $V(t,x) = V_1(t,x) + V_2(t,x)$. Assume that the following conditions hold*:

(V1)$'$ $V_1 \in C^1$ *and there are positive constants a_1, a_2, α_1, μ, and r with $\mu < 2$ and $\alpha_1 \le \frac{\mu}{2}$ such that*

$$|V_{1,x}| \le a_1|x|^{\alpha_1} + a_2 \ \forall (t,x),$$

$$0 < V_{1,x}(t,x)\cdot x \le \mu V_1(t,x), \qquad |x| \ge r,$$

$$\lim_{|x|\to+\infty} V_1(t,x) = +\infty \text{ uniformly in } t.$$

(V2)′ $V_2 \in C^2$, $\int_0^1 V_{2,x}(t,x)\,dt = 0$, *and there are positive constants* a_3 *and* α_2 *with* $\alpha_2 \le \frac{\mu}{2}$ *such that*

$$|V_{2,xx}(t,x)| \le a_3|x|^{\alpha_2 - 1}, \qquad |x| \gg 1,$$

$$\lim_{|x|\to+\infty} \frac{|V_{2,x}(s,x)|^2 + |V_{2,xx}(s,x)\cdot x|^2}{V_1(t,x)} = 0 \text{ uniformly in } s, t.$$

Then the same conclusion as that of Theorem 1.1 *holds*.

Conditions (V1) and (V2) (or (V1)′ and (V2)′) imply that $V = V_1 + V_2$ is subquadratic in x as $|x| \to +\infty$ and V_1 is positive for $|x|$ large. But the function V_2 can change sign as t varies.

As a consequence of the above theorems, for a potential $(a + b(t))V(x)$, we have

THEOREM 1.3. *Let* $V(x) \in C^2$ *and* $a > 0$ *be a positive number, let* $b(t)$ *be continuous, and let* $\int_0^1 b(t)\,dt = 0$. *Assume that* V *satisfies*

(V) *There are constants* α_1, α_2, $0 < \alpha_1 < 1$, $\alpha_2 \le (1 + \alpha_1)/2$, a_2, *and* a_3 *such that*

$$|V_x(x)| \le a_1|x|^{\alpha_1} + a_2, \qquad |V_{xx}(x)| \le a_3|x|^{\alpha_2 - 1} \qquad |x| \gg 1.$$

$$\frac{|V_x(x)|^2}{V(x)} \to 0, \qquad \frac{V(x)}{|x|^{(3\alpha_1+1)/2}} \to +\infty \qquad \text{as } |x| \to \infty.$$

Then the same conclusions as in Theorem 1.1 *hold*.

Remark 1.1. A typical potential which satisfies the above conditions is $V(x) = |x|^{1+\alpha}$, $|x| \gg 1$, for $-1 < \alpha < 1$.

The proof of the theorems is based on two elementary symplectic transformations and an existence result of subharmonic solutions by Silva. The first transformation is time dependent, which eliminates the sign changing part V_2 of the potential V. The second one is time independent, which transforms a subquadratic second order Hamiltonian system into a subquadratic first order Hamiltonian system.

2. PROOF OF THEOREMS

We first write (1.1) in Hamiltonian form,

$$-\dot{x} = H_y(t,x,y), \qquad \dot{y} = H_x(t,x,y), \tag{2.1}$$

with $H(t,x,y) = \frac{1}{2}|y|^2 + V(t,x)$. It is well known that (2.1) is invariant with respect to symplectic transformations. Thus, let ϕ be a diffeomorphism of $\mathbf{R}^{2n}$ such that

$$\phi\colon (X,Y) \to (x,y), \qquad \sum_1^n dy_i \wedge dx_i = \sum_1^n dY_i \wedge dX_i.$$

Then (x,y) is a solution of (2.1) if and only if (X,Y) solves

$$-\dot{X} = K_Y(t,X,Y), \qquad \dot{Y} = K_X(t,X,Y) \tag{2.2}$$

with $K(t,X,Y) = H(t, x(X,Y), y(X,Y))$. If ϕ_t is a family of symplectic transformations, then we need to add the new Hamiltonian K, a correction term, because of the time dependence of ϕ_t.

