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Asymptotic Behavior and Positive Solutions of a Chemical Reaction Diffusion System

W. H. RUAN

Department of Mathematical Sciences, Purdue University Calumet, Hammond, Indiana 46323

AND

C. V. PAO

Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27695

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This paper is concerned with some qualitative analysis for a coupled system of three reaction diffusion equations which arises from certain chemical reactions first discovered by Belousov and Zhabotinskii. The analysis includes the existence of a bounded global time-dependent solutions, the stability and instability of the zero solution, and the existence and nonexistence of a positive steady-state solution, including a global attractor of the system. The global existence and stability problem is determined by the method of upper and lower solutions, and the existence of a positive steady-state solution is based on the fixed point index and bifurcation theory. This analysis leads to a necessary and sufficient condition for the existence and nonexistence of a positive steady-state solution in relation to the various physical parameters of the system.

1. INTRODUCTION

The Belousov–Zhabotinskii reaction is a class of oscillatory metal-ion-catalized oxidations of organic compounds by bromate ion. The reaction was first discovered by B. P. Belousov and A. M. Zhabotinskii, and since then the reaction system has been extensively studied both from chemical and mathematical points of view (cf. [2–6, 9, 10, 13]). The original mathematical model consists of ten chemical reactions with seven intermediates. Due to the complicated chemical kinetics of the system Field and Noyes later abstracted a modified model of three variable equations, called oregonator (cf. [2]). This simplified system retains most of the important

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features of the original mechanism yet much more tractable mathematically. The chemical reaction scheme of this simplification is given by

$$A+Y\to X,\qquad X+Y\to P,\qquad B+X\to 2X+Z$$
$$2X\to Q,\qquad Z\to\gamma Y$$

where A and B are reactants, P and Q are products, γ is a reaction constant, and X, Y, and Z are concentrations of the intermediates $HBrO_2$ (bromous acid), Br^- (bromide ion), and Ce(IV) (Cerium), respectively. Under the condition that the concentrations of reactants are held constant and uniform the dimensionless form of the equations are given by

$$\begin{aligned} u_t - D_1\nabla^2 u &= u(a_1' - b'u - c_1'v) + q_1'v \\ v_t - D_2\nabla^2 v &= -c_2'uv + d_2'w - q_2'v \qquad (t>0,\, x\in\Omega) \\ w_t - D_3\nabla^2 w &= b_3'u - d_3'w, \end{aligned} \tag{1.1}$$

where u, v, and w represent the respective concentrations X, Y, and Z, the D_i's are the diffusion coefficients, and Ω is the diffusion medium which is a bounded domain in $\mathbb{R}^n$. The constants a_i', b_i', c_i', d_i', and q_i', $i=1,2,3$, are all positive.

Considerable attention has been given to the system (1.1) but it is mostly devoted either to the well-stirred system in which problem (1.1) is reduced to a system of ordinary differential equations or to the case of one spatial dimension for the traveling wave solution (cf. [2–6]). In this paper we use the method of upper-lower solutions and bifurcation theory to treat some qualitative aspects of the system in an arbitrary bounded domain Ω in $\mathbb{R}^n$ (cf. [7–9, 11, 12]). The boundary condition under consideration is given by

$$Bu=0,\qquad Bv=0,\qquad Bw=0\quad (t>0,\, x\in\partial\Omega) \tag{1.2}$$

and the initial condition is

$$u(o,x)=u_o(x),\qquad v(o,x)=v_o(x),\qquad w(o,x)=w_o(x)\quad (x\in\Omega), \tag{1.3}$$

where $\partial\Omega$ is the boundary of Ω, $B\equiv\alpha_o\partial/\partial\nu+\beta_o$ is a linear boundary operator, and $\partial/\partial\nu$ is the outward normal derivative on $\partial\Omega$. This qualitative analysis includes the existence of a bounded global solution of (1.1)–(1.3), the asymptotic behavior of the solution, and the existence and nonexistence of a positive solution of the steady-state problem

$$\begin{aligned} -\nabla^2 u &= u(a_1 - b_1u - c_1v) + q_1v, &\quad Bu=0 \\ -\nabla^2 v &= -c_2uv + d_2w - q_2v, &\quad Bv=0 \\ -\nabla^2 w &= b_3u - d_3w, &\quad Bw=0, \end{aligned} \tag{1.4}$$

where

$$a_1 = a'_1/D_1, \qquad b_i = b'_i/D_i, \qquad c_i = c'_i/D_i, \qquad d_i = d'_i/D_i, \qquad q_i = q'_i/D_i, \quad i = 1, 2, 3. \tag{1.5}$$

Of special concern is the bifurcation of the trivial solution to a positive solution of (1.4). It is assumed that Ω is of class $C^{2+\alpha}$, α_o and β_o are nonnegative constants with $\alpha_o + \beta_o > 0$, and u_o, v_o, and w_o are nonnegative functions in $C^\alpha(\Omega)$. The consideration of the boundary operator B includes the Dirichlet condition ($\alpha_o = 0$, $\beta_o = 1$), the Neumann condition ($\alpha_o = 1$, $\beta_o = 0$), and the Robin condition ($\alpha_o = 1$, $\beta_o > 0$).

The plan of the paper is as follows: In Section 2 we use the method of upper and lower solutions to show the existence of a bounded global solution to (1.1)–(1.3) and establish its convergence property as $t \to \infty$. It is shown that under the condition (2.15) every time-dependent solution converges to the trivial solution as $t \to \infty$ while under the reversed condition (2.19) the solution tends to a global attractor. Based on the theory of bifurcation and fixed point index we show in Section 3 that condition (2.19) is a necessary and sufficient condition for the existence of a positive solution to the steady-state problem (1.4). This condition yields a bifurcation point in relation to the physical parameters where the solution bifurcates from the trivial solution to a unique positive solution.

2. The Time-Dependent System

In this section we use the method of upper and lower solutions to show the existence of a bounded global solution to (1.1)–(1.3). The construction of upper-lower solutions yields a necessary and sufficient condition for the asymptotic convergence of the time-dependent solution to the zero steady-state solution. To obtain this result it is convenient to set $\mathscr{D} \equiv [0, \infty) \times \Omega$, $\mathscr{S} \equiv [0, \infty) \times \partial\Omega$, and write Eq. (1.1) in the form

$$\begin{aligned} \sigma_1 u_t - \nabla^2 u &= u(a_1 - b_1 u - c_1 v) + q_1 v \\ \sigma_2 v_t - \nabla^2 v &= -c_2 uv + d_2 w - q_2 v \qquad \text{in } \mathscr{D} \\ \sigma_3 w_t - \nabla^2 w &= b_3 u - d_3 w, \end{aligned} \tag{2.1}$$

where $\sigma_i = D_i^{-1}$ and a_i, b_i, etc.., $i = 1, 2, 3$, are given by (1.5). Since the reaction function in (2.1) does not possess a quasimonotone property when $q_1 > 0$ it is necessary to modify the definition of upper and lower solutions as follow.

DEFINITION 2.1. A pair of smooth functions $(\tilde{u}, \tilde{v}, \tilde{w})$, $(\hat{u}, \hat{v}, \hat{w})$ are called coupled upper and lower solutions of (2.1), (1.2), (1.3) if $(\tilde{u}, \tilde{v}, \tilde{w}) \geqslant (\hat{u}, \hat{v}, \hat{w})$ and if they satisfy the differential inequalities

$$\begin{aligned}
&\sigma_1 \tilde{u}_t - \nabla^2 \tilde{u} \geqslant \tilde{u}(a_1 - b_1 \tilde{u} - c_1 \hat{v}) + q_1 \tilde{v} \\
&\sigma_2 \tilde{v}_t - \nabla^2 \tilde{v} \geqslant -c_2 \hat{u}\tilde{v} + d_2 \tilde{w} - q_2 \tilde{v} \\
&\sigma_3 \tilde{w}_t - \nabla^2 \tilde{w} \geqslant b_3 \tilde{u} - d_3 \tilde{w} \\
&\sigma_1 \hat{u}_t - \nabla^2 \hat{u} \leqslant \hat{u}(a_1 - b_1 \hat{u} - c_1 \tilde{v}) + q_1 \hat{v} \\
&\sigma_2 \hat{v}_t - \nabla^2 \hat{v} \leqslant -c_2 \tilde{u}\hat{v} + d_2 \hat{w} - q_2 \hat{v} \\
&\sigma_3 \hat{w}_t - \nabla^2 \hat{w} \leqslant b_3 \hat{u} - d_3 \hat{w}
\end{aligned} \tag{2.2}$$

and the boundary and initial inequalities

$$\begin{aligned}
&B\tilde{u} \geqslant 0 \geqslant B\hat{u}, \qquad B\tilde{v} \geqslant 0 \geqslant B\hat{v}, \qquad B\tilde{w} \geqslant 0 \geqslant B\hat{w} \qquad \text{on } \mathscr{S} \\
&\tilde{u} \geqslant u_o \geqslant \hat{u}, \qquad \tilde{v} \geqslant v_o \geqslant \hat{v}, \qquad \tilde{w} \geqslant w_o \geqslant \hat{w} \qquad \text{at} \quad t = 0, \quad x \in \Omega.
\end{aligned} \tag{2.3}$$

