Asymptotic behavior for almost-orbits of a reversible semigroup of non-Lipschitzian mappings in a metric space

Behzad Djafari Rouhani a,*,1 and Jong Kyu Kim b,2

a Institute for Studies in Nonlinear Analysis, School of Mathematical Sciences, Shahid Beheshti University, P.O. Box 19395-4716, Evin, 19834 Tehran, Iran
b Department of Mathematics, Kyungnam University, Masan, Kyungnam 631-701, Republic of Korea

Received 13 January 2002
Submitted by W.A. Kirk

Abstract

Let $(M, \rho)$ be a metric space and $\tau$ a Hausdorff topology on $M$ such that $\{M, \tau\}$ is compact. Let $S$ be a right reversible semitopological semigroup and $\mathcal{I} = \{T(s): s \in S\}$ a representation of $S$ as $\rho$-asymptotically nonexpansive type self-mappings of $M$ and $u$ a $\rho$-bounded almost-orbit of $\mathcal{I}$. We study the $\tau$-convergence of the net $\{u(s): s \in S\}$ in $M$ when the triplet $\{M, \rho, \tau\}$ satisfies various types of $\tau$-Opial conditions. Our results extend and unify many previously known results.

Keywords: Almost-orbit; Reversible; Semitopological semigroup; Asymptotically nonexpansive type; Opial condition; $\tau$-asymptotically regular; Fixed point

* Corresponding author.
E-mail addresses: b-rohani@cc.sbu.ac.ir (B.D. Rouhani), jongkyuk@kyungnam.ac.kr (J.K. Kim).
1 Supported in part by a grant from Shahid Beheshti University.
2 Supported by grant No. R05-2001-000-0001-0 from Korea Science and Engineering Foundation.
1. Introduction

The first weak convergence theorem for the sequence of iterates \( \{T^n x\} \) of a nonexpansive mapping was proved by Opial [19] in a Hilbert space. It states that if \( M \) is a nonempty bounded closed and convex subset of a Hilbert space \( H \) and \( T \) is a nonexpansive self-mapping of \( M \), i.e., \( \|Tx - Ty\| \leq \|x - y\| \) for all \( x, y \in M \), then for each \( x \in M \), \( \{T^n x\} \) converges weakly to a fixed point of \( T \) if and only if \( T \) is weakly asymptotically regular, i.e., \( w\lim_{n \to \infty} (T^{n+1} x - T^n x) = 0 \) for each \( x \in M \). This result was subsequently extensively studied and extended in many directions [1–9,11,12,14–20,24,25].

In [5] we gave an extension of some of these results to the case of an asymptotically nonexpansive type mapping defined on a metric space \((M, \rho)\) with a Hausdorff topology \( \tau \) such that the triplet \( \{M, \rho, \tau\} \) satisfies various types of \( \tau \)-Opial conditions.

In this paper we further extend these results by studying the \( \tau \)-convergence of a \( \rho \)-bounded almost-orbit of a right reversible semitopological semigroup of asymptotically nonexpansive type self-mappings of \( M \). Our results extend in particular recent results of Li [16] and Kim and Li [11,17] from Banach space to metric space, and from nonexpansive to right reversible semigroup of asymptotically nonexpansive type mappings. We note that our results are new even in a Banach space \( X \), since compared to [16], no other requirement than the appropriate Opial condition is assumed for the norm of \( X \). Moreover, since in our case the \( \tau \)-limit of the almost-orbit is not necessarily a common fixed point for the semigroup, a new method of proof is required by introducing the notion of an asymptotic almost-orbit for the semigroup. On the other hand, our results answer affirmatively an open question of Reich [23, p. 550] even in this very general context.