LEMMA 2.1. *Let $V \in C^2(\mathbf{R}^1 \times \mathbf{R}^n)$ such that $V(t,x) = V(t+1,x)$, $\int_0^1 V_x(t,x)\,dt = 0$. For fixed t, let*

$$\phi_t\colon (X,Y) \to (x,y)$$

be defined by

$$x = X, \qquad y = Y + \int_0^t V_X(s,X)\,ds. \tag{2.3}$$

Then ϕ_t is a symplectic transformation of $(\mathbf{R}^{2n}, \sum_1^n dx_i \wedge dy_i)$ which transforms the Hamiltonian system

$$-\dot{x} = H_y(t,x,y), \qquad \dot{y} = H_x(t,x,y) + V_x(t,x) \tag{2.4}$$

into

$$-\dot{X} = K_Y(t,X,Y), \qquad \dot{Y} = K_X(t,X,Y) \tag{2.5}$$

with $K(t,X,Y) = H(t,X,Y + \int_0^t V_X(s,X)\,ds)$; i.e., $(X(t),Y(t))$ is a solution of (2.5) if and only if $(x(t),y(t)) = \phi_t(X(t),Y(t))$ is a solution of (2.4). Moreover $(x(t),y(t))$ is k-periodic if and only if $(X(t),Y(t))$ is k-periodic.

Proof. The fact that ϕ_t is a symplectic transformation follows from direct computation. Suppose $(X(t),Y(t))$ is a solution of (2.5). Then we have

$$-\dot{x} = \dot{X}(t) = K_Y(t, X, Y)$$
$$= H_y\left(t, X, Y + \int_0^t V_X(s, X)\, ds\right) = H_y(t, x, y).$$

$$\dot{y} = \dot{Y} + V_X(t, X(t)) + \int_0^t V_{XX}(s, X(t))\, ds \cdot \dot{X}(t)$$
$$= H_x\left(t, X, Y + \int_0^t V_X(t, X)\, ds\right) + V_X(t, X)$$
$$+ \int_0^t V_{XX}(s, X)\, ds\, H_y\left(t, X, Y + \int_0^t V_X(t, X)\, ds\right)$$
$$+ \int_0^t V_{XX}(s, X)\, ds\, \dot{X}(t)$$
$$= H_x(t, x, y) + V_x(t, x).$$

Hence $(x(t), y(t))$ solves (2.4). The proof of the other part is the same. The periodicity is a consequence of $\int_0^1 V_x(t, x)\, dt = 0$.

Now we turn to the proof of the main theorems. Let

$$H(t, x, y) = \tfrac{1}{2}|y|^2 + V_1(t, x) + V_2(t, x).$$

Assume that $\int_0^1 V_{2,x}(t, x)\, dt = 0$; then by Lemma 2.1,

$$-\dot{x} = H_y(t, x, y), \qquad \dot{y} = H_x(t, x, y)$$

is equivalent to

$$-\dot{x} = K_y(t, x, y), \qquad \dot{y} = K_x(t, x, y) \tag{2.6}$$

with $K(t, x, y) = \frac{1}{2}|y + \int_0^t V_{2,x}(s, x)\, ds|^2 + V_1(t, x)$. Thus the sign changing part $V_2(t, x)$ disappears. However, the new Hamiltonian K is not subquadratic in (x, y). Using a symplectic transformation given by the following proposition, which is proved in [9], we will show that (2.6) is equivalent to a Hamiltonian system which is subquadratic in the usual sense.

PROPOSITION 2.1. *For any* $0 < \lambda < 2$, *there is a symplectic transformation* ψ *of* $(\mathbf{R}^{2n}, \sum_0^n dx_i \wedge dy_i)$, $\psi(X, Y) = (x(X, Y), y(X, Y))$ *such that*

$$x(cX, cY) = c^{2-\lambda} x(X, Y), \qquad y(cX, cY) = c^{\lambda} y(X, Y)$$

for $c \geq 1$, $|X|^2 + |Y|^2 \geq 1$.

The role of this proposition is to adjust the growth rate of Hamiltonian $H(t, x, y)$. It is useful in dealing with Hamiltonian $H(t, x, y)$ which has different growth rates in x and y. There is another approach to dealing with periodic solutions for such Hamiltonian systems (see [5]). It is shown in [9] that some Hamiltonian systems can be transformed into subquadratic (or superquadratic) Hamiltonian systems in the usual sense. In particular, (1.1) is equivalent to a first order Hamiltonian system (2.1) with H being subquadratic (or superquadratic) if V is subquadratic (or superquadratic).