Notice that the above definition is different from the usual definition for mixed quasimonotone functions since both $\tilde{v}$ and $\hat{v}$ are involved in the differential inequalities for $\tilde{u}$ and $\hat{u}$. The purpose for the above definition is to obtain two monotone sequences from the following iteration processes

$$\begin{aligned}
&L_1 \bar{u}^{(k)} = \gamma_1 \bar{u}^{(k-1)} + \bar{u}^{(k-1)}(a_1 - b_1 \bar{u}^{(k-1)} - c_1 \underline{v}^{(k-1)}) + q_1 \bar{v}^{(k-1)} \\
&L_2 \bar{v}^{(k)} = (\gamma_2 - q_2)\bar{v}^{(k-1)} - c_2 \underline{u}^{(k-1)} \bar{v}^{(k-1)} + d_2 \bar{w}^{(k-1)} \\
&L_3 \bar{w}^{(k)} = b_3 \bar{u}^{(k-1)}
\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
&L_1 \underline{u}^{(k)} = \gamma_1 \underline{u}^{(k-1)} + \underline{u}^{(k-1)}(a_1 - b_1 \underline{u}^{(k-1)} - c_1 \bar{v}^{(k-1)}) + q_1 \underline{v}^{(k-1)} \\
&L_2 \underline{v}^{(k)} = (\gamma_2 - q_2)\underline{v}^{(k-1)} - c_2 \bar{u}^{(k-1)} \underline{v}^{(k-1)} + d_2 \underline{w}^{(k-1)} \\
&L_3 \underline{w}^{(k)} = b_3 \underline{u}^{(k-1)}
\end{aligned} \tag{2.5}$$

for $k = 1, 2, \ldots$, where L_1, L_2, and L_3 are the operators given by

$$L_i = (\sigma_i \partial/\partial t - \nabla^2 + \gamma_i), \qquad i = 1, 2, 3$$

and the γ_i's are any positive constants satisfying

$$\gamma_1 \geqslant \max\{2b_1 u + c_1 v - a_1 ;\ \hat{u} \leqslant u \leqslant \tilde{u},\ \hat{v} \leqslant v \leqslant \tilde{v}\}$$

$$\gamma_2 \geqslant c_2 \tilde{u} + q_2 \qquad \text{and} \qquad \gamma_3 = d_3.$$

The initial iterations are $(\bar{u}^{(o)}, \bar{v}^{(o)}, \bar{w}^{(o)}) = (\tilde{u}, \tilde{v}, \tilde{w})$ and $(\underline{u}^{(o)}, \underline{v}^{(o)}, \underline{w}^{(o)}) = (\hat{u}, \hat{v}, \hat{w})$, and the boundary and initial conditions are given by

$$\begin{aligned} &B\bar{u}^{(k)} = B\underline{u}^{(k)} = 0, \qquad B\bar{v}^{(k)} = B\underline{v}^{(k)} = 0, \qquad B\bar{w}^{(k)} = B\underline{w}^{(k)} = 0 \\ &\bar{u}^{(k)} = \underline{u}^{(k)} = u_o, \qquad \bar{v}^{(k)} = \underline{v}^{(k)} = v_o, \qquad \bar{w}^{(k)} = \underline{w}^{(k)} = w_o \qquad \text{at} \quad t = 0. \end{aligned} \tag{2.6}$$

We show that the two sequences given by (2.4)–(2.6) both converge monotonically to a unique solution of (2.1), (1.2), (1.3).

THEOREM 2.1. *Let $(\tilde{u}, \tilde{v}, \tilde{w})$ and $(\hat{u}, \hat{v}, \hat{w})$ be a pair of nonnegative coupled upper-lower solutions of* (2.1), (1.2), (1.3). *Then the two sequences* $\{\bar{u}^{(k)}, \bar{v}^{(k)}, w^{(k)}\}$, $\{\underline{u}^{(k)}, \underline{v}^{(k)}, \underline{w}^{(k)}\}$ *given by* (2.4)–(2.6) *converge monotonically to a unique solution* (u, v, w) *and*

$$(\hat{u}, \hat{v}, \hat{w}) \leqslant (u, v, w) \leqslant (\tilde{u}, \tilde{v}, \tilde{w}) \qquad \text{in } \mathscr{D}. \tag{2.7}$$

Proof. Let $(y_1, y_2, y_3) = (\bar{u}^{(o)}, \bar{v}^{(o)}, \bar{w}^{(o)}) - (\bar{u}^{(1)}, \bar{v}^{(1)}, \bar{w}^{(1)})$. By (2.2), (2.4), and $(\bar{u}^{(o)}, \bar{v}^{(o)}, \bar{w}^{(o)}) = (\tilde{u}, \tilde{v}, \tilde{w})$,

$$\begin{aligned} L_1 y_1 &= (\sigma_1 \bar{u}_t^{(o)} - \nabla^2 \bar{u}^{(o)} + \gamma_1 \bar{u}^{(o)}) \\ &\quad - [\gamma_1 \bar{u}^{(o)} + \bar{u}^{(o)}(a_1 - b_1 \bar{u}^{(o)} - c_1 \underline{v}^{(o)}) + q_1 \bar{v}^{(o)}] \\ &= \sigma_1 \tilde{u}_t - \nabla^2 \tilde{u} - \tilde{u}(a_1 - b_1 \tilde{u} - c_1 \hat{v}) - q_1 \tilde{v} \geqslant 0 \\ L_2 y_2 &= (\sigma_2 \bar{v}_t^{(o)} - \nabla^2 \bar{v}^{(o)} + \gamma_2 \bar{v}^{(o)}) \\ &\quad - [(\gamma_2 - q_2)\bar{v}^{(o)} - c_2 \underline{u}^{(o)} \bar{v}^{(o)} + d_2 \bar{w}^{(o)}] \\ &= \sigma_2 \tilde{v}_t - \nabla^2 \tilde{v} - (-q_2 \tilde{v} - c_2 \hat{u} \tilde{v} + d_2 \tilde{w}) \geqslant 0 \\ L_3 y_3 &= (\sigma_3 \bar{w}_t^{(o)} - \nabla^2 \bar{w}^{(o)} + \gamma_3 \bar{w}^{(o)}) - b_3 \bar{u}^{(o)} \\ &= \sigma_3 \tilde{w}_t - \nabla^2 \tilde{w} + d_3 \tilde{w} - b_3 \tilde{u} \geqslant 0. \end{aligned}$$

Since by (2.3), (2.6),

$$\begin{gathered} By_1 = B\tilde{u} \geqslant 0, \qquad By_2 = B\tilde{v} \geqslant 0, \qquad By_3 = B\tilde{w} \geqslant 0 \\ y_1(0, x) = \tilde{u}(0, x) - u_o \geqslant 0, \qquad y_2(0, x) = \tilde{v}(0, x) - v_o \geqslant 0, \\ y_3(0, x) = \tilde{w}(0, x) - w_o \geqslant 0 \end{gathered}$$

the maximal principle implies that $y_1 \geqslant 0$, $y_2 \geqslant 0$, and $y_3 \geqslant 0$ in $\mathscr{D}$. This proves the relation $(\bar{u}^{(1)}, \bar{v}^{(1)}, \bar{w}^{(1)}) \leqslant (\bar{u}^{(o)}, v^{(o)}, w^{(o)})$. A similar argument, using the property of a lower solution, gives $(\underline{u}^{(1)}, \underline{v}^{(1)}, \underline{w}^{(1)}) \geqslant$

$(\underline{u}^{(o)}, \underline{v}^{(o)}, \underline{w}^{(o)})$. Let $(z_1, z_2, z_3) = (\bar{u}^{(1)}, \bar{v}^{(1)}, \bar{w}^{(1)}) - (\underline{u}^{(1)}, \underline{v}^{(1)}, \underline{w}^{(1)})$. By (2.4), (2.5), and the property of γ_1, γ_2, and γ_3,

$$\begin{aligned} L_1 z_1 &= [\gamma_1 + a_1 - b_1(\bar{u}^{(o)} + \underline{u}^{(o)}) - c_1 \underline{v}^{(o)}](\bar{u}^{(o)} - \underline{u}^{(o)}) \\ &\quad + (c_1 \underline{u}^{(o)} + q_1)(\bar{v}^{(o)} - \underline{v}^{(o)}) \geqslant 0 \\ L_2 z_2 &= [(\gamma_2 - q_2) - c_2 \underline{u}^{(o)}](\bar{v}^{(o)} - \underline{v}^{(o)}) + c_2 \underline{v}^{(o)}(\bar{u}^{(o)} - \underline{u}^{(o)}) \\ &\quad + d_2(\bar{w}^{(o)} - \underline{w}^{(o)}) \geqslant 0 \\ L_3 z_3 &= b_3(\bar{u}^{(o)} - \underline{u}^{(o)}) \geqslant 0. \end{aligned}$$