2. Preliminaries

Throughout the paper \((M, \rho)\) is a metric space and \( \tau \) is a Hausdorff topology on \( M \). \( S \) is a semitopological semigroup, that is, \( S \) is a semigroup with a Hausdorff topology such that for each \( s \in S \) the mappings \( s \mapsto t \cdot s \) and \( s \mapsto s \cdot t \) from \( S \) to \( S \) are continuous. We assume that \( S \) is right reversible, that is, any two closed left ideals of \( S \) have nonempty intersection. In this case, \((S, \preceq)\) is a directed system when the binary relation \( \preceq \) on \( S \) is defined by \( s \preceq t \) if and only if \( \{s\} \cup \overline{ss} \supset \{t\} \cup \overline{st} \), for \( s, t \in S \). Right reversible semigroups include all commutative semigroups and all semitopological semigroups which are right amenable as discrete semigroups. Asymptotically nonexpansive type mappings were introduced by Kirk [13]. We assume that \( \mathcal{S} = \{T(t): t \in S\} \) is a \( \rho \)-asymptotically nonexpansive type semigroup on \( M \), that is, for each \( t \in S \), \( T(t) \) is a self-
mapping of $M$, $T(st)x = T(s)T(t)x$ for all $s, t \in S$ and $x \in M$, and for each $x \in M$,

$$\limsup_{t \in S} \sup_{y \in M} \left[ \rho(T(t)y, T(t)x) - \rho(x, y) \right] \leq 0;$$

i.e.,

$$\rho(T(t)y, T(t)x) \leq \rho(x, y) + \epsilon(t, x)$$

for all $x, y \in M$ and $t \in S$,

where $\epsilon(t, x) \geq 0$ and for each $x \in M$, $\lim_{t \in S} \epsilon(t, x) = 0$. $\mathcal{S}$ is called a nonexpansive semigroup if $\epsilon(t, x) = 0$ for all $t \in S$ and $x \in M$. We denote by $F(\mathcal{S})$ the common fixed point set of $\mathcal{S}$ and $AF(\mathcal{S}) := \{ p \in M : \lim_{t \in S} \rho(T(t)p, p) = 0 \}$. When there is no confusion these sets will be denoted by $F$ and $AF$, respectively.

It is clear that $F \subset AF$. A function $u : S \rightarrow M$, to which we shall refer as a curve in $M$ in the sequel, is called an almost-orbit of $\mathcal{S}$ if

$$\lim_{t \in S} \left[ \sup_{x \in S} \rho(u(st), T(s)u(t)) \right] = 0.$$

Definition 2.1. A curve $u = \{ u(t) : t \in S \}$ is called an asymptotic almost-orbit of $\mathcal{S}$ if

$$\lim_{t \in S} \left[ \limsup_{s \in S} \rho(u(st), T(s)u(t)) \right] = 0.$$

Obviously every orbit of $\mathcal{S}$ is an almost-orbit for $\mathcal{S}$, which itself is an asymptotic almost-orbit of $\mathcal{S}$.

Definition 2.2. A curve $u$ in $M$ is said to be $\tau$-asymptotically regular if for each $x \in M$ and each $\tau$-neighborhood $V$ of $x$ containing an infinite subnet $\{ u(t_{\alpha}) : \alpha \in A \}$ of $u$, and for each $h \in S$, there exists $t_{0} \in S$ such that $u(ht_{\alpha}) \in V$ for all $t_{\alpha} \geq t_{0}$. $u$ is said to be $\rho$-asymptotically regular if for each $h \in S$, $\lim_{t \in S} \rho(u(ht), u(t)) = 0$.

Now we define various types of $\tau$-Opial conditions; see [10,12,19,21,22].

Definition 2.3. The triplet $\{ M, \rho, \tau \}$ is said to satisfy the $\tau$-Opial condition if for each $\rho$-bounded net $\{ x_{\alpha} : \alpha \in A \}$ in $M$ that $\tau$-converges to some $x \in M$, we have

$$\limsup_{\alpha \in A} \rho(x_{\alpha}, x) < \limsup_{\alpha \in A} \rho(x_{\alpha}, y)$$

for all $y \neq x$.