Set

$$K(t, x, y) = \tfrac{1}{2}\left| y + \int_0^t V_{2,x}(s, x)\, ds \right|^2 + V_1(t, x).$$

Since $\int_0^1 V_{2,x}(s, x)\, ds = 0$, K is 1-periodic in t. Now let $\lambda = 2(1 + \alpha_1)/(3 + \alpha_1) < 1$ and $\psi(X, Y) = (x(X, Y), y(X, Y))$ be the transformation given by Proposition 2.1. Then (2.6) is transformed into

$$-\dot{X} = K_{1,Y}(t, X, Y), \qquad \dot{Y} = K_{1,X}(t, X, Y) \tag{2.7}$$

with $K_1(t, X, Y) = K(t, x(X, Y), y(X, Y))$. Set $Z = (X, Y)$, $\alpha = 2\lambda - 1 = (1 + 3\alpha_1)/(3 + \alpha_1)$, as $0 < \alpha_1 < 1$, we have $0 < \alpha < 1$. In order to prove Theorem 1.1, it suffices to prove the same conclusion for (2.7).

LEMMA 2.2. *Under the assumptions of Theorem* 1.1, *there are positive constants* b_1 *and* b_2 *such that*

$$|K_{1,Z}(t, Z)| \le b_1 |Z|^\alpha + b_2 \ \forall (t, Z).$$

$$\lim_{|Z| \to \infty} \frac{K_1(t, Z)}{|Z|^{2\alpha}} = +\infty \text{ uniformly in } t.$$

Proof. Let $K_1(t, X, Y) = K_2(t, X, Y) + h(t, X, Y)$ with

$$K_2(t, X, Y) = \tfrac{1}{2} |y(X, Y)|^2 + V_1(t, x(X, Y)),$$

$$h(t, X, Y) = y(X, Y) \cdot \int_0^t V_{2,x}(s, x(X, Y))\, ds$$

$$+ \tfrac{1}{2} \left| \int_0^t V_{2,x}(s, x(X, Y))\, ds \right|^2.$$

For $|Z| \ge 1$, the homogeneity of $x(X, Y)$, $y(X, Y)$ implies

$$|x_Z| \le c|Z|^{1-\lambda}, \qquad |y_Z| \le c|Z|^{\lambda - 1}$$

for some constant $c > 0$. Hence conditions (V1) and (V2) imply

$$\begin{aligned}|K_{2,Z}(t,X,Y)| &\le |y|\,|y_Z| + |V_{1,x}(t,x(X,Y))|\,|x_Z| \\ &\le b_1|Z|^{\alpha} + b_2,\\ |h_Z(t,Z)| &\le b_1|Z|^{\alpha} + b_2\end{aligned}$$

for some constants b_1 and b_2. From condition (V2), we know there is a constant $b > 0$ such that

$$\begin{aligned}K_1(t,Z) &\ge \tfrac{1}{4}|y|^2 - \tfrac{1}{2}\left|\int_0^t V_{2,x}(s,x(X,Y))\,ds\right|^2 + V_1(t,x)\\ &\ge \tfrac{1}{2}K_2(t,Z) - b.\end{aligned}$$

The homogeneity of $x(X,Y)$, $y(X,Y)$ and condition (V1) then lead to the conclusion that

$$\frac{K_2(t,Z)}{|Z|^{2\alpha}} \to +\infty \qquad \text{as } |Z| \to \infty.$$

This finishes the proof of the lemma.

With this lemma, Theorem 1.1 is an immediate consequence of the following result of Silva [19].

PROPOSITION 2.2. *Let $z = (x, y)$ and $H \in C^1$ satisfy*

(H1) $H(t,z) = H(t+1,z)$.

(H2) *There are constants $c_1, c_2 > 0$ and $\alpha \in (0,1)$ such that*

$$|H_z(t,z)| \le c_1|z|^{\alpha} + c_2 \;\forall (t,z).$$

(H3) *Either* $\lim_{|z|\to\infty} \dfrac{H(t,z)}{|z|^{2\alpha}} = -\infty$ *or* $\lim_{|z|\to\infty} \dfrac{H(t,z)}{|z|^{2\alpha}} = +\infty$

uniformly in t. Then for any positive integer k, there is a k-periodic solution z_k of

$$\dot{x} = H_y(t,x,y), \qquad -\dot{y} = H_x(t,x,y)$$

such that $\|z_k\|_\infty \to +\infty$ as $k \to +\infty$.