Since $Bz_i = 0$ and $z_i(0, x) = 0$ for each $i = 1, 2, 3$, it follows that $z_i \geqslant 0$ in $\mathscr{D}$. This shows that the sequences

$$\{\bar{\mathbf{u}}^{(k)}\} \equiv \{\bar{u}^{(k)}, \bar{v}^{(k)}, \bar{w}^{(k)}\}, \qquad \{\underline{\mathbf{u}}^{(k)}\} \equiv \{\underline{u}^{(k)}, \underline{v}^{(k)}, \underline{w}^{(k)}\}$$

possess the property $\underline{\mathbf{u}}^{(o)} \leqslant \underline{\mathbf{u}}^{(1)} \leqslant \bar{\mathbf{u}}^{(1)} \leqslant \bar{\mathbf{u}}^{(o)}$ in $\mathscr{D}$. It is easily shown by an induction argument that

$$\underline{\mathbf{u}}^{(k)} \leqslant \underline{\mathbf{u}}^{(k+1)} \leqslant \bar{\mathbf{u}}^{(k+1)} \leqslant \bar{\mathbf{u}}^{(k)} \qquad \text{in } \mathscr{D} \tag{2.8}$$

for every $k = 0, 1, 2, \ldots$. This implies that the pointwise limits

$$\left.\begin{aligned} \lim(\bar{u}^{(k)}, \bar{v}^{(k)}, \bar{w}^{(k)}) &= (\bar{u}, \bar{v}, \bar{w}) \\ \lim(\underline{u}^{(k)}, \underline{v}^{(k)}, \underline{w}^{(k)}) &= (\underline{u}, \underline{v}, \underline{w}) \end{aligned}\right\} \quad \text{as} \quad k \to \infty \tag{2.9}$$

exist and $(\bar{u}, \bar{v}, \bar{w}) \geqslant (\underline{u}, \underline{v}, \underline{w})$ in $\mathscr{D}$. By the same regularity argument as given in [7, 8] the limits in (2.9) coincide and yield a unique solution to (2.1), (1.2), (1.3). This leads to the conclusion of the theorem. ∎

It is seen from Theorem 2.1 that to ensure the existence of a global solution to (2.1), (1.2), (1.3) it suffices to find a pair of coupled upper-lower solutions. For this purpose we consider the coupled system

$$\begin{aligned} \sigma_1 U_t - \nabla^2 U &= U(a_1 - b_1 U) + q_1 V \\ \sigma_2 V_t - \nabla^2 V &= d_2 W - q_2 V \qquad \text{in } \mathscr{D} \\ \sigma_3 W_t - \nabla^2 W &= b_3 U - d_3 W \end{aligned} \tag{2.10}$$

under the boundary and initial conditions

$$\begin{aligned} BU = 0, \qquad BV = 0, \qquad BW = 0 \qquad \text{on } \mathscr{S} \\ U(0, x) = M_o, \qquad V(0, x) = M_1, \qquad W(0, x) = M_2 \qquad \text{in } \Omega, \end{aligned} \tag{2.11}$$

where M_o, M_1, and M_2 are any constants satisfying the condition

$$\begin{gathered} M_o \geqslant a_1/b_1 + b_3 d_2 q_1 / b_1 d_3 q_2 \\ M_1 = (b_3 d_2 / d_3 q_2) M_o, \qquad M_2 = (b_3/d_3) M_o. \end{gathered} \tag{2.12}$$

We first show that a unique positive solution to (2.10), (2.11) exists and is uniformly bounded in $\mathscr{D}$.

LEMMA 2.1. *Given any positive constants* M_o, M_1, *and* M_2 *satisfying condition* (2.12) *the coupled system* (2.10), (2.11) *has a unique solution* (U, V, W) *such that*

$$(0, 0, 0) \leqslant (U, V, W) \leqslant (M_o, M_1, M_2) \quad \text{in } \mathscr{D}. \tag{2.13}$$

Proof. Since the nonlinear function at the right side of (2.10) is quasimonotone nondecreasing in $(U, V, W) \in \mathbb{R}^3$ the existence of a unique solution to (2.10), (2.11) is assured if there exists a pair of ordered upper-lower solutions in the usual sense. Here $(\tilde{U}, \tilde{V}, \tilde{W})$ is an upper solution if it satisfies all the relations in (2.10), (2.11) when the equality sign $=$ is replaced by the inequality sign $\geqslant$. Similarly, $(\hat{U}, \hat{V}, \hat{W})$ is a lower solution if it satisfies the reversed inequality $\leqslant$. It is clear from this definition that $(\hat{U}, \hat{V}, \hat{W}) = (0, 0, 0)$ is a lower solution. Moreover, since the constant function $(\tilde{U}, \tilde{V}, \tilde{W}) \equiv (M_o, M_1, M_2)$ satisfies the boundary and initial inequalities it is a positive upper solution if

$$\begin{aligned} 0 &\geqslant M_o(a_1 - b_1 M_o) + q_1 M_1 \\ 0 &\geqslant d_2 M_2 - q_2 M_1 \\ 0 &\geqslant b_3 M_o - d_3 M_2. \end{aligned}$$

The above inequalities are clearly satisfied by the relation (2.12). It follows from the existence-comparison theorem for quasimonotone nondecreasing functions that the system (2.10), (2.11) has a unique solution (U, V, W) which satisfies the relation (2.13) (cf. [7, 8]). This proves the lemma. ∎

Using the result of the above lemma we obtain the following global existence result for the problem (1.1)–(1.3).

THEOREM 2.2. *Given any* $(u_o, v_o, w_o) \geqslant (0, 0, 0)$ *there exist positive constants* M_o, M_1, *and* M_2 *such that the problem* (1.1)–(1.3) *has a unique global solution* (u, v, w) *which satisfies*

$$(0, 0, 0) \leqslant (u, v, w) \leqslant (U, V, W) \leqslant (M_o, M_1, M_2) \quad \text{in } \mathscr{D}, \tag{2.14}$$

where (U, V, W) *is the unique solution of* (2.10), (2.11).

Proof. Let M_o, M_1, M_2 be any positive constants such that $(M_o, M_1, M_2) \geqslant (u_o, v_o, w_o)$ and satisfy the relation (2.12). Then by direct computation the pair

$$(\tilde{u}, \tilde{v}, \tilde{w}) = (U, V, W) \quad \text{and} \quad (\hat{u}, \hat{v}, \hat{w}) = (0, 0, 0)$$

satisfy all the differential and boundary-initial inequalities in (2.2), (2.3), and therefore they are coupled upper-lower solutions of (2.1), (1.2), (1.3). The conclusion of the theorem follows from Theorem 2.1, Lemma 2.1, and the equivalence between the systems (1.1) and (2.1). ∎

We next show that the solution (u, v, w) converges to $(0, 0, 0)$ as $t \to \infty$ if

$$(\lambda_o - a_1)(\lambda_o + d_3)(\lambda_o + q_2) \geqslant b_3 d_2 q_1, \tag{2.15}$$

where λ_o is the principle eigenvalue of the problem

$$\nabla^2 \phi + \lambda \phi = 0 \quad \text{in } \Omega, \qquad B\phi = 0 \quad \text{on } \partial\Omega. \tag{2.16}$$

In view of the relation (2.14) it suffices to show that $(U, V, W) \to (0, 0, 0)$ as $t \to \infty$. Consider the coupled system

$$\begin{aligned} -\nabla^2 U &= U(a_1 - b_1 U) + q_1 V, \qquad & BU &= 0 \\ -\nabla^2 V &= d_2 W - q_2 V, & BV &= 0 \\ -\nabla^2 W &= b_3 U - d_3 W, & BW &= 0 \end{aligned} \tag{2.17}$$

which is the steady-state problem of (2.10), (2.11). Clearly this problem has the trivial solution $(0, 0, 0)$. The following lemma gives the uniqueness property of the trivial solution.

LEMMA 2.2. *Under the condition* (2.15) *the only nonnegative solution of the problem* (2.17) *is the trivial solution* $(0, 0, 0)$.