It is said to satisfy the locally uniform $\tau$-Opial condition if for each $\rho$-bounded net $\{ x_{\alpha} \}$ in $M$ that $\tau$-converges to some $x \in M$ and every $\epsilon > 0$, there exists $\eta(\{ x_{\alpha} \}, \epsilon) > 0$ such that for each $y \in M$ with $\rho(x, y) \geq \epsilon$, we have

$$\limsup_{\alpha \in A} \rho(x_{\alpha}, x) + \eta \leq \limsup_{\alpha \in A} \rho(x_{\alpha}, y).$$
It is said to satisfy the uniform $\tau$-Opial condition if for every $R > 0$ and every $\epsilon > 0$, there exists $\eta(R, \epsilon) > 0$ such that for every net $\{x_\alpha : \alpha \in A\}$ in $M$ that $\tau$-converges to some $x \in M$ with $\limsup_{\alpha \in A} \rho(x_\alpha, x) \leq R$, and for every $y \in M$ with $\rho(x, y) \geq \epsilon$, we have

$$\limsup_{\alpha \in A} \rho(x_\alpha, x) + \eta \leq \limsup_{\alpha \in A} \rho(x_\alpha, y).$$

The uniform Opial condition in Banach spaces was introduced by Prus [22]. It is clear that the uniform $\tau$-Opial condition implies the locally uniform $\tau$-Opial condition which in turn implies the $\tau$-Opial condition. It is also clear that in Definition 2.3 all the $\limsup_{\alpha \in A}$ can be replaced by $\liminf_{\alpha \in A}$.

The following lemma which gives an equivalent condition to the locally uniform $\tau$-Opial condition is well-known; see [12,19].

**Lemma 2.4.** The triplet $\{M, \rho, \tau\}$ satisfies the locally uniform $\tau$-Opial condition if and only if for every $\rho$-bounded net $\{x_\alpha : \alpha \in A\}$ in $M$ that $\tau$-converges to some $x \in M$, and for every net $\{y_\beta : \beta \in B\}$ in $M$ that satisfies

$$\limsup_{\beta \in B} \left[ \limsup_{\alpha \in A} \rho(x_\alpha, y_\beta) \right] \leq \limsup_{\alpha \in A} \rho(x_\alpha, x),$$

we have $\lim_{\beta \in B} \rho(y_\beta, x) = 0$.

3. Asymptotic behavior

In this section, unless otherwise stated, $S$ is a right reversible semitopological semigroup and $\mathcal{S} = \{T(t) : t \in S\}$ is a representation of $S$ as $\rho$-asymptotically nonexpansive type self-mappings of $M$ and $u = \{u(t) : t \in S\}$ is an almost-orbit of $\mathcal{S}$. We study the $\tau$-convergence of $u$ in $M$. We denote the $\tau$-convergence of a net $\{x_\alpha : \alpha \in A\}$ to $x \in M$ by $x_\alpha \xrightarrow{\tau} x$, or by $x_\alpha \xrightarrow{\tau} x$, and the $\rho$-convergence by $\rho(x_\alpha) \xrightarrow{\rho} x$. $\omega_\tau(u)$ denotes the $\tau$-limit set of $u$, i.e.,

$$\omega_\tau(u) = \{x \in M; \exists \text{ subnet } u(t_\alpha) \xrightarrow{\tau} x\}.$$

$\omega_\tau(u) \neq \emptyset$ if $\{M, \tau\}$ is compact.

Let $L(u) := \{p \in M; \lim_{t \in S} \rho(u(t), p) \text{ exists}\}$.

**Lemma 3.1.** If $u$ and $v$ are asymptotic almost-orbits of $T$, then $\lim_{t \in S} \rho(u(t), v(t))$ exists. In particular, $F \subset AF \subset L(u)$.

