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A continuity theorem for Stinespring's dilation

Dennis Kretschmann [a,b], Dirk Schlingemann [a,c,*], Reinhard F. Werner [a]

[a] *Institut für Mathematische Physik, Technische Universität Braunschweig, Mendelssohn Strasse 3, 38106 Braunschweig, Germany*

[b] *Quantum Information Theory Group, Dipartimento di Fisica A. Volta, Università di Pavia, via Bassi 6, 27100 Pavia, Italy*

[c] *ISI Foundation, Quantum Information Theory Unit, Viale S. Severo 65, 10133 Torino, Italy*

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Abstract

We show a continuity theorem for Stinespring's dilation: two completely positive maps between arbitrary C^*-algebras are close in cb-norm if and only if we can find corresponding dilations that are close in operator norm. The proof establishes the equivalence of the cb-norm distance and the Bures distance for completely positive maps. We briefly discuss applications to quantum information theory.

Keywords: Completely positive maps; Dilation theorems; Stinespring representation; Bures distance; Completely bounded norms; Quantum information theory

1. Introduction and overview

Completely positive maps (cp maps, for short) describe the dynamics of open quantum systems. Stinespring's dilation theorem [13,20] is the basic structure theorem for such maps. It states that any cp map $T:\mathcal{A}\to\mathcal{B}$ between two C^*-algebras $\mathcal{A}$ and $\mathcal{B}\subset\mathcal{B}(\mathcal{H})$ can be written as a concatenation of two basic cp maps: a *-homomorphism $\pi:\mathcal{A}\to\mathcal{B}(\mathcal{K})$ into a larger (*dilated*) algebra $\mathcal{B}(\mathcal{K})$ (the bounded operators on some Hilbert space $\mathcal{K}$), followed by a compression $V^*(\cdot)V$ into the range algebra $\mathcal{B}$:

* Corresponding author at: ISI Foundation, Quantum Information Theory Unit, Viale S. Severo 65, 10133 Torino, Italy.

E-mail addresses: d.kretschmann@tu-bs.de (D. Kretschmann), d.schlingemann@tu-bs.de (D. Schlingemann), r.werner@tu-bs.de (R.F. Werner).

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$$T(a) = V^*\pi(a)V \quad \forall a \in \mathcal{A}. \tag{1}$$

Stinespring's theorem provides a neat characterization of the set of permissible quantum operations and is also a most useful tool in the theory of open quantum systems and quantum information [12,22]. In a way, the increased system size is the price one has to pay for a simpler description of the map T in terms of just two basic operations.

A triple $(\pi, V, \mathcal{K})$ such that Eq. (1) holds is called a *Stinespring representation* for T. Stinespring's representation is unique up to partial isometries on the dilation spaces: given two representations $(\pi_1, V_1, \mathcal{K}_1)$ and $(\pi_2, V_2, \mathcal{K}_2)$ for a completely positive map $T : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, there exists a partial isometry $U : \mathcal{K}_1 \to \mathcal{K}_2$ such that

$$UV_1 = V_2, \qquad U^*V_2 = V_1 \quad \text{and} \quad U\pi_1(a) = \pi_2(a)U \tag{2}$$

for all $a \in \mathcal{A}$. A Stinespring representation $(\pi, V, \mathcal{K})$ of a cp map $T : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is called *minimal* if and only if the set $\{\pi(a)V|\psi\rangle \mid a \in \mathcal{A}, |\psi\rangle \in \mathcal{H}\}$ is dense in $\mathcal{K}$. If $(\pi_1, V_1, \mathcal{K}_1)$ and $(\pi_2, V_2, \mathcal{K}_2)$ are two minimal dilations for the cp map T, then U in Eq. (2) is unitary. Hence, any two minimal dilations are unitarily equivalent. In particular, the Stinespring dilation V for a unit-preserving completely positive map T is an isometry since then we have $T(\mathbb{1}) = V^*V = \mathbb{1}$.

Our contribution is a continuity theorem for Stinespring's dilation: two cp maps, T_1 and T_2, are close in cb-norm if and only if there exist corresponding dilations, V_1 and V_2, that are close in operator norm:

$$\frac{\|T_1 - T_2\|_{\text{cb}}}{\sqrt{\|T_1\|_{\text{cb}}} + \sqrt{\|T_2\|_{\text{cb}}}} \leqslant \inf_{V_1, V_2} \|V_1 - V_2\| \leqslant \sqrt{\|T_1 - T_2\|_{\text{cb}}}. \tag{3}$$

This result generalizes the uniqueness clause in Stinespring's theorem to cp maps that differ by a finite amount. As we have seen, uniqueness holds only up to partial isometries on the dilation spaces. So we cannot expect that any two dilations satisfy such a norm bound, only that they can be *chosen* in a suitable way. Hence the infimum in Eq. (3).

The *norm of complete boundedness* (*cb-norm,* for short) $\|\cdot\|_{\text{cb}}$ that appears in the continuity bound equation (3) is a stabilized version of the standard operator norm: For a linear map $R : \mathcal{A} \to \mathcal{B}$ between C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, we set $\|R\|_{\text{cb}} := \sup_{n \in \mathbb{N}} \|R \otimes \text{id}_n\|$, where id_n denotes the identity map on the $(n \times n)$ matrices, and $\|R\| := \sup_{\|a\| \leqslant 1} \|R(a)\|$. Maps R for which $\|R\|_{\text{cb}}$ is finite are usually called *completely bounded*. In particular, any completely positive map R is completely bounded, and we have $\|R\|_{\text{cb}} = \|R\| = \|R(\mathbb{1}_{\mathcal{A}})\| = \|V^*V\| = \|V\|^2$, where V is a Stinespring dilation for R. Obviously, $\|R\| \leqslant \|R\|_{\text{cb}}$ for every completely bounded map R. If the range algebra is Abelian, we even have equality: $\|R\| = \|R\|_{\text{cb}}$. An Abelian domain is still enough to ensure that positive maps are completely positive, but not sufficient to guarantee that bounded maps are completely bounded [13]. Quantum systems typically show a separation between stabilized and unstabilized norms [10,11]. Hence, Eq. (3) will in general fail to hold if the cb-norm $\|\cdot\|_{\text{cb}}$ is replaced by the standard operator norm $\|\cdot\|$.

The cb-norm is dual to the *diamond norm* which is frequently used in the realm of quantum computing [1]. Namely, if $R^* : \mathcal{B}^* \to \mathcal{A}^*$ denotes the dual map of R, then the identity $\|R\|_{\text{cb}} = \|R^*\|_\diamond$ holds. Obviously, by replacing the maps in Eq. (3) by its dual counterparts, the analogous bounds hold for the diamond norm as well.