Proof. Let (U_s, V_s, W_s) be any nonnegative solution of (2.17) and let ϕ be the positive eigenfunction of (2.16) corresponding to λ_o. Since by Green's identity and the boundary condition in (2.17),

$$\int_\Omega (Z_s \nabla^2 \phi - \phi \nabla^2 Z_s)\, dx = \int_{\partial\Omega} \left(Z_s \frac{\partial \phi}{\partial \nu} - \phi \frac{\partial Z_s}{\partial \nu} \right) dS = 0,$$

where Z_s stands for U_s, V_s, or W_s, the relations (2.16) and (2.17) imply that

$$\lambda_o \int_\Omega \phi U_s\, dx = \int_\Omega \phi[U_s(a_1 - b_1 U_s) + q_1 V_s]\, dx$$

$$\lambda_o \int_\Omega \phi V_s\, dx = \int_\Omega \phi(d_2 W_s - q_2 V_s)\, dx$$

$$\lambda_o \int_\Omega \phi W_s\, dx = \int_\Omega \phi(b_3 U_s - d_3 W_s)\, dx.$$

Elimination of the integral for ϕW_s from the second and third equations gives

$$\int_\Omega \phi V_s\, dx = (b_3 d_2/(\lambda_o + d_3)(\lambda_o + q_2)) \int_\Omega \phi U_s\, dx.$$

By substituting this relation into the first equation we obtain

$$[-(\lambda_o - a_1) + b_3 d_2 q_1/(\lambda_o + d_3)(\lambda_o + q_2)] \int_\Omega \phi U_s\, dx = b_1 \int \phi U_s^2\, dx.$$

In view of condition (2.15), the above relation can hold only when $U_s \equiv 0$ in Ω. It follows from the last equation in (2.17) that $W_s \equiv 0$ in Ω and consequently $V_s \equiv 0$ in Ω. This shows that every nonnegative solution of (2.17) is necessarily the trivial solution (0, 0, 0). ∎

The uniqueness result of Lemma 2.2 leads to the following global stability of the trivial solution.

THEOREM 2.3. *Let the condition* (2.15) *hold. Then for any* $(u_o, v_o, w_o) \geqslant (0, 0, 0)$ *the solution* (u, v, w) *of* (1.1)–(1.3) *satisfies the relation*

$$\lim(u(t, x), v(t, x), w(t, x)) = (0, 0, 0) \qquad as \quad t \to \infty. \tag{2.18}$$

Proof. Let (U, V, W) be the nonnegative solution of (2.10), (2.11) and for any constant $\delta > 0$ let

$$(U_\delta, V_\delta, W_\delta) = (U(t, x) - U(t+\delta, x), V(t, x) - V(t+\delta, x), W(t, x) - W(t+\delta, x)).$$

By the mean-value theorem, $(U_\delta, V_\delta, W_\delta)$ satisfies the relation

$$\sigma_1 (U_\delta)_t - \nabla^2 U_\delta = (a_1 - 2b_1 \eta) U_\delta + q_1 V_\delta$$

$$\sigma_2 (V_\delta)_t - \nabla^2 V_\delta + q_2 V_\delta = d_2 W_\delta$$

$$\sigma_3 (W_\delta)_t - \nabla^2 W_\delta + d_3 W_\delta = b_3 U_\delta$$

and the boundary condition in (2.11), where $\eta \equiv \eta(t, x)$ is an intermediate value between $U(t, x)$ and $U(t+\delta, x)$. Since q_1, d_2, and b_3 are positive and

$$U_\delta = M_o - U(\delta, x) \geqslant 0, \quad V_\delta = M_1 - V(\delta, x) \geqslant 0, \quad W_\delta = M_2 - W(\delta, x) \geqslant 0$$

at $t = 0$ the well-known comparison theorem for linear systems implies that $(U_\delta, V_\delta, W_\delta) \geqslant (0, 0, 0)$ (cf. [8, 14]). Hence for each $x \in \bar{\Omega}$, (U, V, W) is monotone nonincreasing in t, and therefore it converges to some nonnegative function (U_s, V_s, W_s) as $t \to \infty$. By the same regularity argument as in [8, 9], (U_s, V_s, W_s) is a solution of the steady-state problem (2.17). It follows from Lemma 2.2 that $(U_s, V_s, W_s) = (0, 0, 0)$. This shows that (U, V, W) converges monotonically to $(0, 0, 0)$ as $t \to \infty$. The conclusion of the theorem follows from the relation (2.14). ∎

The result of Theorem 2.3 states that under the condition (2.15) the trivial solution (0, 0, 0) is globally asymptotically stable (with respect to nonnegative perturbations), and therefore the steady-state problem (1.4) cannot sustain a positive solution. However, if the reversed inequality of (2.15), that is,

$$(\lambda_o - a_1)(\lambda_o + d_3)(\lambda_o + q_2) < b_3 d_2 q_1 \tag{2.19}$$

holds, then a straight forward application of the linearization method shows that the trivial solution (0, 0, 0) is unstable. We show in this situation that the problem (1.1)–(1.3) has a global attractor which is given by

$$\mathscr{S} \equiv [0, U_s] \times [0, V_s] \times [0, W_s], \tag{2.20}$$

where (U_s, V_s, W_s) is the unique positive solution of (2.17). The existence of the positive solution (U_s, V_s, W_s) will be proven in the next section (see Theorem 3.1 with $c_1 = c_2 = 0$). In the following lemma, we show its uniqueness.

LEMMA 2.3. *Let the condition* (2.19) *hold. Then the problem* (2.17) *has at most one nontrivial nonnegative solution.*

Proof. Let (U_1, V_1, W_1) and (U_2, V_2, W_2) be any two nontrivial nonnegative solutions of (2.17). It is easily seen by the maximum principle that if any one of the components U_i, V_i, and W_i is not identically zero, then they are all positive in Ω. This implies that any nontrivial nonnegative solution of (2.17) is necessarily positive in Ω. Since the function at the right side of (2.17) is quasimonotone nondecreasing for $(U, V, W) \geqslant (0, 0, 0)$, and for any positive constants M_0, M_1, and M_2 satisfying (2.12), $(\tilde{U}_s, \tilde{V}_s, \tilde{W}_s) = (M_0, M_1, M_2)$ is a positive upper solution, it follows that there exists a maximal solution $(\bar{U}_s, \bar{V}_s, \bar{W}_s)$ such that $(0, 0, 0) \leqslant$

$(\bar{U}_s, \bar{V}_s, \bar{W}_s) \leqslant (M_0, M_1, M_2)$ (cf. [7, 8]). By choosing $(M_0, M_1, M_2) \geqslant (U_i, V_i, W_i)$ and considering (U_i, V_i, W_i) as a lower solution, the maximal property of $(\bar{U}_s, \bar{V}_s, \bar{W}_s)$ ensures that

$$(\bar{U}_s, \bar{V}_s, \bar{W}_s) \geqslant (U_i, V_i, W_i) \qquad \text{for} \quad i = 1, 2. \tag{2.21}$$

It is clear that if either $d_2 = 0$ or $b_3 = 0$, then $\bar{V}_s = 0$ and the inequality (2.19) is reduced to $a_1 > \lambda_o$. Since for $a_1 > \lambda_o$ the scalar boundary value problem

$$-\nabla^2 \bar{U} = \bar{U}(a_1 - b_1 \bar{U}), \qquad B\bar{U} = 0$$

has a unique positive solution $\bar{U}$, it follows that $U_1 = U_2 = \bar{U}$ when $\bar{V}_s = 0$. This implies that $V_1 = V_2 = 0$ and $W_1 = W_2$ which shows the uniqueness result when $b_3 d_2 = 0$. Assume $b_3 d_2 \neq 0$. By (2.17) and Green's identity,

$$\begin{aligned}
0 &= \int_\Omega (\bar{U}_s \nabla^2 U_i - U_i \nabla^2 \bar{U}_s)\, dx \\
&= \int_\Omega [U_i \bar{U}_s(-b_1(\bar{U}_s - U_i)) + q_1(U_i \bar{V}_s - V_i \bar{U}_s)]\, dx \\
0 &= \int_\Omega (\bar{V}_s \nabla^2 V_i - V_i \nabla^2 \bar{V}_s)\, dx = \int_\Omega d_2(\bar{W}_s V_i - W_i \bar{V}_s)\, dx \\
0 &= \int_\Omega (\bar{V}_s \nabla^2 W_i - W_i \nabla^2 \bar{V}_s)\, dx \\
&= \int_\Omega (d_2 \bar{W}_s W_i - q_2 \bar{V}_s W_i - b_3 \bar{V}_s U_i + d_3 \bar{V}_s W_i)\, dx \\
0 &= \int_\Omega (\bar{W}_s \nabla^2 V_i - V_i \nabla^2 \bar{W}_s)\, dx \\
&= \int_\Omega (-d_2 \bar{W}_s W_i + q_2 \bar{W}_s V_i + b_3 V_i \bar{U}_s - d_3 V_i \bar{W}_s)\, dx,
\end{aligned} \tag{2.22}$$

where $i = 1, 2$. Addition of the last two equations gives

$$0 = \int_\Omega [q_2(\bar{W}_s V_i - \bar{V}_s W_i) + b_3(V_i \bar{U}_s - \bar{V}_s U_i) + d_3(\bar{V}_s W_i - V_i \bar{W}_s)]\, dx.$$

In view of the second equation in (2.22), the above relation is reduced to

$$\int_\Omega (V_i \bar{U}_s - \bar{V}_s U_i)\, dx = 0.$$

Using this relation in the first equation in (2.22) yields

$$b_1 \int_\Omega U_i \bar{U}_s(\bar{U}_s - U_u)\, dx = 0. \tag{2.23}$$

It follows from the positive property of U_i and $\bar{U}_s$ that $\bar{U}_s = U_i$, and by (2.17), $\bar{V}_s = V_i$ and $\bar{W}_s = W_i$ for $i = 1, 2$. This shows that $(U_1, V_2, W_3) = (U_2, V_2, W_2)$ which proves the lemma. ∎

As a consequence of Lemma 2.3 we show that the set S in (2.20) is a global attractor of the problem (1.1)–(1.3).