**Proof.** Let

$$a(t) = \limsup_{s \in S} \rho(u(st), T(s)u(t))$$
and

\[ b(t) = \limsup_{s \in S} \rho(v(st), T(s)v(t)). \]

Then \( \lim_{t \in S} a(t) = \lim_{t \in S} b(t) = 0 \), and we have

\[
\rho(u(st), v(st)) \leq \rho(u(st), T(s)u(t)) + \rho(T(s)u(t), T(s)v(t)) \\
+ \rho(T(s)v(t), v(st)) \\
\leq \rho(u(st), T(s)u(t)) + \rho(T(s)v(t), v(st)) + \rho(u(t), v(t)) \\
+ \epsilon(s, u(t)).
\]

Keeping \( t \) fixed and taking the limit over \( s \in S \), we get

\[
\limsup_{s \in S} \rho(u(s), v(s)) \leq a(t) + b(t) + \rho(u(t), v(t)).
\]

Now taking the limit over \( t \in S \) we get

\[
\limsup_{s \in S} \rho(u(s), v(s)) \leq \liminf_{t \in S} \rho(u(t), v(t))
\]

which implies that \( \lim_{t \in S} \rho(u(t), v(t)) \) exists.

Now to complete the proof of the lemma, the inclusion \( F \subset AF \) is obvious, and we have \( AF \subset L(u) \) since every element of \( AF \) is clearly an asymptotic almost-orbit of \( \mathcal{I} \).

\[\Box\]

Lemma 3.2. Assume \( \{M, \tau\} \) is compact and \( \{M, \rho, \tau\} \) satisfies the \( \tau \)-Opial condition. Then an almost-orbit \( u = \{u(t): t \in S\} \) of \( \mathcal{I} \) is \( \tau \)-convergent in \( M \) if \( \omega_{\tau}(u) \subset L(u) \).

Proof. Since \( \{M, \tau\} \) is compact, \( \omega_{\tau}(u) \neq \emptyset \) and hence \( L(u) \neq \emptyset \); therefore without loss of generality, we can assume that \( u \) is \( \rho \)-bounded. Assume \( u(t_\alpha) \xrightarrow{\tau} p \) and \( u(s_\beta) \xrightarrow{\tau} q \). Then by assumption \( p, q \in L(u) \). If \( p \neq q \), by using the \( \tau \)-Opial condition we have

\[
\lim_{t \in S} \rho(u(t), p) = \limsup_{\alpha} \rho(u(t_\alpha), p) < \limsup_{\alpha} \rho(u(t_\alpha), q) \\
= \lim_{t \in S} \rho(u(t), q) = \limsup_{\beta} \rho(u(s_\beta), q) \\
< \limsup_{\beta} \rho(u(s_\beta), p) = \lim_{t \in S} \rho(u(t), p)
\]

which is a contradiction. Therefore we must have \( p = q \) which implies that \( \omega_{\tau}(u) \) is a singleton. Since \( \{M, \tau\} \) is compact, this implies the \( \tau \)-convergence of \( u \) in \( M \). \[\Box\]

The following proposition plays a crucial role in the proof of our main result.
Proposition 3.3. Assume \( \{ M, \tau \} \) is compact and \( u \) is a \( \rho \)-bounded and \( \tau \)-asymptotically regular almost-orbit of \( \mathcal{S} \). Then \( \omega_\tau(u) \subset A F \) if either one of the following (i) or (ii) holds:

(i) \( \{ M, \rho, \tau \} \) satisfies the uniform \( \tau \)-Opial condition.

(ii) \( \{ M, \rho, \tau \} \) satisfies the locally uniform \( \tau \)-Opial condition and \( u \) is moreover \( \rho \)-asymptotically regular.

If \( \mathcal{S} \) is a nonexpansive semigroup, then we even have \( \omega_\tau(u) \subset F \) if \( \{ M, \rho, \tau \} \) satisfies the \( \tau \)-Opial condition.