The continuity bound equation (3) shows that the distance between two cp maps can equivalently be evaluated in terms of their dilations. We call this distance measure the *Bures distance*, since it generalizes Bures' metric [7] from positive functionals to general cp maps. In Section 2 we will formally introduce the Bures distance between general cp maps and state the continuity theorem. The remainder of the article is devoted to the proof of the theorem. Section 3 gives the lower bound on the Bures distance in terms of the cb-norm, which is elementary. The upper bound is established in Section 4; it relies on Bures' corresponding result for positive functionals [7] and on Ky Fan's minimax theorem. We first discuss cp maps with range $\mathcal{B}(\mathcal{H})$, and then extend the results to cp maps with injective range in Section 5. We conclude with a pair of appendices: In Appendix A we show that the Bures distance is indeed a metric on the set of completely positive maps, and in Appendix B for completeness we reproduce Bures's proof of the upper bound for positive functionals.

Building on earlier work by Belavkin et al. [6], the continuity theorem has appeared in [11] for the special case of unital cp maps (i. e., quantum channels) between finite-dimensional matrix algebras and has been applied to derive bounds on the tradeoff between information gain and disturbance in quantum physics, to establish a continuity bound for the no-broadcasting theorem, and to improve security bounds for quantum key distribution with faulty devices. A generalization to channels between direct sums of finite-dimensional matrix algebras has been used to derive a strengthened impossibility proof for quantum bit commitment [8].

2. Main results

The *Bures distance* evaluates the distance between two cp maps in terms of their dilations. We first discuss maps with range algebra $\mathcal{B} = \mathcal{B}(\mathcal{H})$, the bounded operators on some Hilbert space $\mathcal{H}$.

Definition 1 *(Bures distance).* Assume a C^*-algebra $\mathcal{A}$, a Hilbert space $\mathcal{H}$, and two cp maps $T_i : \mathcal{A} \to \mathcal{B}(\mathcal{H})$.

(1) The *π-distance between T_1 and T_2* is defined as

$$\beta_\pi(T_1, T_2) := \inf\bigl\{\|V_1 - V_2\| \bigm| V_i \in S(T_i, \pi)\bigr\}, \tag{4}$$

where the *π-fiber* $S(T, \pi)$ of a cp map $T : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is defined as the set of all operators $V : \mathcal{H} \to \mathcal{K}$ such that $(\pi, V, \mathcal{K})$ dilates T. If one or both of the fibers are empty, we set $\beta_\pi(T_1, T_2) := 2$.

(2) The *Bures distance between T_1 and T_2* is the smallest such π-distance

$$\beta(T_1, T_2) := \inf_\pi \beta_\pi(T_1, T_2), \tag{5}$$

with β_π as in Eq. (4).

For cp maps with one-dimensional range algebra, i.e. positive functionals, β coincides with Bures' distance function, as introduced in his seminal 1969 paper [7]. Our definition is the natural generalization to arbitrary cp maps; we hence choose the same name. The statement of the continuity theorem amounts to showing that the cb-norm and the Bures distance are equivalent distance measures for cp maps.

Theorem 1 *(Continuity of Stinespring's dilation). Let $\mathcal{A}$ be a C*-algebra, and let $T_i : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be completely positive maps such that $T_i \neq 0$ for at least one $i \in \{1, 2\}$. With $\beta(T_1, T_2)$ defined as in Eq.* (5)*, we then have the following inequality*:

$$\frac{\|T_1 - T_2\|_{\mathrm{cb}}}{\sqrt{\|T_1\|_{\mathrm{cb}}} + \sqrt{\|T_2\|_{\mathrm{cb}}}} \leqslant \beta(T_1, T_2) \leqslant \sqrt{\|T_1 - T_2\|_{\mathrm{cb}}}. \tag{6}$$

Moreover, there exist a common representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ for T_1 and T_2 and two corresponding Stinespring dilations $V_i : \mathcal{H} \to \mathcal{K}$ such that

$$\|V_1 - V_2\| = \beta_\pi(T_1, T_2) = \beta(T_1, T_2). \tag{7}$$

If $(\hat{\pi}_i, \hat{V}_i, \hat{\mathcal{K}}_i)$ is the minimal Stinespring dilation for the cp map T_i, we can choose $\pi := \hat{\pi}_1 \oplus \hat{\pi}_2$ as the common representation in Theorem 1. Even more is known for positive functionals: in that case the Bures distance can be evaluated in *any* common representation [2–4]. We do not yet know whether this result extends to general cp maps.

What about general range algebras $\mathcal{B} \neq \mathcal{B}(\mathcal{H})$? Since any C^*-algebra $\mathcal{B}$ can be faithfully embedded into a norm-closed self-adjoint algebra $\mathcal{B}(\mathcal{H})$ with a suitably chosen Hilbert space $\mathcal{H}$, it may appear natural to define the Bures distance for cp maps $T_i : \mathcal{A} \to \mathcal{B}$ in terms of the concatenated maps $\sigma \circ T_i$, with a faithful representation $\sigma : \mathcal{B} \to \mathcal{B}(\mathcal{H})$. However, $\beta(\sigma \circ T_1, \sigma \circ T_2)$ might possibly depend on the embedding representation σ. We instead choose an intrinsic definition of the Bures distance—and show that it reduces to Definition 1 if $\mathcal{B} = \mathcal{B}(\mathcal{H})$.

Definition 2 *(Bures distance for general range algebras).* Given two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ and two cp maps $T_i : \mathcal{A} \to \mathcal{B}$, the *Bures distance* is defined as

$$\beta(T_1, T_2) := \inf_{\hat{T}} \left\| \hat{T}_{11}(\mathbb{1}_\mathcal{A}) + \hat{T}_{22}(\mathbb{1}_\mathcal{A}) - \hat{T}_{12}(\mathbb{1}_\mathcal{A}) - \hat{T}_{21}(\mathbb{1}_\mathcal{A}) \right\|^{\frac{1}{2}}. \tag{8}$$

The infimum in Eq. (8) is taken over all completely positive extensions $\hat{T} : \mathcal{A} \to \mathcal{B} \otimes \mathcal{B}(\mathbb{C}^2) \simeq \mathcal{M}_2(\mathcal{B})$ of the form

$$\hat{T} \simeq \begin{pmatrix} \hat{T}_{11} & \hat{T}_{12} \\ \hat{T}_{21} & \hat{T}_{22} \end{pmatrix} \tag{9}$$

with completely bounded maps $\hat{T}_{ij} : \mathcal{A} \to \mathcal{B}$ satisfying $\hat{T}_{ii} = T_i$.