If (H3) *is replaced by*

(H4) *either* $\lim_{|z|\to\infty}[H_z(t,z)\dot{z} - 2H(t,z)] = -\infty$ *or*

$$\lim_{|z|\to\infty}\left[H_z(t,z)\dot{z} - 2H(t,z)\right] = +\infty,$$

uniformly in t, then the same conclusion holds.

The proof of Theorem 1.2 is similar to that of Theorem 1.1; it follows from Lemma 2.3 and the second part of Proposition 2.2. Set $\lambda = 2\mu/(\mu + 2)$. As $\mu < 2$, we have $\lambda < 1$. Let $(x(X,Y), y(X,Y))$ be the symplectic transformation given by Proposition 2.1 and let $K_1(t, X, Y) = K(t, x(X,Y), y(X,Y))$ be as in Lemma 2.2.

LEMMA 2.3. *Let V_1 and V_2 satisfy the conditions* (V1)′ *and* (V2)′. *Then there are positive constants b_1, b_2, and α with $0 < \alpha < 1$ such that*

(1) $|K_{1,Z}(t, Z)| \le b_1|Z|^\alpha + b_2$.

(2) $0 < K_{1,Z}(t, Z) \cdot Z \le \mu' K_1(t, Z)$, $|Z| \gg 1$, *for* μ' *with* $4\mu/(2 + \mu) < \mu' < 2$.

(3) $K_1(t, Z) \to +\infty$ *as* $|Z| \to \infty$ *uniformly in* t.

Proof. (1) is similar to that of Lemma 2.2. (3) is easy to verify. Now we prove (2). As before, let

$$K_2(t, X, Y) = \tfrac{1}{2}|y(X,Y)|^2 + V_1(t, x(X,Y)),$$

$$h(t, X, Y) = y(X,Y) \cdot \int_0^t V_{2,x}(s, x(X,Y))\, ds + \tfrac{1}{2}\left|\int_0^t V_{2,x}(s, x(X,Y))\, ds\right|^2.$$

Then we have for $|Z| \ge 1$,

$$K_{2,Z}(t, Z) \cdot Z \le \frac{2\mu}{\mu + 2} K_2(t, Z), \qquad |Z| \gg 1.$$

$$\begin{aligned} h_Z(t, Z) \cdot Z &= y_Z \cdot Z \cdot \int_0^t V_{2,x}(s, x(X,Y))\, ds + y \cdot \int_0^t V_{2,xx}(s, x)\, ds \cdot x_Z \cdot Z \\ &\quad + \int_0^t V_{2,x}(t, x) \cdot \int_0^t V_{2,xx}(s, x)\, ds \cdot x_Z \cdot Z \\ &= \lambda y \cdot \int_0^t V_{2,x}(s, x)\, ds + (2 - \lambda) y \cdot \int_0^t V_{2,xx}(s, x)\, ds \cdot x \\ &\quad + (2 - \lambda) \int_0^t V_{2,x}(s, x)\, ds \cdot \int_0^t V_{2,xx}(s, x)\, ds \cdot x. \end{aligned}$$

Now conclusion (2) is an easy consequence of (V2)′.

Remark 2.1. Our results hold for more general Hamiltonian systems. Indeed, what is important in our approach is the asymptotical behavior of the Hamiltonian $H(t, x, y) = \frac{1}{2}|y|^2 + V_1(t, x) + V_2(t, x)$ near infinity. For example, $|y|^2$ can be replaced by a function $g(y)$ that looks like $|y|^2$ as $|y|$ goes to infinity. Similar results hold for $H(t, x, y) = g(y) + V_1(t, x) + V_2(t, x)$, where $g(y) = \frac{1}{\beta}|y|^{\beta}$ for $|y| \gg 1$ and $\beta > 1$.

3. AUBRY–MATHER SETS IN DIMENSION 2

In this section, we show that, under some additional conditions on the potential V, for small and positive rotation numbers, there exist Aubry–Mather sets for 2-dimensional Hamiltonian system

$$-\dot{x} = y, \qquad \dot{y} = V_x(t, x) + h_x(t, x), \tag{3.1}$$

where $(x, y) \in \mathbf{R}^2$. An Aubry–Mather set for (3.1) with rotation number ω is an invariant set of the Poincare map of (3.1), which is homeomorphic either to S^1 or to a Cantor set of S^1, and the Poincare map restricts to this invariant set is the rotation with rotation number ω. Such an invariant set corresponds to a family of quasiperiodic solutions with the same rotation number for (3.1). The existence of Aubry–Mather sets for monotone twist area preserving maps of the cylinder $S^1 \times \mathbf{R}^1$ was developed by Aubry and Mather independently in the early eighties. This is now called the Aubry–Mather theory (see [15]).