Proof. We know \( \omega_\tau(u) \neq \emptyset \). Let \( p \in \omega_\tau(u) \) and \( \tau \)-lim_{\alpha \in A} u(t_\alpha) = p \); the \( \tau \)-asymptotic regularity of \( u \) implies that \( \tau \)-lim_{\alpha \in A} u(st_\alpha) = p \) for each \( s \in S \). Let \( a(t) = \sup_{s \in S} \rho(u(st), T(s)u(t)) \), then \( \lim_{t \in S} a(t) = 0 \); let \( c(s) = \limsup_{\alpha \in A} \rho(u(st_\alpha), p) \) and \( c = \inf_{s \in S} c(s) \). First assume that \( \{ M, \rho, \tau \} \) satisfies the \( \tau \)-Opial condition. Then we have

\[
c(hs) = \limsup_{\alpha \in A} \rho(u(hst_\alpha), p) \leq \limsup_{\alpha \in A} \rho(u(hst_\alpha), T(h)p) \leq \limsup_{\alpha \in A} \rho(u(hst_\alpha), T(h)u(st_\alpha)) + \limsup_{\alpha \in A} \rho(T(h)u(st_\alpha), T(h)p) \leq \limsup_{\alpha \in A} a(st_\alpha) + \limsup_{\alpha \in A} \rho(u(st_\alpha), p) + \epsilon(h, p) = c(s) + \epsilon(h, p).
\]

Keeping \( s \) fixed and taking the limit over \( h \in A \), we get \( \liminf_{t \in S} c(t) = c(s) \) for all \( s \in S \). This implies that \( \lim_{t \in S} c(t) = \inf_{s \in S} c(s) = c \).

Now let \( \{ \epsilon_n \} \) be an arbitrary sequence of positive numbers tending to zero (e.g., \( \epsilon_n = 1/n \)), and let \( \{ O_\gamma : \gamma \in \Gamma \} \) be the family of all \( \tau \)-open neighborhoods of \( p \). Let \( h \in S \) fixed. For each integer \( l \geq 1 \) we choose \( s_l \in S \) and \( \alpha^1_l \in A \) so that \( c(s_l) \leq c + \epsilon_l \) and \( a(t_{\alpha^1_l}) \leq \epsilon_l \) for all \( t \in S \) and \( \alpha \in A \) with \( \alpha \geq \alpha^1_l \).

Now we choose \( \alpha^2_l \in A \) with \( \alpha^2_l \geq \alpha^1_l \) so that \( \rho(u(s_l t_{\alpha^1_l}), p) \leq c + 2\epsilon_l \) and \( \rho(u(hs_l t_{\alpha^1_l}), p) \geq c - \epsilon_l \) for all \( t \in S \) with \( \alpha \geq \alpha^2_l \). Now for each \( \tau \)-neighborhood \( O_\gamma \) of \( p \) we choose \( \alpha^{h,l}_{\gamma} \in A \) with \( \alpha^{h,l}_{\gamma} \geq \alpha^2_l \) so that \( u(hs_l t_{\alpha^1_l}) \in O_\gamma \) for all \( \alpha \in A \) with \( \alpha \geq \alpha^{h,l}_{\gamma} \). This is possible, since for \( h \in S \) and \( l \geq 1 \) fixed, we have \( \tau \)-lim_{\alpha \in A} u(hs_l t_{\alpha^1_l}) = p \). We now consider the set \( I := N \times \Gamma \) directed by the relation

\[
(n_1, \gamma_1) \leq (n_2, \gamma_2) \quad \text{if and only if} \quad n_1 \leq n_2 \quad \text{and} \quad O_{\gamma_2} \subset O_{\gamma_1}.
\]

Then from our construction above, it is clear that for each \( h \in S \) fixed, we have \( \tau \)-lim_{(l, \gamma) \in I} u(hs_l t_{\alpha^{h,l}_{\gamma}}) = p \) and for each \( h \in S \), \( l \geq 1 \) and \( \gamma \in \Gamma \) we have the following inequalities:

\[
\rho(u(hs_l t_{\alpha^{h,l}_{\gamma}}), T(h)p) \leq \rho(u(hs_l t_{\alpha^{h,l}_{\gamma}}), T(h)s_l t_{\alpha^{h,l}_{\gamma}})) + \rho(T(h)s_l t_{\alpha^{h,l}_{\gamma}}), T(h)p).
\]
\[
\leq a(slt_{a_{h,l}}, p) + \rho(u(slt_{a_{h,l}}, p) + \epsilon(h, p) \\
\leq \epsilon_I + c + 2\epsilon_I + \epsilon(h, p) = c + 3\epsilon_I + \epsilon(h, p) \\
\leq \rho(u(slt_{a_{h,l}}, p) + 4\epsilon_I + \epsilon(h, p).
\]