THEOREM 2.4. *Let the condition* (2.19) *hold and let* (U_s, V_s, W_s) *be the positive solution of* (2.17). *Then for any* $(u_o, v_o, w_o) \geq (0, 0, 0)$, *the solution* (u, v, w) *of* (1.1)–(1.3) *satisfies the relation*

$$\overline{\lim_{t \to \infty}}\, (u(t, x), v(t, x), w(t, x)) \leq (U_s, V_s, W_s). \tag{2.24}$$

Proof. Let (U, V, W) be the nonnegative solution of (2.10), (2.11) where (M_0, M_1, M_2) satisfies (2.12) and the relation

$$(M_0, M_1, M_2) \geq \max\{(\|U_s\|, \|V_s\|, \|W_s\|), (\|u_o\|, \|v_o\|, \|w_o\|)\}.$$

Using the same argument as in the proof of Theorem 2.3 and the uniqueness property of (U_s, V_s, W_s), the solution (U, V, W) is monotone nonincreasing in t and converges to (U_s, V_s, W_s) as $t \to \infty$. Therefore, by Theorem 2.2,

$$\overline{\lim_{t \to \infty}}\, (u, v, w) \leq \lim_{t \to \infty} (U, V, W) = (U_s, V_s, W_s)$$

which gives the relation (2.24). ∎

3. Positive Steady-State Solutions

In view of Theorem 2.3, a necessary condition for the existence of a nontrivial-nonnegative solution (u_s, v_s, w_s) to (1.4) is (2.19), that is,

$$(\lambda_o - a_1)(\lambda_o + d_3)(\lambda_o + q_2) < b_3 d_2 q_1. \tag{3.1}$$

Clearly, $u_s \not\equiv 0$, for otherwise, $v_s = 0$ and $w_s = 0$, which is absurd. Knowing $u_s \not\equiv 0$, the maximum principle implies that $w_s > 0$ in Ω and consequently $v_s > 0$, $u_s > 0$ in Ω. This shows that condition (3.1) is a necessary condition for the existence of a positive solution. In this section we show that condition (3.1) is also sufficient for the existence of a positive solution. To this end, we first consider a_1 as a parameter and establish an existence result in relation to a_1. This will be achieved by a bifurcation theorem from a simple eigenvalue of the corresponding operator L which is defined by

$$L\mathbf{u} = \nabla^2\mathbf{u} - \sigma\mathbf{u} \qquad (\mathbf{u} \in X), \tag{3.2}$$

where $\mathbf{u} = (u, v, w)$, σ is a positive constant satisfying $\sigma \leqslant \min\{q_2, d_3\}$, and

$$X = \{(\eta_1, \eta_2, \eta_3); \eta_i \in C^2(\Omega) \text{ and } B\eta_i = 0, i = 1, 2, 3\}. \tag{3.3}$$

It is clear from $\sigma > 0$ that the operator L has a compact inverse L^{-1} on $R(L)$, the range of L. Set

$$\begin{gathered} \alpha = a_1 - a_0 + \sigma, \qquad \lambda^* = \lambda_o + \sigma, \qquad q = q_2 - \sigma, \qquad d = d_3 - \sigma, \\ a_o = \lambda^* - b_3 d_2 q_1/(\lambda^* + d)(\lambda^* + q). \end{gathered} \tag{3.4}$$

Then λ^* is the principle eigenvalue of the equation

$$-L\phi = \lambda\phi, \qquad B\phi = 0 \tag{3.5}$$

and condition (3.1) is equivalent to $\alpha > 0$. Let $F(\cdot; \alpha)$ be the Nemytskii operator given by

$$\begin{aligned} F(\mathbf{u}; \alpha) = (&-u(\alpha + a_o - b_1 u - c_1 v) - q_1 v, \\ &c_2 uv + qv - d_2 w, \ -b_3 u + dw). \end{aligned} \tag{3.6}$$

In terms of F, the steady-state problem (1.4) becomes

$$L\mathbf{u} - F(\mathbf{u}; \alpha) = \mathbf{o} \qquad (\mathbf{u} \in X). \tag{3.7}$$

It is clear that $(L - F) \in C^2(X \times \mathbb{R}; Y)$ where $Y = [C(\Omega)]^3$. Define operators $\mathscr{L}_1$, $\mathscr{L}_2$ by

$$\begin{aligned} \mathscr{L}_1 &= D_u(L - F)(\mathbf{o}; 0) \\ \mathscr{L}_2 &= D_\alpha D_u(L - F)(\mathbf{o}; 0), \end{aligned} \tag{3.8}$$

where $D_\mathbf{u}$, D_α are the Fréchet derivatives with respect to $\mathbf{u}$ and α, respectively. By direct computation

$$D_\mathbf{u}(L - F)(\mathbf{u}; \alpha)) = L - J(F)(\mathbf{u}; \alpha),$$

where $J(F)$ is the Jacobian of F which takes the form

$$J(F)(\mathbf{u};\alpha)=\begin{bmatrix} -\alpha-a_o+2b_1u+c_1v & c_1u-q_1 & 0 \\ c_2v & c_2u+q & -d_2 \\ -b_3 & 0 & d \end{bmatrix}. \tag{3.9}$$

This implies that

$$\mathscr{L}_2=-D_\alpha J(F)(\mathbf{o};0)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.10}$$

Denote by $N(\mathscr{L}_1)$ the null space of $\mathscr{L}_1$, and set

$$A_\alpha\equiv -J(F)(\mathbf{o};\alpha)=\begin{bmatrix} \alpha+a_o & q_1 & 0 \\ 0 & -q & d_2 \\ b_3 & 0 & -d \end{bmatrix}, \tag{3.11}$$

where α is considered as a parameter. Then

$$\mathscr{L}_1\equiv \mathscr{D}_\mathbf{u}(L-F)(\mathbf{o};0)\equiv L+A_o$$

and A_o is given by (3.11) with $\alpha=0$. The following bifurcation result from [12] will be needed.

PROPOSITION 3.1. *Let Z be any closed subspace of X such that $X=\mathscr{N}(\mathscr{L}_1)\oplus Z$ and let the following conditions hold*: (i) $\mathscr{N}(\mathscr{L}_1)$ *has dimension one*, (ii) $R(\mathscr{L}_1)$ *has co-dimension one, and* (iii) $\mathscr{L}_2\mathbf{u}_o\notin R(\mathscr{L}_1)$, *where* $\mathbf{u}_o$ *is any spanning vector of* $\mathscr{N}(\mathscr{L}_1)$. *Then there is a* $\delta>0$ *and a* C^1*-curve* $(\phi(s);\alpha(s))\colon(-\delta,\delta)\to Z\times\mathbb{R}$ *such that* $\phi(0)=\mathbf{o}$, $\alpha(0)=0$, *and*

$$(L-F)(s(\mathbf{u}+\phi(s));\alpha(s))=\mathbf{o} \qquad \text{for} \quad |s|<\delta.$$

Furthermore, there is a neighborhood of $(\mathbf{o},0)$ *such that any zero of* $(L-F)$ *either lies on this curve or is of the form* $(\mathbf{o},\alpha)$ *for any* α.

In view of Proposition 3.1, to show the existence of a nontrivial steady-state solution to (1.4), it suffices to verify the conditions (i)–(iii). We first determine a spanning vector $\mathbf{u}_o$ of $\mathscr{N}(\mathscr{L}_1)$. It is easily seen that the characteristic equation of A_o is given by

$$(a_o-\lambda)(\lambda+q)(\lambda+d)+b_3d_2q_1=0. \tag{3.12}$$

By (3.4), $\lambda=\lambda^*$ is an eigenvalue of A_o. Hence there exists an invertible matrix P such that $PA_oP^{-1}=\Lambda$ where Λ is the canonical form of A_o with

λ^* at its (1, 1) position. We show that the vector $\zeta^* \equiv P^{-1}\mathbf{z}_o$ is a spanning vector of $\mathcal{N}(\mathcal{L}_1)$, where $\mathbf{z}_o = (\phi_o, 0, 0)$ and ϕ_o is the positive eigenfunction of (3.5) corresponding to λ^*. The notation $P^{-1}\mathbf{z}_o$ is the usual product between matrices and column vectors.

LEMMA 3.1. *Let* $\mathbf{z}_o = (\phi_o, 0, 0)$ *and* $\zeta^* \equiv P^{-1}\mathbf{z}_o$. *Then* $\mathcal{N}(\mathcal{L}_1)$ *has dimension one and is spanned by* ζ^*.