The existence of Aubry–Mather sets for (3.1) has been investigated in the last few years, see [10, 11, 16, 17] for the case that V is positive near infinity, and the Aubry–Mather sets are found near infinity. In [14], sign changing potential is considered, and it is shown that Aubry–Mather sets, as well as invariant curves, exist near the origin for small positive rotation numbers if $V(t, x) = (a + b(t))|x|^{2n+2} + h(t, x)$, $a > 0$, $\int_0^1 b(t)\,dt = 0$, $n \geq 1$, h and b are 1-periodic in t, and h is a higher order term as $|x| \to 0$. This leads to the stability of the equilibrium $(0, 0)$. Indeed, $a > 0$ is also a necessary condition for $(0, 0)$ being stable, see [14].

In the following we consider subquadratic potential and will show that there exist Aubry–Mather sets for small positive rotation number near the infinity. The proof is based on following proposition which is proved in [10]. It is a consequence of the Aubry–Mather theory for monotone twist maps on the cylinder with finite twist.

PROPOSITION 3.1. *Let $K(t, r, \theta) \in C^2$ satisfy*

(1) $K(t+1, r, \theta) = K(t, r, \theta) = K(t, r, \theta + 1)$.

(2) $K_{rr}(t, r, \theta) < 0$, $\lim_{r \to +\infty} K_r(t, r, \theta) = 0$.

(3) $\lim_{r \to +\infty} K(t, r, \theta) = +\infty$.

Then there is an $\epsilon > 0$ *such that for* $\omega \in (0, \epsilon)$, *there exists an Aubry–Mather set* $M(\omega)$ *for*

$$-\dot{r} = K_\theta(t, r, \theta), \qquad \dot{\theta} = K_r(t, r, \theta),$$

provided the maximal existence interval of the above equation is $\mathbf{R}$ *for any initial data* (r_0, θ_0). *Moreover,*

$$\inf\{r \mid (r, \theta) \in M(\omega)\} \to +\infty \qquad \text{as } \omega \to 0+ .$$

We will prove Theorem 3.1 in the manner along the idea of [12], [13].

THEOREM 3.1. *Let* $V \in C^3$ *and satisfy*:

(V3) *There are positive constants* c, c', *and* $\alpha < 2$ *such that*

$$2V_{xx}(x)x^2 - \alpha V_x(x)x \le -cV(x), \qquad |x| \gg 1,$$

$$V(x) \le c'|x|^\alpha, \qquad |x| \gg 1, \qquad \lim_{|x|\to\infty} V(x) = +\infty.$$

(V4) $$\lim_{|x|\to\infty} \frac{|V_x(x)|^2 + |V_{xx}(x)x|^2 + |V_{xxx}(x)x^2|^2}{V(x)} = 0.$$

Then there is an $\epsilon > 0$ *such that for* $\omega \in (0, \epsilon)$, *there is an Aubry–Mather set* $M(\omega)$ *with rotation number* ω *for*

$$-\dot{x} = y, \qquad \dot{y} = (a + b(t))V_x(x) \tag{3.2}$$

provided $a > 0$ *and* $b(t)$ *is continuous satisfying* $\int_0^1 b(t)\,dt = 0$.

Remark 3.1. It is easy to see that $V(x) = |x|^\alpha$, for $|x| \gg 1$ and $0 < \alpha < 2$, satisfies the conditions.