Introducing the cp map $\eta : \mathcal{B}(\mathbb{C}^2) \to \mathbb{C}$ by setting $\eta(x) := \mathrm{tr}\left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)x$, Eq. (8) can be rewritten more compactly,

$$\beta(T_1, T_2) = \inf_{\hat{T}} \left\| (\mathrm{id}_\mathcal{B} \otimes \eta) \circ \hat{T}(\mathbb{1}_A) \right\|^{\frac{1}{2}}. \tag{10}$$

While this definition of the Bures distance admittedly looks quite different from Definition 1, we will show in Section 5.1 that the definitions coincide if $\mathcal{B} = \mathcal{B}(\mathcal{H})$, and hence it is justified to use the same symbol for both.

With this definition of the Bures distance, Theorem 1 can now be generalized to cp maps with injective range algebras. Recall that a C^*-algebra $\mathcal{B}$ is called *injective* if for every C^*-algebra $\mathcal{A}$ and operator system $\mathcal{S}$ contained in $\mathcal{A}$, every completely positive map $R : \mathcal{S} \to \mathcal{B}$ can be extended to a completely positive map on all of $\mathcal{A}$ (cf. the work by Arveson [5] and [13, Chapter 7]). In fact, in order to show that $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$ is injective it is enough to find a completely positive map $P : \mathcal{B}(\mathcal{H}) \to \mathcal{B}$ such that $P(b) = b$ for all $b \in \mathcal{B}$. P is usually called a *completely positive conditional expectation.* Connes has shown that a von Neumann algebra $\mathcal{B}$ is injective if and only if it is *hyperfinite,* which means that $\mathcal{B}$ contains an ascending sequence of finite-dimensional subalgebras with dense union. We refer to Chapter XVI in Takesaki's textbook [21] for this and further equivalent conditions for injectivity of von Neumann algebras. A characterization of injective C^*-algebras has been given by Robertson et al. [15,16]. For cp maps with non-injective range, we only have a lower bound on $\beta(T_1, T_2)$, though we could always apply Theorem 1 to the concatenated maps $\sigma \circ T_1$ with some faithful embedding σ.

Theorem 2 *(Continuity for general range algebras). Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, and let $T_i : \mathcal{A} \to \mathcal{B}$ be completely positive. With $\beta(T_1, T_2)$ defined as in Definition 2 above, we have*

$$\frac{\|T_1 - T_2\|_{\mathrm{cb}}}{\sqrt{\|T_1\|_{\mathrm{cb}}} + \sqrt{\|T_2\|_{\mathrm{cb}}}} \leqslant \beta(T_1, T_2). \tag{11}$$

If in addition $\mathcal{B}$ is injective, we also have

$$\beta(T_1, T_2) \leqslant \sqrt{\|T_1 - T_2\|_{\mathrm{cb}}}, \tag{12}$$

and $\beta(T_1, T_2) = \beta(\sigma \circ T_1, \sigma \circ T_2)$ for any faithful representation $\sigma : \mathcal{B} \to \mathcal{B}(\mathcal{H})$.

The remainder of the article is devoted to the proof of Theorems 1 and 2. We start in Section 3 with a lower bound on the Bures distance in terms of the cb-norm.

3. Lower bound

A lower bound on the Bures distance $\beta(T_1, T_2)$ in terms of the cb-norm distance $\|T_1 - T_2\|_{\mathrm{cb}}$ easily follows from the standard properties of the operator norm.

Proposition 3 *(Lower bound). Let $\mathcal{A}$ be a C^*-algebra, and $T_1, T_2 : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be completely positive maps. We then have*

$$\|T_1 - T_2\|_{\mathrm{cb}} \leqslant \left(\sqrt{\|T_1\|_{\mathrm{cb}}} + \sqrt{\|T_2\|_{\mathrm{cb}}}\right)\beta(T_1, T_2). \tag{13}$$

Proof. Let π be a common representation for the cp maps T_i with corresponding dilations $(\pi, V_i, \mathcal{K})$. Given $n \in \mathbb{N}$ and $x \in \mathcal{A} \otimes \mathcal{B}(\mathbb{C}^n)$, we can then apply the triangle inequality to conclude that

$$\begin{aligned}
&\left\| T_1 \otimes \mathrm{id}_n(x) - T_2 \otimes \mathrm{id}_n(x) \right\| \\
&\quad = \left\| \left(V_1^* \otimes \mathbb{1}_n\right)(\pi \otimes \mathrm{id}_n)(x)\left(V_1 \otimes \mathbb{1}_n\right) - \left(V_2^* \otimes \mathbb{1}_n\right)(\pi \otimes \mathrm{id}_n)(x)\left(V_2 \otimes \mathbb{1}_n\right) \right\| \\
&\quad \leqslant \left\| \left(\left(V_1^* - V_2^*\right) \otimes \mathbb{1}_n\right)(\pi \otimes \mathrm{id}_n)(x)\left(V_1 \otimes \mathbb{1}_n\right) \right\|
\end{aligned}$$

$$
\begin{aligned}
&+ \left\| \left(V_2^* \otimes \mathbb{1}_n\right)(\pi \otimes \mathrm{id}_n)(x)\left((V_1 - V_2) \otimes \mathbb{1}_n\right) \right\| \\
&\leqslant \|V_1 - V_2\| \|V_1\| \|x\| + \|V_1 - V_2\| \|V_2\| \|x\| \\
&= \left(\sqrt{\|T_1\|_{\mathrm{cb}}} + \sqrt{\|T_2\|_{\mathrm{cb}}}\right) \|V_1 - V_2\| \|x\|,
\end{aligned} \tag{14}
$$

where we have used that the operator norm is preserved under both the adjoint operation and tensoring with the identity $\mathbb{1}_n$, as well as $\|\pi\|_{\mathrm{cb}} = 1$. The statement then immediately follows from the definition of the cb-norm and the Bures distance. □

4. Upper bound

In this section we will complement Proposition 3 with an upper bound on the Bures distance $\beta(T_1, T_2)$ in terms of the cb-norm $\|T_1 - T_2\|_{\mathrm{cb}}$. We start by investigating several alternative ways to evaluate the Bures distance—a useful tool for our proof but also a result of independent interest.