Proof. Let $q' = \lambda^* + q$, $d' = \lambda^* + d$, and $s' = d_2 b_3 q_1/d'q'$. It can be shown by direct computation that the solutions of (3.12) are given by $\lambda_1 = \lambda^*$ and

$$\lambda_2, \lambda_3 = \lambda^* - \tfrac{1}{2}(s' + d' + q') \pm \tfrac{1}{2}[(s' + q' + d')^2 - 4(s'q' + s'd' + d'q')]^{1/2}.$$

Since $s'q' + s'd' + d'q' \geqslant 0$, it follows that $\operatorname{Re}(\lambda_2) < \operatorname{Re}(\lambda^*)$ and $\operatorname{Re}(\lambda_3) < \operatorname{Re}(\lambda^*)$. Hence λ_2 and λ_3 are not eigenvalues of the operator L. This ensures that the canonical form Λ of A_o can be written in the form

$$\Lambda \equiv PA_oP^{-1} = \begin{bmatrix} \lambda^* & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & \gamma & \lambda_3 \end{bmatrix}, \tag{3.13}$$

where $\gamma = 0$ if A_o is diagonalizable and $\gamma = 1$ if A_o is not diagonalizable. Let $\mathbf{z} \equiv (z_1, z_2, z_3)$ be any solution of the equation

$$L\mathbf{z} + \Lambda\mathbf{z} = 0. \tag{3.14}$$

Since λ_2 and λ_3 are not eigenvalues of L, $z_2 = 0$ and consequently $z_3 = 0$. Therefore, it is necessary that $\mathbf{z} = c\mathbf{z}_o \equiv c(\phi_o, 0, 0)$ for some constant c. Now, for any $\mathbf{u} \in \mathcal{N}(\mathcal{L}_1)$,

$$L\mathbf{u} + A_o\mathbf{u} = 0 \tag{3.15}$$

which implies that $\mathbf{z} = P\vec{\mathbf{u}}$ is a solution of (3.14). This shows that $\mathbf{u} = P^{-1}\mathbf{z}$ and $\mathbf{z} = c(\phi_o, 0, 0)$ for some constant c. Hence $\mathcal{N}(\mathcal{L}_1)$ is spanned by $P^{-1}\mathbf{z}_o$. ∎

We next verify the conditions (ii) and (iii) in Proposition 3.1.

LEMMA 3.2. *Let* $\mathcal{L}_1$, $\mathcal{L}_2$ *be defined by* (3.8). *Then* $R(\mathcal{L}_1)$ *has co-dimension one and* $\mathcal{L}_2\zeta^* \notin R(\mathcal{L}_1)$.

Proof. Let $\vec{\boldsymbol{\eta}} = (\eta_1, \eta_2, \eta_3) \in X$ and let η_0 and $\hat{\eta}$, be defined by

$$\eta_0 = \int_\Omega \eta_1 \phi_0 \, dx \Big/ \int_\Omega \phi_0 \, dx,$$

$$\hat{\eta}_1 = \eta_1 - \eta_0.$$

Then from the relation

$$\int_\Omega \hat{\eta}_1(x) \phi_o(x) \, dx = \int_\Omega \eta_1 \phi_o \, dx - \int_\Omega \eta_o \phi_o \, dx = 0$$

the function $\hat{\eta}_1$ is orthogonal to ϕ_o. By the Fredholm alternative theorem, the equation

$$Ly_1 + \lambda^* y_1 + \hat{\eta}_1 = 0$$

has a solution $y_1 \in C^2(\Omega)$. Since λ_2 and λ_3 are not eigenvalues of A_o, the equations

$$Ly_2 + \lambda_2 y_2 + \eta_2 = 0, \qquad Ly_3 + \lambda_3 y_3 + \gamma y_2 + \eta_3 = 0$$

possess solutions y_2 and y_3 in $C^2(\Omega)$. This implies that the vector $\mathbf{y} \equiv (y_1, y_2, y_3)$ is a solution of the equation

$$L\mathbf{y} + \Lambda \mathbf{y} = -(\hat{\eta}_1, \eta_2, \eta_3) = -\boldsymbol{\eta} + \eta_o \mathbf{e}_1$$

in X where $\vec{\mathbf{e}_1} = (1, 0, 0)$. Let $\mathbf{u} = P^{-1}\mathbf{y}$ and $\boldsymbol{\eta} = P\boldsymbol{\xi}$, where $\boldsymbol{\xi}$ is any vector in X. Then

$$\begin{aligned} L\mathbf{u} + A_o \mathbf{u} &= P^{-1}(L\mathbf{y} + \Lambda \mathbf{y}) = -P^{-1}\boldsymbol{\eta} + \eta_o P^{-1}\mathbf{e}_1 \\ &= -\boldsymbol{\xi} + \eta_o P^{-1}\mathbf{e}_1. \end{aligned}$$

This leads to

$$\boldsymbol{\xi} = -\mathscr{L}_1 \mathbf{u} + \eta_o P^{-1}\mathbf{e}_1 \in R(\mathscr{L}_1) + \operatorname{span}\{P^{-1}\mathbf{e}_1\}$$

which shows that $R(\mathscr{L}_1)$ has co-dimension either one or zero. To prove co-dim $R(\mathscr{L}_1) = 1$, it suffices to show that there exists $\mathbf{z} \in Y$ such that $\mathbf{z} \notin R(\mathscr{L}_1)$. We do this by showing that $\mathscr{L}_2 \boldsymbol{\zeta}^* \notin R(\mathscr{L}_1)$, where $\boldsymbol{\zeta}^* = P^{-1}\mathbf{z}_o$. Since $\mathbf{z}_o = (\phi_o, 0, 0)$, $\boldsymbol{\zeta}^* = (c_1 \phi_o, c_2 \phi_o, c_3 \phi_o)$ for some constants c_i, $i = 1, 2, 3$. In view of $\boldsymbol{\zeta}^* \in N(\mathscr{L}_1)$ and (3.15),

$$-c_3 L\phi_o = (b_3 c_1 - d c_3)\phi_o, \qquad -c_2 L\phi_o = (d_2 c_3 - q c_2)\phi_o.$$

Using $-L\phi_o = \lambda^* \phi_o$, the above relations yield

$$c_2 = c_1 b_3 d_2 / (\lambda^* + d)(\lambda^* + q), \qquad c_3 = c_1 b_3 / (\lambda^* + d). \tag{3.16}$$

This shows that $c_1 \neq 0$.

Let $P=(p_{ij})$. From (3.11) and (3.13), the first row of the relation $PA_o=\Lambda P$ gives

$$\begin{aligned}(a_o-\lambda^*)\,p_{11}+b_3\,p_{13}&=0\\ -q_1\,p_{11}+(q+\lambda^*)\,p_{12}&=0\\ d_2\,p_{12}-(d+\lambda^*)\,p_{13}&=0.\end{aligned}\tag{3.17}$$

It is obvious that $p_{11}\neq 0$, for otherwise, the second and third equations would imply $p_{12}=p_{13}=0$, which contradicts the nonsingular property of P. We show that $(c_1\phi_o, 0, 0)\notin R(\mathscr{L}_1)$. If this were not true, then there exists $\mathbf{u}\in X$ such that $\mathscr{L}_1\mathbf{u}=(c_1\phi_o, 0, 0)$, or equivalently, $L\vec{\mathbf{u}}+A_o\mathbf{u}=(c_1\phi_o, 0, 0)$. Let $\mathbf{z}=P\mathbf{u}$. Then $\mathbf{z}$ is a solution of the equation

$$L\mathbf{z}+\Lambda\mathbf{z}=c_1(P\mathbf{e}_1)\phi_o.$$

The first equation of the above system has the form

$$Lz_1+\lambda^*z_1=\theta_1,$$

where θ_1 is the first component of $c_1(P\mathbf{e}_1)\phi_o$ which is $c_1\,p_{11}\phi_o$. In view of $p_{11}\neq 0$ and $c_1\neq 0$, $\theta_1\neq 0$, and

$$\int_\Omega \theta_1\phi_o\,dx=p_{11}c_1\int_\Omega \phi_o^2(x)\,dx\neq 0.$$

Hence, by the Fredholm alternative theorem, (3.18) has no solution. This leads to a contradiction which shows that $\mathscr{L}_2\zeta^*\notin R(\mathscr{L}_1)$. ∎

The results of Lemmas 3.1 and 3.2 yield the following conclusion.