Proof. With conditions (V3) and (V4), it is easy to see that any solution of (3.2) is defined on $\mathbf{R}$. By Lemma 2.1, (3.2) can be transformed into

$$-\dot{x} = H_y(t, x, y), \qquad \dot{y} = H_x(t, x, y) \tag{3.3}$$

with $H(t, x, y) = \frac{1}{2}|y + \int_0^t b(s)\,ds\, V_x(x)|^2 + aV(x)$. Now let (R, θ) be the polar coordinate of $\mathbf{R}^2 \setminus \{0\}$ and let

$$x(R, \theta) = \left(\frac{R}{2\pi}\right)^{2/(2+\alpha)} a^2(\theta)\cos 2\pi\theta,$$

$$y(R, \theta) = \left(\frac{R}{2\pi}\right)^{\alpha/(2+\alpha)} a^\alpha(\theta)\sin 2\pi\theta,$$

where

$$a(\theta) = \left(\frac{2 + \alpha}{2\cos^2 2\pi\theta + \alpha \sin^2 2\pi\theta} \right)^{1/(2+\alpha)}.$$

Then (R, θ) satisfies

$$dR \wedge d\theta = dx \wedge dy.$$

Thus (3.3) in $\mathbf{R}^2 \setminus \{0\}$ is equivalent to

$$-\dot{R} = K_\theta(t, R, \theta), \qquad \dot{\theta} = K_R(t, R, \theta), \tag{3.4}$$

where $K(t, R, \theta) = \frac{1}{2}|y + \int_0^t b(s)\, ds\, V_x(x)|^2 + aV(x)$, $R > 0$.

LEMMA 3.1. *The function K satisfies*

(1) $\lim_{R\to\infty} K(t, R, \theta) = +\infty$.

(2) $K_{RR}(t, R, \theta) < 0$ *if* $R \gg 1$.

(3) $\lim_{R\to\infty} K_R(t, R, \theta) = 0$ *uniformly in* (t, θ).

Now we pick up a function $K_1(t, R, \theta)$, which is defined on $\mathbf{R} \times \mathbf{R} \times S^1$ and 1-periodic in t, and, for $R \gg 1$,

$$K_{1,RR}(t, R, \theta) < 0, \qquad K_1(t, R, \theta) = K(t, R, \theta).$$

By Proposition 3.1, we know that for $\omega > 0$ and small,

$$-\dot{R} = K_{1,\theta}(t, R, \theta), \qquad \dot{\theta} = K_{1,R}(t, R, \theta) \tag{3.5}$$

has an Aubry–Mather set $M(\omega)$ with rotation number ω located near $R = +\infty$. Hence it is also an Aubry–Mather set for (3.4) and (3.2).

Now we prove Lemma 3.1, which consists of elementary computations. (1) immediately follows from the conditions. For simplicity, let $a = 1$. First, we have

$$\begin{aligned}
&\left(\frac{1}{2}y^2 + V(x)\right)_{RR} \\
&\quad = -\frac{2}{(2+\alpha)^2} R^{-2}\left((2-\alpha)\alpha y^2 + \left(\alpha V_x(x)x - 2V_{xx}(x)x^2\right)\right) \\
&\quad \le -\frac{2}{(2+\alpha)^2} R^{-2}\left((2-\alpha)\alpha y^2 + cV(x) - a_1\right)
\end{aligned}$$

for some constant a_1.

$$\begin{aligned}(yV_x(x))_{RR} &= y_{RR}V_x(x) + 2y_R x_R V_{xx}(x) + y\big(V_{xxx}(x)x_R^2 + V_{xx}x_{RR}\big)\\ &= R^{-2}\big(c_1 yV_x(x) + c_2 yV_{xx}(x)\cdot x + c_3 yV_{xxx}(x)\cdot x^2\big),\end{aligned}$$

where c_i, $i = 1, 2, 3$ are constants depending only on α. Similarly, there are such constants b_i, $i = 1, 2, 3$ satisfying

$$\begin{aligned}&(V_x(x))_{RR}^2\\ &\quad = R^{-2}\big(b_1 V_{xx}^2(x) + b_2 V_x(x)\cdot V_{xx}(x)\cdot x + b_3 V_x(x)\cdot V_{xxx}(x)\cdot x^2\big).\end{aligned}$$

Thus conditions (V3) and (V4) show that the sign of $K_{RR}(t, R, \theta)$ is determined by that of $(\frac{1}{2}y^2 + V(x))_{RR}$, which is negative since

$$\lim_{R\to+\infty}\big((2-\alpha)\alpha y^2 + cV(x) - a_1\big) = +\infty \text{ uniformly in } (t, \theta).$$

Finally,

$$\lim_{R\to\infty} K_R(t, R, \theta) = \lim_{R\to\infty} \frac{K(t, R, \theta)}{R} = 0$$

by (V3) and (V4). This concludes the proof of Lemma 3.1, hence that of Theorem 3.1.

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