Given two cp maps $T_i : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$, we set

$$
\mathcal{N}_\pi(T_1, T_2) := \left\{ V_1^* V_2 \;\middle|\; V_i \in S(T_i, \pi) \right\} \subset \mathcal{B}(\mathcal{H}). \tag{15}
$$

The π-distance $\beta_\pi(T_1, T_2)$ can now be calculated in terms of $\mathcal{N}_\pi(T_1, T_2)$ as follows:

Lemma 4. *For cp maps $T_i : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$, we have*

$$
\beta_\pi^2(T_1, T_2) = \inf_{N \in \mathcal{N}_\pi(T_1, T_2)} \sup_{\varrho \in \mathcal{B}_{*,1}^+(\mathcal{H})} \left\{ \operatorname{tr} \varrho T_1(\mathbb{1}_\mathcal{A}) + \operatorname{tr} \varrho T_2(\mathbb{1}_\mathcal{A}) - 2 \operatorname{Re}(\operatorname{tr} \varrho N) \right\}, \tag{16}
$$

where $\mathcal{B}_{*,1}^+(\mathcal{H})$ denotes the positive trace class operators of unit trace on the Hilbert space $\mathcal{H}$.

Proof. The map $x \mapsto \operatorname{tr}((\cdot)x)$ defines an isometric isomorphism from $\mathcal{B}(\mathcal{H})$ to the normalized trace class operators $\mathcal{B}_{*,1}(\mathcal{H})$ (cf. Section VI.6 in [14]). Since in addition $(V_1 - V_2)^*(V_1 - V_2)$ is positive, we can write

$$
\begin{aligned}
\|V_1 - V_2\|^2 &= \left\| (V_1 - V_2)^* (V_1 - V_2) \right\| \\
&= \sup_{\varrho \in \mathcal{B}_{*,1}^+(\mathcal{H})} \operatorname{tr} \varrho (V_1 - V_2)^*(V_1 - V_2) \\
&= \sup_{\varrho \in \mathcal{B}_{*,1}^+(\mathcal{H})} \left\{ \operatorname{tr} \varrho T_1(\mathbb{1}_\mathcal{A}) + \operatorname{tr} \varrho T_2(\mathbb{1}_\mathcal{A}) - 2 \operatorname{Re}\left(\operatorname{tr} \varrho V_1^* V_2\right) \right\}
\end{aligned} \tag{17}
$$

for $V_i \in S(T_i, \pi)$ and any given representation π. The result then immediately follows from the definition of $\beta_\pi(T_1, T_2)$ in Eq. (4) and $\mathcal{N}_\pi(T_1, T_2)$ in Eq. (15). □

The following lemma allows to replace the infimum over representations π and corresponding $N \in \mathcal{N}_\pi$ in Lemma 4 with an infimum over intertwiners $W : \mathcal{K}_2 \to \mathcal{K}_1$ between any two fixed Stinespring representations. As advertised in Section 2, we will also show how to find a common representation π such that $\beta(T_1, T_2) = \beta_\pi(T_1, T_2)$.

Lemma 5 *(Evaluation of the Bures distance). Let $\mathcal{A}$ be a C^*-algebra, $\mathcal{H}$ a Hilbert space, and $T_1, T_2 : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be two completely positive maps.*

(1) *Assuming Stinespring dilations $(\pi_i, V_i, \mathcal{K}_i)$ for T_i, we define*

$$\mathcal{M}(T_1, T_2) := \{ V_1^* W V_2 \mid W\pi_2(a) = \pi_1(a)W \ \forall a \in \mathcal{A},\ \|W\| \leqslant 1 \}. \tag{18}$$

The set $\mathcal{M}(T_1, T_2) \subset \mathcal{B}(\mathcal{H})$ depends only on the cp maps T_i, not on the dilations $(\pi_i, V_i, \mathcal{K}_i)$.

(2) *The set $\mathcal{M}(T_1, T_2)$ can be represented alternatively as*

$$\mathcal{M}(T_1, T_2) = \bigcup_{\pi} \mathcal{N}_\pi(T_1, T_2) =: \mathcal{N}(T_1, T_2), \tag{19}$$

where the union is over all representations π admitting a common Stinespring representation for T_1 and T_2, and $\mathcal{N}_\pi(T_1, T_2)$ is defined in Eq. (15).

(3) *There exists a representation π such that $\beta(T_1, T_2) = \beta_\pi(T_1, T_2)$. We can choose $\pi := \hat{\pi}_1 \oplus \hat{\pi}_2$, where $\hat{\pi}_i$ is a minimal representation for T_i.*

Proof. (1) For the first part, our strategy is to show that $\mathcal{M}(T_1, T_2)$, defined via some dilations $(\pi_i, V_i, \mathcal{K}_i)$, coincides with $\hat{\mathcal{M}}(T_1, T_2)$ defined via the minimal dilations $(\hat{\pi}_i, \hat{V}_i, \hat{\mathcal{K}}_i)$. Given two dilations $(\pi_i, V_i, \mathcal{K}_i)$ for T_1 and T_2, respectively, we know from the uniqueness clause in Stinespring's theorem that there exist isometries $U_i : \hat{\mathcal{K}}_i \to \mathcal{K}_i$ such that $U_i \hat{V}_i = V_i$ and $U_i^* V_i = \hat{V}_i$. Since $U_i U_i^*$ is a projector onto the closed linear span of $\{\pi_i(a) V_i |\psi\rangle\}$, we have $U_i U_i^* V_i = V_i$, and hence

$$V_1^* W V_2 = V_1^* U_1 U_1^* W U_2 U_2^* V_2 = \hat{V}_1^* U_1^* W U_2 \hat{V}_2 = \hat{V}_1^* \hat{W} \hat{V}_2 \tag{20}$$

for all $W : \mathcal{K}_2 \to \mathcal{K}_1$, where we have set $\hat{W} := U_1^* W U_2 : \hat{\mathcal{K}}_2 \to \hat{\mathcal{K}}_1$. The intertwining relations $U_i \hat{\pi}_i(a) = \pi_i(a) U_i$ and $W\pi_2(a) = \pi_1(a) W$ imply that

$$\begin{aligned} \hat{W} \hat{\pi}_2(a) &= U_1^* W U_2 \hat{\pi}_2(a) \\ &= U_1^* W \pi_2(a) U_2 \\ &= U_1^* \pi_1(a) W U_2 \\ &= \hat{\pi}_1(a) U_1^* W U_2 \\ &= \hat{\pi}_1(a) \hat{W} \end{aligned} \tag{21}$$

for all $a \in \mathcal{A}$. Moreover, $\|\hat{W}\| = \|U_1^* W U_2\| \leqslant \|W\| \leqslant 1$, since the U_i are isometric. Hence, $\mathcal{M}(T_1, T_2) \subset \hat{\mathcal{M}}(T_1, T_2)$. The converse is completely analogous, starting with $\hat{W}$ and setting $W := U_1 \hat{W} U_2^*$.