LEMMA 3.3. *There exists a positive number* $\delta>0$ *such that for all* $a_1\in(a_o-\sigma, \alpha_o-\sigma+\delta)$ *the problem* (1.4) *has a unique positive solution.*

Proof. Using the notations in (3.4), the problem (1.4) is reduced to (3.7) and the relation $a_1\in(a_o-\sigma,\, a_o-\sigma+\delta)$ is equivalent to $\alpha\in(0,\delta)$, In view of Proposition 3.1 and Lemmas 3.1 and 3.2, there exist a $\delta>0$ and a C^1-curve $(\phi(s),\alpha(s))\in X\times\mathbb{R}$ such that for each $s\in(-\delta,\delta)$, Eq. (3.7) with $\alpha=\alpha(s)$ has a solution $\mathbf{u}(s)\equiv s(\zeta^*+\phi(s))$, where $\phi(o)=\mathbf{o}$. We show that by a suitable choice of P, the solution $\mathbf{u}(s)$ is positive for $s\in(o,\delta)$. In view of $\phi(o)=\mathbf{o}$, it suffices to show that $\zeta^*>0$. Since $\zeta^*=(c_1\phi_o, c_2\phi_o, c_3\phi_o)$ and $c_1\neq 0$, the relation (3.16) implies that the constants c_1, c_2, and c_3 have the same sign. Hence ζ^* is positive when $c_1>0$. In case $c_1<0$, a replacement of P by $-P$ gives the same conclusion. Since by Theorem 2.3, the only nonnegative solution of (3.7) is the trivial solution when $\alpha\leqslant 0$, it follows that there exists a positive $\delta_1 < \delta$ such that $\alpha(s) > 0$ for $s \in (o, \delta_1)$. This proves the existence of a positive solution. The uniqueness of the positive solution follows from Proposition 3.1. ∎

The result of Lemma 3.3 ensures the existence of a positive steady-state solution to (3.7) when α is in a positive neighborhood of 0. Our main goal is to show that Eq. (3.7) has a positive solution for all $\alpha > 0$. Let $\mathscr{P}$ and $\mathscr{B}$ be the positive cone and the unit ball in the Banach space Y, respectively, and let $\dot{\mathscr{P}} = \mathscr{P} \setminus \{\mathbf{o}\}$ and $\mathscr{P}_\rho = \mathscr{P} \cap (\rho\mathscr{B})$, where ρ is a positive constant and $\rho\mathscr{B} \equiv \{z \in Y;\ \|\mathbf{z}\| \leqslant \rho\}$. In view of (3.9) and (3.11), $D_{\mathbf{u}}F(\mathbf{o}; \alpha) = -A_\alpha$ and

$$\mathscr{D}_{\mathbf{u}} L^{-1} F(\mathbf{o}; \alpha) = -L^{-1} A_\alpha. \tag{3.19}$$

Since L^{-1} is a compact operator in Y, Eq. (3.19) implies that $\mathscr{D}_{\mathbf{u}} L^{-1} F(\mathbf{o}; \alpha)$ is also a compact operator in Y. The following proposition from [1] gives some results about the fixed point index $i(L^{-1}F(\cdot, \alpha), \mathscr{P}_\rho)$, which will be needed for the proof of our existence theorem.

PROPOSITION 3.2. *If α is a positive constant such that*

(i) *every nonnegative eigenvector of $-L^{-1}A_\alpha$ has its eigenvalue not equal to one, and*

(ii) *there is a nonnegative eigenvector of $-L^{-1}A_\alpha$ whose corresponding eigenvalue is greater than one,*

then there exists a positive constant ρ_o such that for every $\rho \in (0, \rho_o]$, $i(L^{-1}F(\cdot, \alpha), \mathscr{P}_\rho) = 0$.

Using this result, we prove the following

LEMMA 3.4. *If $\alpha > 0$, then there exists a constant $\rho_o > 0$ such that for every $\rho \in (0, \rho_o]$, $i(L^{-1}F(\cdot; \alpha), \mathscr{P}_\rho) = 0$.*

Proof. In view of Proposition 3.2, it suffices to show the properties (i) and (ii). Assume by contradiction that there exists $\mathbf{u} \equiv (u_1, u_2, u_3) \in \dot{\mathscr{P}}$ such that $\mathbf{u} = -L^{-1}A_\alpha \mathbf{u}$. Since u_2, u_3 satisfy the relations

$$-Lu_2 = -qu_2 + d_2 u_3, \qquad -Lu_3 = b_3 u_1 - du_3$$

it follows that $u_1 \not\equiv 0$, for otherwise, $u_2 = u_3 = 0$ which contradicts $\mathbf{u} \in \dot{\mathscr{P}}$. This implies that the vector $\mathbf{z}^*$ defined by $\mathbf{z}^* \equiv (\alpha u_1, 0, 0)$ is in $\dot{\mathscr{P}}$, and by (3.11)

$$\mathbf{u} + L^{-1} A_o \mathbf{u} = -L^{-1} \mathbf{z}^*.$$

Since $-L^{-1}\mathbf{z}^*$ is also in $\dot{\mathscr{P}}$, the spectral radius of $-L^{-1}A_o$ is less than one and is the only eigenvalue which has a nonnegative eigenvector (cf. [1]).

However, since $\zeta^* \in \dot{\mathscr{P}}$ and satisfies $-L\zeta^* = A_o\zeta^*$, it follows that $\lambda = 1$ is an eigenvalue of $-L^{-1}A_o$ whose corresponding eigenvector is nonnegative. This leads to a contradiction, which proves the property (i).

To show the property (ii), we observe from (3.4) that the function

$$h(\lambda) \equiv (\lambda^{-1}(\alpha + a_o) - \lambda^*)(\lambda^* + d/\lambda)(\lambda^* + q/\lambda) + b_3 d_2 q_1/\lambda^3$$

is positive when $\lambda = 1$. Since $h(\lambda) \to (-\lambda^*)^3 < 0$ as $\lambda \to \infty$, there is a $\lambda_1 > 1$ such that $h(\lambda_1) = 0$. Define

$$\psi = (\phi_o, b_3 d_2 \phi_o/(d + \lambda_1 \lambda^*)(q + \lambda_1 \lambda^*), b_3 \phi_o/(d + \lambda_1 \lambda^*)).$$

It is easily seen by direct computation that ψ satisfies the relation $\lambda_1 \psi = -L^{-1}A_\alpha \psi$. This shows that ψ is a nonnegative eigenvector of $-L^{-1}A_\alpha$ whose eigenfalue is $\lambda_1 > 1$. Hence the property (ii) is proven. ∎

The next lemma gives the boundedness of the steady-state solution of (3.7) with respect to the parameter α.

LEMMA 3.5. *There is a positive increasing function* $M(\alpha)$ *in* $\mathbb{R}_+$ *such that any solution* $\mathbf{u} \equiv (u_1, u_2, u_3)$ *of* (3.7) *satisfies* $u_i(x) \leqslant M(\alpha)$ $(i = 1, 2, 3)$.

Proof. Let $\mathbf{u}(\cdot, \alpha) = (u_1(\cdot, \alpha), u_2(\cdot, \alpha), u_3(\cdot, \alpha))$ be a solution of (3.7) and let (U, V, W) be the solution of (2.10), (2.11) with $(M_o, M_1, M_2) \geqslant \mathbf{u}(x; \alpha)$ where $\alpha = a_1 - a_o + \sigma$ is considered as a parameter. Since Eq. (3.7) is equivalent to (1.4), the solution $\mathbf{u}(\cdot; \alpha)$ may be considered as a solution of (1.1)–(1.3) with $(u_o, v_o, w_o) = (u_1, u_2, u_3)$. By using $(\tilde{u}, \tilde{v}, \tilde{w}) = (U, V, W)$ and $(\hat{u}, \hat{v}, \hat{w}) = (0, 0, 0)$ in Theorem 2.1, the uniqueness property of the time-dependent solution ensures that

$$(0, 0, 0) \leqslant (u_1, u_2, u_3) \leqslant (U, V, W) \qquad \text{for} \quad t \geqslant 0, \quad x \in \bar{\Omega}.$$

From the proof of Theorem 2.3, (U, V, W) is nonincreasing and converges to a nonnegative steady-state solution $\mathbf{U}_s \equiv (U_s, V_s, W_s)$. This leads to $(u_1, u_2, u_3) \leqslant (U_s, V_s, W_s)$ in Ω. Define $M(\alpha) = \max\{\|U_s\|, \|V_s\|, \|W_s\|\}$. Then $u_i(x, \alpha) \leqslant M(\alpha)$ for $i = 1, 2, 3$. To complete the proof, it suffices to show that $M(\alpha)$ is an increasing function of α for $\alpha \geqslant 0$. Let $\alpha_2 > \alpha_1 \geqslant 0$ and let $\mathbf{U}_s(\cdot, \alpha)$ be the positive solution of (2.17) with $a_1 = \alpha + a_o - \sigma$. Then by the quasimonotone nondecreasing property of the reaction function, $\mathbf{U}_s(\cdot, \alpha_1)$ is a lower solution of (2.17) when $a_1 = \alpha_2 + a_o - \sigma$. Since (M_o, M_1, M_2) is an upper solution, the uniqueness property of the positive solution implies that $\mathbf{U}_s(\cdot, \alpha_2) \geqslant \mathbf{U}_s(\cdot, \alpha_1)$. This leads to $M(\alpha_2) \geqslant M(\alpha_1)$ which proves the lemma. ∎

We are now in a position to prove our main result for the steady-state problem (1.4).