(1)

First we note that if \( \mathcal{I} \) is a nonexpansive semigroup, then \( \epsilon(h, p) = 0 \) for all \( h \in S \). Therefore, taking in (1) the limit over \((l, \gamma) \in I\), we get

\[
\limsup_{(l, \gamma) \in I} \rho(u(slt_{a_{h,l}}, p), T(h)p) \leq \limsup_{(l, \gamma) \in I} \rho(u(slt_{a_{h,l}}, p), p) + \epsilon(h, p).
\]

which implies by the \( \tau \)-Opial condition that \( T(h)p = p \). Since \( h \in S \) was arbitrary, we therefore have \( p \in F \) and hence \( \omega_\tau(u) \subset F \), completing the proof to the last assertion of the proposition. Assume now that (i) holds. Then for fixed \( h \in S \), we get from (1) that

\[
\limsup_{(l, \gamma) \in I} \rho(u(slt_{a_{h,l}}, p), T(h)p) \leq \limsup_{(l, \gamma) \in I} \rho(u(slt_{a_{h,l}}, p), p) + \epsilon(h, p).
\]

Since \( \lim_{h \in S} \epsilon(h, p) = 0 \), the uniform \( \tau \)-Opial condition for \( \{M, \rho, \tau\} \) implies that \( \lim_{h \in S} \rho(T(h)p, p) = 0 \), i.e., \( p \in AF \). Hence \( \omega_\tau(u) \subset AF \) and the proof of (i) is now complete.

Now assume that (ii) holds. By the triangle inequality we have

\[
\rho(u(slt_{a_{h,l}}, p), p) \leq \rho(u(slt_{a_{h,l}}, p), p) + \rho(u(slt_{a_{h,l}}, u(slt_{a_{h,l}})), u(slt_{a_{h,l}}))
\]

and

\[
\rho(u(slt_{a_{h,l}}, T(h)p), T(h)p) \geq \rho(u(slt_{a_{h,l}}, p), T(h)p) \\
- \rho(u(slt_{a_{h,l}}, u(slt_{a_{h,l}}))).
\]

Hence for fixed \( h \in S \), we get from (1) and the \( \rho \)-asymptotic regularity of \( u \) that

\[
\limsup_{(l, \gamma) \in I} \rho(u(slt_{a_{h,l}}, p), T(h)p) \leq \limsup_{(l, \gamma) \in I} \rho(u(slt_{a_{h,l}}, p), p) + \epsilon(h, p).
\]

Now taking the limit over \( h \in S \), we get

\[
\limsup_{h \in S} \left[ \limsup_{(l, \gamma) \in I} \rho(u(slt_{a_{h,l}}, p), T(h)p) \right] \leq \limsup_{(l, \gamma) \in I} \rho(u(slt_{a_{h,l}}, p), p).
\]

Since \( \{M, \rho, \tau\} \) satisfies the locally uniform \( \tau \)-Opial condition, by Lemma 2.4 we conclude that \( \lim_{h \in S} \rho(T(h)p, p) = 0 \), i.e., \( p \in AF \). Hence \( \omega_\tau(u) \subset AF \) and the proof of the proposition is now complete. \( \square \)

Now we can state our main result.

**Theorem 3.4.** Assume \( \{M, \tau\} \) is compact and \( u \) is a \( \rho \)-bounded and \( \tau \)-asymptotically regular almost-orbit of \( \mathcal{I} \). Then \( u \) is \( \tau \)-convergent in \( M \) if either one of the following (i), (ii) or (iii) holds:

\[
\limsup_{h \in S} \left[ \limsup_{(l, \gamma) \in I} \rho(u(slt_{a_{h,l}}, p), T(h)p) \right] \leq \limsup_{(l, \gamma) \in I} \rho(u(slt_{a_{h,l}}, p), p).
\]
(i) $\mathcal{I}$ is a nonexpansive semigroup and $\{M, \rho, \tau\}$ satisfies the $\tau$-Opial condition.