(2) In order to show that $\mathcal{M}(T_1, T_2) \subset \mathcal{N}(T_1, T_2)$, it is sufficient to find a common representation π such that $\mathcal{M}(T_1, T_2) \subset \mathcal{N}_\pi(T_1, T_2)$. Since $\mathcal{M}(T_1, T_2)$ is independent of the dilations according to part (1), we can assume it to be defined via the minimal dilations $(\hat{\pi}_i, \hat{V}_i, \hat{\mathcal{K}}_i)$. Given $\hat{W} : \hat{\mathcal{K}}_2 \to \hat{\mathcal{K}}_1$ such that $\|\hat{W}\| \leqslant 1$, we define the bounded operators $V_i : \mathcal{H} \to \hat{\mathcal{K}}_1 \oplus \hat{\mathcal{K}}_2$ by setting

$$V_1|\psi\rangle := \hat{V}_1|\psi\rangle \oplus 0, \tag{22}$$

$$V_2|\psi\rangle := \hat{W}\hat{V}_2|\psi\rangle \oplus \sqrt{\mathbb{1}_{\hat{\mathcal{K}}_2} - \hat{W}^*\hat{W}}\,\hat{V}_2|\psi\rangle. \tag{23}$$

Making use of the intertwining relation $\hat{W}\hat{\pi}_2(a) = \hat{\pi}_1(a)\hat{W}$, it is then straightforward to verify that $\hat{\pi}_1 \oplus \hat{\pi}_2$ is indeed a common representation for the cp maps T_1 and T_2, with Stinespring dilations $(\hat{\pi}_1 \oplus \hat{\pi}_2, V_i, \hat{\mathcal{K}}_1 \oplus \hat{\mathcal{K}}_2)$. Moreover, $\hat{V}_1^*\hat{W}\hat{V}_2 = V_1^*V_2 \in \mathcal{N}_{\hat{\pi}_1\oplus\hat{\pi}_2}(T_1, T_2) \subset \mathcal{N}(T_1, T_2)$, as suggested. In particular, the direct sum construction shows that we can always find a common representation for the cp maps T_i, and hence $\mathcal{N}(T_1, T_2)$ is always non-empty. For the converse implication, $\mathcal{N}(T_1, T_2) \subset \mathcal{M}(T_1, T_2)$, let π be any such common representation and $V_1^*V_2 \in \mathcal{N}_\pi(T_1, T_2)$. Defining $\mathcal{M}(T_1, T_2)$ via the dilations $(\pi, V_i, \mathcal{K})$ and choosing $W = \mathbb{1}_{\mathcal{K}}$, we have $\mathcal{N}_\pi(T_1, T_2) \subset \mathcal{M}(T_1, T_2)$.

(3) From the proof of part (2) we have $\mathcal{N}_\pi(T_1, T_2) \subset \mathcal{M}(T_1, T_2) \subset \mathcal{N}_{\hat{\pi}_1\oplus\hat{\pi}_2}(T_1, T_2)$ for any common representation π. We can then immediately conclude from Lemma 4 that

$$\beta_{\hat{\pi}_1\oplus\hat{\pi}_2}(T_1, T_2) \leqslant \beta_\pi(T_1, T_2), \tag{24}$$

implying $\beta(T_1, T_2) = \beta_{\hat{\pi}_1\oplus\hat{\pi}_2}(T_1, T_2)$. Consequently, the Bures distance can always be evaluated in the direct sum representation of the minimal representations. □

Lemmas 4 and 5 can now be applied to derive the desired upper bound on the Bures distance in terms of the cb-norm. For the special case of positive functionals, this result was obtained by Bures [7] (cf. Proposition 11 in Appendix B), and will now be lifted to cp maps with the help of Ky Fan's minimax theorem [9].

Proposition 6 *(Upper bound). Let $\mathcal{A}$ be a C^*-algebra, and $T_1, T_2 : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be completely positive maps. We can then find a common representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ and corresponding dilations $(\pi, V_i, \mathcal{K})$ for T_i such that*

$$\|V_1 - V_2\| = \beta_\pi(T_1, T_2) = \beta(T_1, T_2) \leqslant \sqrt{\|T_1 - T_2\|_{\text{cb}}}. \tag{25}$$

Proof. Spelling out $\beta_\pi(T_1, T_2)$ as in Lemma 4 and then making use of the relation $\mathcal{N}(T_1, T_2) = \mathcal{M}(T_1, T_2)$ from Lemma 5, we have

$$\begin{aligned}
\beta^2(T_1, T_2) &= \inf_\pi \beta_\pi^2(T_1, T_2) \\
&= \inf_{N\in\mathcal{N}(T_1,T_2)} \sup_{\varrho\in\mathcal{B}^+_{*,1}(\mathcal{H})} \bigl\{\operatorname{tr}\varrho T_1(\mathbb{1}_{\mathcal{A}}) + \operatorname{tr}\varrho T_2(\mathbb{1}_{\mathcal{A}}) - 2\operatorname{Re}(\operatorname{tr}\varrho N)\bigr\} \\
&= \inf_{M\in\mathcal{M}(T_1,T_2)} \sup_{\varrho\in\mathcal{B}^+_{*,1}(\mathcal{H})} \bigl\{\operatorname{tr}\varrho T_1(\mathbb{1}_{\mathcal{A}}) + \operatorname{tr}\varrho T_2(\mathbb{1}_{\mathcal{A}}) - 2\operatorname{Re}(\operatorname{tr}\varrho M)\bigr\} \\
&= \inf_{\|W\|\leqslant 1} \sup_{\varrho\in\mathcal{B}^+_{*,1}(\mathcal{H})} \bigl\{\operatorname{tr}\varrho T_1(\mathbb{1}_{\mathcal{A}}) + \operatorname{tr}\varrho T_2(\mathbb{1}_{\mathcal{A}}) - 2\operatorname{Re}\bigl(\operatorname{tr}\varrho V_1^*WV_2\bigr)\bigr\}
\end{aligned} \tag{26}$$

with $W \in \mathcal{B}(\mathcal{K}_2, \mathcal{K}_1)$, where $(\pi_1, V_1, \mathcal{K}_1)$ and $(\pi_2, V_2, \mathcal{K}_2)$ are now any two *fixed* dilations for the cp maps T_1 and T_2, respectively. The target functional in Eq. (26) is affine in both inputs. Since the state $\varrho \in \mathcal{B}^+_{*,1}(\mathcal{H})$ is trace-class, so is $V_2\varrho V_1^*$, and hence the functional is weakly continuous in W. Moreover, we know from the Banach–Alaoglu theorem (cf. Section IV.5 in [14]) that the unit ball $\|W\| \leqslant 1$ is weakly compact, and hence the infimum is attained. In addition, both optimizations in Eq. (26) are performed over convex sets. Under these conditions, Ky Fan's minimax theorem [9,19] guarantees that the order of the optimizations in Eq. (26) can be interchanged to yield