THEOREM 3.1. *A necessary and sufficient condition for the existence of a positive solution to problem* (1.4) *is that condition* (3.1) *holds.*

Proof. The necessary part of the theorem is a consequence of Theorem 2.3. To prove the sufficient part, it suffices to show that under the condition $\alpha > 0$, Eq. (3.7) has a solution in $\dot{\mathscr{P}}$. Assume by contradiction that Eq. (3.7) has no solution in $\dot{\mathscr{P}}$ for some $\bar{\alpha} > 0$. Let $\bar{\rho} = M(\bar{\alpha}) + 1$ and $\partial\mathscr{P}_{\bar{\rho}} = \{\xi \in \mathscr{P};\ \|\xi\|_Y = \bar{\rho}\}$, and for each positive $\rho < \bar{\rho}$, define subsets Σ, S_ρ of $Y \times \mathbb{R}_+$ by

$$\Sigma = \{(\mathbf{u}; \alpha);\ \mathbf{u} \text{ is a solution of (3.7) with } \alpha \leqslant \bar{\alpha}\}$$

$$S_\rho = (\partial\mathscr{P}_\rho \times [o, \bar{\alpha}]) \cup (\overline{\mathscr{P}_{\bar{\rho}} \backslash \mathscr{P}_\rho} \times \{\bar{\alpha}\}),$$

where $M(\alpha)$ is given by Lemma 3.6. It is obvious that S_ρ is a closed bounded subset of $Y \times \mathbb{R}_+$. Moreover Σ is a closed subset of $Y \times \mathbb{R}_+$, for if $\{(\mathbf{u}_n, \alpha_n)\}$ is a sequence in Σ such that $(\mathbf{u}_n, \alpha_n) \to (\mathbf{u}, \alpha)$ in $Y \times \mathbb{R}_+$ as $n \to \infty$, then by the continuity of $L^{-1}F$ and $\mathbf{u}_n = L^{-1}F(\mathbf{u}_n; \alpha_n)$, the limit $(\mathbf{u}; \alpha)$ satisfies $\mathbf{u} = L^{-1}F(\mathbf{u}; \alpha)$. This shows that $(\mathbf{u}; \alpha) \in \Sigma$, and therefore Σ is closed. Let

$$K \equiv \max\{|F(\mathbf{u}; \alpha)|;\ \|\mathbf{u}\| \leqslant M(\bar{\alpha}),\ \alpha \leqslant \bar{\alpha}\}$$

and let $\mathscr{B}_K = K\mathscr{B}$ where $\mathscr{B}$ is the unit ball in Y. Then for any $(\mathbf{u}, \alpha) \in \Sigma$, the relation $\mathbf{u} = L^{-1}F(\mathbf{u}; \alpha)$ implies that $\mathbf{u} \in L^{-1}\mathscr{B}_K$. Hence Σ is a subset of $L^{-1}\mathscr{B}_K \times [o, \bar{\alpha}]$ which shows that it is a compact subset of $Y \times \mathbb{R}_+$. Now by Lemma 3.5 each $(\mathbf{u}, \alpha) \in \Sigma$ satisfies the relation $\|\mathbf{u}\| \leqslant M(\bar{\alpha}) < \bar{\rho}$. This leads to $\Sigma \cap (\partial\mathscr{P}_{\bar{\rho}} \times [o, \bar{\alpha}]) = \phi$ which together with $\Sigma \cap \{\mathscr{P} \times \{\bar{\alpha}\}\} = \{(\mathbf{o}; \bar{\alpha})\}$ ensures that

$$\Sigma \cap S_\rho = \phi \qquad \text{for each} \quad \rho \in (o, \bar{\rho}). \tag{3.20}$$

It follows from the above properties of Σ and S_ρ that there is a bounded open set $Q \subset Y \times \mathbb{R}_+$ such that $\Sigma \subset Q$ and $Q \cap S_\rho = \phi$.

Let $Z_\alpha \equiv \{\zeta; (\zeta, \alpha) \in Q\}$ be the slice of Q at α. By the homotopy invariance and the excision property of the fixed point index

$$\begin{aligned} i(L^{-1}F(\cdot\,; \bar{\alpha}), \mathscr{P}_\rho) &= i(L^{-1}F(\cdot\,; \bar{\alpha}), Z_{\bar{\alpha}}) \\ &= i(L^{-1}F(\cdot\,; o), Z_o) = i(L^{-1}F(\cdot\,; o), \mathscr{P}_1), \end{aligned} \tag{3.21}$$

where $\mathscr{P}_1 = \mathscr{P} \cap \mathscr{B}$ (cf. [1, Corollary 11.2 and Theorem 11.3]). Since by Theorem 2.3, the problem (1.4) has only the trivial solution when $\alpha = 0$, we see that $tL^{-1}F(\cdot\,; 0)$ has only the trivial fixed point in $\mathscr{P}$ for each $t \in [0, 1]$. This means that

$$tL^{-1}F(\mathbf{u}; 0) \neq \mathbf{u} \qquad \text{for all} \quad t \in [0, 1], \quad \mathbf{u} \in \partial\mathscr{P}_1.$$

It follows from the homotopy invariance property that the index $i(tL^{-1}F(\cdot;0), \mathscr{P}_1)$ is independent of $t \in [0, 1]$. Moreover by the normalization property (with $t=1$ and $t=0$)

$$i(L^{-1}F(\cdot, 0); \mathscr{P}_1 = i(0, \mathscr{P}_1) = 1.$$

This relation and (3.21) leads to

$$i(L^{-1}F(\cdot, \bar{\alpha}), \mathscr{P}_\rho) = 1 \qquad \text{for} \quad 0 < \rho < \bar{\rho}. \tag{3.22}$$

However, by Lemma 3.4 and $\bar{\alpha} > 0$, there exists a $\rho_o > 0$ such that

$$i(L^{-1}F(\cdot; \bar{\alpha}); \mathscr{P}_\rho) = 0 \qquad \text{for all} \quad \rho \leqslant \rho_o$$

we obtain a contradiction. This shows for all $\alpha \in (0, \infty)$, Eq. (3.7) has at least one solution in $\dot{\mathscr{P}}$. The equivalence between (1.4) and (3.7) and the fact that every solution of (3.7) in $\dot{\mathscr{P}}$ is necessarily positive in Ω imply that problem (1.4) has at least one positive solution. ∎

It is to be noted that since the problem (2.17) is a special case of (1.4) with $c_1 = c_2 = 0$ and the condition (3.1) is independent of c_1 and c_2 the existence of a positive solution in Theorem 3.1 is directly applicable to problem (2.17). This fact has been used in the proof of Theorem 2.4.

References

1. H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.* **18** (1976), 620–709.
2. R. J. Field and R. M. Noyes, Oscillations in chemical systems. Part 4. Limit cycle behavior in a model of a real chemical reaction, *J. Chem. Phys.* **60** (1974), 1877–1884.
3. R. J. Field and M. Burger, "Oscillations and Traveling Waves in Chemical Systems," Wiley–Interscience, New York, 1985.
4. P. C. Fife, On the question of existence and nature of homogeneous-center target patterns in the Belousov–Zhabotinskii reagent, *in* "Proceedings, Conf. on Analytical and Numerical Approaches to Asymptotic Problems in Analysis, Nijmegen, 1980."
5. R. G. Gibbs, Traveling waves in the Belousov–Zhabotinskii reaction, *SIAM J. Appl. Math.* **38** (1980), 422–444.
6. S. P. Hastings and J. D. Murray, The existence of oscillatory solutions in the Field–Noyes model of the Belousov–Zhabotinskii reaction, *SIAM J. Appl. Math.* **28** (1975), 678–688.
7. G. S. Ladde, V. Lakshmikantham, and A. S. Vatsala, "Monotone Iterative Techniques for Nonlinear Differential Equations," Pitman, Boston, 1985.
8. C. V. Pao, On nonlinear reaction-diffusion systems, *J. Math. Anal. Appl.* **87** (1982), 165–198.
9. C. V. Pao, Asymptotic behavior of solutions for the Belousov–Zhabotinskii reaction diffusion system, *J. Partial Differential Equations* **1** (1988), 61–66.

10. J. RINZEL AND W. C. TROY, A one-variable map analysis of bursting in the Belousov–Zhabotinskii reaction, *in* "Nonlinear Partial Differential Equations" (J. A. Smoller, Ed.), AMS Summer Res. Conf., Durham, NH, 1982.
11. W. RUAN, Bounded solutions for reaction-diffusion systems with nonlinear boundary conditions, *Nonlinear Anal.* **14** (12) (1990), 1051–1077.
12. J. A. SMOLLER, "Shock Waves and Reaction-Diffusion Equations," Springer-Verlag, New York/Berlin, 1982.
13. J. J. TYSON AND P. C. FIFE, Target patterns in a realistic model of the Belousov–Zhabotinskii reaction, *J. Chem. Phys.* **73** (1980), 2224–2237.
14. H. F. WEINBERGER, Invariant sets for weakly coupled parabolic and elliptic systems, *Rend. Mat. Appl. (6)* **8** (1975), 295–310.