(ii) $\{M, \rho, \tau\}$ satisfies the uniform $\tau$-Opial condition.

(iii) $\{M, \rho, \tau\}$ satisfies the locally uniform $\tau$-Opial condition and $u$ is moreover $\rho$-asymptotically regular.

In (i), the $\tau$-limit of $u$ belongs to $F$. In (ii) and (iii), the $\tau$-limit of $u$ belongs to $AF$; it belongs to $F$ if either $T(t)$ is $\rho$-continuous for each $t \in S$, or $T(t)$ is $\rho$-nonexpansive for some $t \in S$ and $S$ satisfies the following property: $\forall h \in S$, $\exists n \geq 1$ such that $h \preceq t^n$.

**Proof.** By Lemma 3.1 we have $F \subset AF \subset L(u)$; by Proposition 3.3 we have $\omega_\tau(u) \subset F \subset L(u)$ in (i), and $\omega_\tau(u) \subset AF \subset L(u)$ in (ii) and (iii). Therefore an application of Lemma 3.2 gives the $\tau$-convergence of $u$ in all these cases.

By Proposition 3.3, we know that in (i) the $\tau$-limit of $u$ belongs to $F$, and in (ii) and (iii) it belongs to $AF$. If $T(t)$ is $\rho$-continuous for each $t \in S$, then clearly $AF = F$, since for $t \in S$ and $p \in AF$ we have

$$p = \rho-\lim_{s \in S} T(s)p = \rho-\lim_{s \in S} T(ts)p = T(t)\left(\rho-\lim_{s \in S} T(s)p\right) = T(t)p,$$

so the result follows in this case.

Now assume that $T(t)$ is $\rho$-nonexpansive for some $t \in S$. Then replacing $h$ by $t$ in the inequality (1) in Proposition 3.3 and noting that $\epsilon(t, p) = 0$, we conclude by using the $\tau$-Opial condition for $\{M, \rho, \tau\}$ that $T(t)p = p$. Now by the property we assumed for $S$, we have $\rho-\lim_{n \to \infty} T(st^n)p = p$, $\forall s \in S$, since $p \in AF$. Therefore for each $s \in S$ we have

$$T(s)p = T(s)T(t^n)p = T(st^n)p = p,$$

i.e., $p \in F$. This completes the proof of the theorem. □

**Remark 3.5.** It is clear that every $\tau$-convergent curve $u$ in $M$ is $\tau$-asymptotically regular.

**Remark 3.6.** Theorem 3.4 gives an affirmative answer to an open question of Reich [23, p. 550] even in this very general context.

**Remark 3.7.** Theorem 3.4 extends recent results of Li [16] and Kim and Li [11, 17], as well as [5]; if $M$ is a weakly (respectively weak*) compact subset of a Banach space and $\tau$ is the weak (respectively weak*) topology on $M$, then it extends many previously known results to asymptotically nonexpansive type mappings and semigroups, as mentioned in the Introduction.
4. Some open problems

Our discussion leaves the following problems open:

(1) In Theorem 3.4(ii) or (iii), does the conclusion hold if we assume only that \( \{M, \rho, \tau\} \) satisfies the locally uniform \( \tau \)-Opial condition?

(2) Is it possible to extend Theorem 3.4 to nonexpansive (respectively almost nonexpansive) curves in \( M \)? See [2–4] and references therein for appropriate definitions and an affirmative answer in the Hilbert space case. In this case, for \( t \in S \), \( T(t) \) is not defined anymore on \( \omega_\tau(u) \).

Acknowledgments

This work started during the first author’s visit to Kyungnam University. He thanks Professor J.K. Kim and the Department of Mathematics for their kind hospitality during his visit.

References