$$\begin{aligned}\beta^2(T_1,T_2) &= \min_{\|W\|\leqslant 1}\sup_{\varrho\in\mathcal{B}^+_{*,1}(\mathcal{H})}\bigl\{\operatorname{tr}\varrho T_1(\mathbb{1}_{\mathcal{A}}) + \operatorname{tr}\varrho T_2(\mathbb{1}_{\mathcal{A}}) - 2\operatorname{Re}\bigl(\operatorname{tr}\varrho V_1^* W V_2\bigr)\bigr\}\\ &= \sup_{\varrho\in\mathcal{B}^+_{*,1}(\mathcal{H})}\min_{\|W\|\leqslant 1}\bigl\{\operatorname{tr}\varrho T_1(\mathbb{1}_{\mathcal{A}}) + \operatorname{tr}\varrho T_2(\mathbb{1}_{\mathcal{A}}) - 2\operatorname{Re}\bigl(\operatorname{tr}\varrho V_1^* W V_2\bigr)\bigr\}\\ &= \sup_{|\psi\rangle\in\mathcal{H}\otimes\mathcal{H}}\min_{\|W\|\leqslant 1}\bigl\{\langle\psi|T_1(\mathbb{1}_{\mathcal{A}})\otimes\mathbb{1}_{\mathcal{H}}|\psi\rangle + \langle\psi|T_2(\mathbb{1}_{\mathcal{A}})\otimes\mathbb{1}_{\mathcal{H}}|\psi\rangle\\ &\quad - 2\operatorname{Re}\bigl(\langle\psi|\bigl(V_1^*\otimes\mathbb{1}_{\mathcal{H}}\bigr)(W\otimes\mathbb{1}_{\mathcal{H}})\bigl(V_2\otimes\mathbb{1}_{\mathcal{H}}\bigr)|\psi\rangle\bigr)\bigr\}.\end{aligned} \tag{27}$$

In the last step of Eq. (27), we have replaced the supremum over the normal states $\varrho \in \mathcal{B}^+_{*,1}(\mathcal{H})$ by a supremum over their respective purifications. Note that $(\pi_i \otimes \mathrm{id}_{\mathcal{B}(\mathcal{H})}, (V_i \otimes \mathbb{1}_{\mathcal{H}})|\psi\rangle, \mathcal{K}_i \otimes \mathcal{H})$ is a Stinespring dilation for the positive functional $\psi \circ (T_i \otimes \mathrm{id}_{\mathcal{B}(\mathcal{H})})$, and that all operators $\tilde{W} : \mathcal{K}_2 \otimes \mathcal{H} \to \mathcal{K}_1 \otimes \mathcal{H}$ that intertwine the representations $\pi_1 \otimes \mathrm{id}_{\mathcal{B}(\mathcal{H})}$ and $\pi_2 \otimes \mathrm{id}_{\mathcal{B}(\mathcal{H})}$ are of the form $\tilde{W} = W \otimes \mathbb{1}_{\mathcal{H}}$, with an intertwiner $W : \mathcal{K}_2 \to \mathcal{K}_1$. Lemma 5 therefore implies that the inner variation in Eq. (27) is just the Bures distance square $\beta^2(\psi \circ (T_1 \otimes \mathrm{id}_{\mathcal{B}(\mathcal{H})}), \psi \circ (T_2 \otimes \mathrm{id}_{\mathcal{B}(\mathcal{H})}))$. We can then apply Bures' bound for positive functionals from Proposition 11 to conclude that

$$\begin{aligned}\beta^2(T_1,T_2) &= \sup_{|\psi\rangle\in\mathcal{H}\otimes\mathcal{H}}\beta^2\bigl(\psi\circ(T_1\otimes\mathrm{id}_{\mathcal{B}(\mathcal{H})}), \psi\circ(T_2\otimes\mathrm{id}_{\mathcal{B}(\mathcal{H})})\bigr)\\ &\leqslant \sup_{|\psi\rangle\in\mathcal{H}\otimes\mathcal{H}}\bigl\|\psi\circ(T_1\otimes\mathrm{id}_{\mathcal{B}(\mathcal{H})}) - \psi\circ(T_2\otimes\mathrm{id}_{\mathcal{B}(\mathcal{H})})\bigr\|\\ &\leqslant \|T_1 - T_2\|_{\mathrm{cb}},\end{aligned} \tag{28}$$

which is the desired result. For the cb-norm bound in the last step we have used that the finite rank operators are dense in $\mathcal{B}_*(\mathcal{H})$. We have seen above that there exists an intertwiner $W : \mathcal{K}_2 \to \mathcal{K}_1$ which attains the infima in Eqs. (26) and (27). Lemma 5 then by construction yields a common representation π and corresponding dilations $(\pi, V_i, \mathcal{K})$ such that

$$\|V_1 - V_2\| = \beta(T_1,T_2) \leqslant \sqrt{\|T_1 - T_2\|_{\mathrm{cb}}}, \tag{29}$$

just as claimed. □

Theorem 1 now immediately follows by combining the bounds from Propositions 3 and 6.

5. Bures distance for general range algebras

So far our discussion has focused on channels with range algebra $\mathcal{B}(\mathcal{H})$. In this section we will investigate completely positive maps $T_i : \mathcal{A} \to \mathcal{B}$ with general range algebra $\mathcal{B}$. Our results are twofold: in Section 5.1 we will justify the intrinsic definition of the Bures distance $\beta(T_1, T_2)$ by showing that it indeed coincides with Definition 1 if $\mathcal{B} = \mathcal{B}(\mathcal{H})$. For general range algebras $\mathcal{B}$, we will then show in Section 5.3 that $\beta(T_1, T_2) \geqslant \beta(\sigma \circ T_1, \sigma \circ T_2)$ for any representation σ. If $\mathcal{B}$ is injective and σ is faithful, we even have equality, and hence the Bures distance does not depend on the details of the embedding and can then be shown to be completely equivalent to the cb-norm distance. For the proof we need a monotonicity result for the Bures distance, which we will present in Section 5.2.

5.1. Consistency

For the moment, we will denote the Bures distance for cp maps $T_i : \mathcal{A} \to \mathcal{B}$ with general range algebra $\mathcal{B}$, as introduced in Definition 2, by $\beta'(T_1, T_2)$. We will show in this section that indeed $\beta'(T_1, T_2) = \beta(T_1, T_2)$ if $\mathcal{B} = \mathcal{B}(\mathcal{H})$. Thus, Definition 2 is a consistent generalization of Definition 1 to general range algebras, and we may henceforth drop the prime.

Proposition 7. *Let $\mathcal{A}$ be a C^*-algebra, and let $T_i : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be completely positive. With $\beta(T_1, T_2)$ defined as in Definition* 1 *and $\beta'(T_1, T_2)$ defined as in Definition* 2*, we then have*

$$\beta(T_1, T_2) = \beta'(T_1, T_2). \tag{30}$$

Proof. We first show that $\beta(T_1, T_2) \leqslant \beta'(T_1, T_2)$. As in Definition 2, let $\hat{T} : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathbb{C}^2)$ be a completely positive extension of the cp maps T_i with Stinespring dilation $(\pi, V, \mathcal{K})$. Starting from $V : \mathcal{H} \otimes \mathbb{C}^2 \to \mathcal{K}$, for $i \in \{1, 2\}$ we define $V_i : \mathcal{H} \to \mathcal{K}$ by setting $V_i|\psi\rangle := V|\psi\rangle \otimes |i\rangle$. Hence, $\hat{T}_{ij}(a) = V_i^* \pi(a) V_j$ for all $a \in \mathcal{A}$. In particular, $(\pi, V_i, \mathcal{K})$ dilates T_i. We may then conclude from Definition 1 that

$$\begin{aligned}
\beta(T_1, T_2) &\leqslant \|V_1 - V_2\| \\
&= \left\| V_1^* V_1 + V_2^* V_2 - V_1^* V_2 - V_2^* V_1 \right\|^{\frac{1}{2}} \\
&= \left\| \left(\mathrm{id}_{\mathcal{B}(\mathcal{H})} \otimes \eta\right) \circ \hat{T}(\mathbb{1}_{\mathcal{A}}) \right\|^{\frac{1}{2}}
\end{aligned} \tag{31}$$

holds independently of $\hat{T}$, and hence $\beta(T_1, T_2) \leqslant \beta'(T_1, T_2)$ follows immediately from Definition 2.

Conversely, we assume a common representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ for the cp maps T_i with corresponding dilations $(\pi, V_i, \mathcal{K})$. We now set $V|\psi\rangle \otimes |i\rangle := V_i|\psi\rangle$. The linear map $V : \mathcal{H} \otimes \mathbb{C}^2 \to \mathcal{K}$ defines a completely positive extension $\hat{T}(a) = V^* \pi(a) V$ in the sense of Definition 1 with $\hat{T}_{ij}(a) = V_i^* \pi(a) V_j$ for all $a \in \mathcal{A}$. Hence,

$$\begin{aligned}
\beta'(T_1, T_2) &\leqslant \left\| \left(\mathrm{id}_{\mathcal{B}(\mathcal{H})} \otimes \eta\right) \circ \hat{T}(\mathbb{1}_{\mathcal{A}}) \right\|^{\frac{1}{2}} \\
&= \left\| V_1^* V_1 + V_2^* V_2 - V_1^* V_2 - V_2^* V_1 \right\|^{\frac{1}{2}} \\
&= \|V_1 - V_2\|,
\end{aligned} \tag{32}$$

implying that $\beta'(T_1, T_2) \leqslant \beta(T_1, T_2)$. □

5.2. Monotonicity of the Bures distance under cp maps

We will now show that the Bures distance $\beta(T_1, T_2)$ decreases under quantum operations, i.e., $\beta(S \circ T_1, S \circ T_2) \leqslant \beta(T_1, T_2)$ holds for all cp maps S with $\|S\|_{\mathrm{cb}} \leqslant 1$. Only Eq. (33) is needed in the proof of Theorem 2 below, but we include Eq. (34) for completeness.

Proposition 8 *(Monotonicity). Given three C^*-algebras $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{D}$ and cp maps $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ and $S : \mathcal{B} \to \mathcal{D}$, we have*

$$\beta(S \circ T_1, S \circ T_2) \leqslant \sqrt{\|S\|}\beta(T_1, T_2). \tag{33}$$

For cp maps T_i as above and $S : \mathcal{D} \to \mathcal{A}$ we have

$$\beta(T_1 \circ S, T_2 \circ S) \leqslant \sqrt{\|S\|}\beta(T_1, T_2). \tag{34}$$

Proof. This is straightforward. Starting with an extension $\hat{T} : \mathcal{A} \to \mathcal{B} \otimes \mathcal{B}(\mathbb{C}^2)$ for the cp maps T_i, $(S \otimes \mathrm{id}_2) \circ \hat{T}$ defines a completely positive extension for the maps $S \circ T_i$, and we have the estimate

$$\begin{aligned} \beta(S \circ T_1, S \circ T_2) &\leqslant \left\|(\mathrm{id}_{\mathcal{D}} \otimes \eta) \circ (S \otimes \mathrm{id}_2) \circ \hat{T}(\mathbb{1}_{\mathcal{A}})\right\|^{\frac{1}{2}} \\ &\leqslant \sqrt{\|S\|}\left\|(\mathrm{id}_{\mathcal{B}} \otimes \eta) \circ \hat{T}(\mathbb{1}_{\mathcal{A}})\right\|^{\frac{1}{2}}. \end{aligned} \tag{35}$$

Since Eq. (35) holds for all extensions $\hat{T}$, Eq. (33) is proven. The proof of Eq. (34) is completely analogous. □

5.3. Equivalence of Bures distance and cb-norm for injective range algebras

The following proposition shows that for cp maps $T_i : \mathcal{A} \to \mathcal{B}$ with injective range algebra $\mathcal{B}$, the Bures distance $\beta(T_1, T_2)$ may be evaluated in any faithful representation $\sigma : \mathcal{B} \to \mathcal{B}(\mathcal{H})$.

Proposition 9. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, and $T_1, T_2 : \mathcal{A} \to \mathcal{B}$ be completely positive maps. We then have*

$$\beta(T_1, T_2) \geqslant \beta(\sigma \circ T_1, \sigma \circ T_2) \tag{36}$$

for any representation $\sigma : \mathcal{B} \to \mathcal{B}(\mathcal{H})$. Moreover, if $\mathcal{B}$ is injective and the representation σ is faithful equality holds in Eq. (36)*.*

Proof. Since $\|\sigma\| = \|\sigma\|_{\mathrm{cb}} = 1$ for any representation σ, Eq. (36) is immediate from Proposition 8. For the converse inequality, assume that $\mathcal{B}$ is injective and $\sigma : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ is faithful. Let $\hat{T} : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathbb{C}^2)$ be a completely positive extension for the cp maps $\sigma \circ T_i : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Since $\mathcal{B}$ is injective, we can find a completely positive conditional expectation $P : \mathcal{B}(\mathcal{H}) \to \sigma(\mathcal{B})$ and then set $\hat{T}' := (\sigma^{-1} \circ P \otimes \mathrm{id}_2) \circ \hat{T}$. This defines a completely positive extension for the cp maps T_i, and from the definition of the Bures distance we then immediately have the estimate

$$\begin{aligned}\beta^2(T_1,T_2) &\leqslant \big\|T_1(\mathbb{1}_{\mathcal{A}})+T_2(\mathbb{1}_{\mathcal{A}})-\hat{T}'_{12}(\mathbb{1}_{\mathcal{A}})-\hat{T}'_{21}(\mathbb{1}_{\mathcal{A}})\big\| \\ &= \big\|\sigma^{-1}\circ P\circ\big(\sigma\circ T_1(\mathbb{1}_{\mathcal{A}})+\sigma\circ T_2(\mathbb{1}_{\mathcal{A}})-\hat{T}_{12}(\mathbb{1}_{\mathcal{A}})-\hat{T}_{21}(\mathbb{1}_{\mathcal{A}})\big)\big\| \\ &\leqslant \big\|\sigma\circ T_1(\mathbb{1}_{\mathcal{A}})+\sigma\circ T_2(\mathbb{1}_{\mathcal{A}})-\hat{T}_{12}(\mathbb{1}_{\mathcal{A}})-\hat{T}_{21}(\mathbb{1}_{\mathcal{A}})\big\|, \qquad (37)\end{aligned}$$

where in the last step we have used that both σ^{-1} and P are completely positive with norm $\leqslant 1$. Since Eq. (37) holds for all extensions of $\sigma\circ T_i$, we conclude that

$$\beta(T_1,T_2)\leqslant\beta(\sigma\circ T_1,\sigma\circ T_2) \qquad (38)$$

for any faithful representation σ, as suggested. □

With the help of Proposition 9, the proof of Theorem 2 can now be obtained directly from Theorem 1.

Proof of Theorem 2. Since the cb-norm is invariant under faithful representations, Eq. (11) immediately follows by choosing the representation σ to be faithful in Eq. (36) and applying the corresponding bound from Theorem 1. If in addition the range algebra $\mathcal{B}$ is injective, Eq. (12) follows in the same way from Proposition 9 and Theorem 1. □

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Appendix A. Properties of the Bures distance

In this section we will show that the Bures distance $\beta(T_1,T_2)$ defined in Eq. (5) indeed has all the properties of a distance measure.

Proposition 10 *(Bures distance). The functional* $(T_1,T_2)\mapsto\beta(T_1,T_2)$ *is a metric on the set of cp maps* $T_i:\mathcal{A}\to\mathcal{B}(\mathcal{H})$.

Proof. Positivity and symmetry are immediate from the definition of $\beta(T_1,T_2)$. Obviously, $\beta(T_1,T_1)=0$. Conversely, Proposition 3 shows that $\beta(T_1,T_2)=0$ entails $\|T_1-T_2\|_{\mathrm{cb}}=0$, and hence $T_1=T_2$. Thus, it only remains to establish the triangle inequality, $\beta(T_1,T_3)\leqslant\beta(T_1,T_2)+\beta(T_2,T_3)$ for all triples of cp maps T_i. To this end, let $(\pi,V_i,\mathcal{K})$ be dilations for the cp maps T_1 and T_2 with a common representation π. Further assume that $(\check{\pi},\check{V}_j,\check{\mathcal{K}})$ are dilations for the pair T_2,T_3 with a common representation $\check{\pi}$. As before, $(\hat{\pi}_i,\hat{V}_i,\hat{\mathcal{K}}_i)$ will denote the corresponding minimal dilations, with intertwiners $U_i:\hat{\mathcal{K}}_i\to\mathcal{K}$ for $i\in\{1,2\}$ and $\check{U}_j:\hat{\mathcal{K}}_j\to\check{\mathcal{K}}$ for $j\in\{2,3\}$. We now set

$$\tilde{V}_1|\psi\rangle := \sqrt{\mathbb{1}_{\hat{\mathcal{K}}_1} - U_1^* U_2 U_2^* U_1}\,\hat{V}_1|\psi\rangle \oplus U_2^* V_1|\psi\rangle \oplus 0, \tag{39}$$

$$\tilde{V}_2|\psi\rangle := 0 \oplus \hat{V}_2|\psi\rangle \oplus 0, \tag{40}$$

$$\tilde{V}_3|\psi\rangle := 0 \oplus \check{U}_2^* \check{V}_3|\psi\rangle \oplus \sqrt{\mathbb{1}_{\hat{\mathcal{K}}_3} - \check{U}_3^* \check{U}_2 \check{U}_2^* \check{U}_3}\,\hat{V}_3|\psi\rangle. \tag{41}$$

Obviously, $\tilde{V}_2 \in S(T_2, \hat{\pi}_1 \oplus \hat{\pi}_2 \oplus \hat{\pi}_3)$. Making use of the intertwining relations equation (2), we also have

$$\begin{aligned}
&\tilde{V}_1^* \big(\hat{\pi}_1(a) \oplus \hat{\pi}_2(a) \oplus \hat{\pi}_3(a)\big) \tilde{V}_1 \\
&= \hat{V}_1^* \sqrt{\mathbb{1}_{\hat{\mathcal{K}}_1} - U_1^* U_2 U_2^* U_1}\,\hat{\pi}_1(a) \sqrt{\mathbb{1}_{\hat{\mathcal{K}}_1} - U_1^* U_2 U_2^* U_1}\,\hat{V}_1 + V_1^* U_2 \hat{\pi}_2(a) U_2^* V_1 \\
&= V_1^* U_1 \hat{\pi}_1(a) \big(\mathbb{1}_{\hat{\mathcal{K}}_1} - U_1^* U_2 U_2^* U_1\big) U_1^* V_1 + V_1^* \pi(a) U_2 U_2^* V_1 \\
&= V_1^* U_1 U_1^* \pi(a) \big(\mathbb{1}_{\mathcal{K}} - U_2 U_2^*\big) U_1 U_1^* V_1 + V_1^* \pi(a) U_2 U_2^* V_1 \\
&= V_1^* \pi(a) \big(\mathbb{1}_{\mathcal{K}} - U_2 U_2^*\big) V_1 + V_1^* \pi(a) U_2 U_2^* V_1 = T_1(a)
\end{aligned} \tag{42}$$

for all $a \in \mathcal{A}$, and thus $\tilde{V}_1 \in S(T_1, \hat{\pi}_1 \oplus \hat{\pi}_2 \oplus \hat{\pi}_3)$. An analogous calculation shows that $\tilde{V}_3 \in S(T_3, \hat{\pi}_1 \oplus \hat{\pi}_2 \oplus \hat{\pi}_3)$. Hence, $\hat{\pi}_1 \oplus \hat{\pi}_2 \oplus \hat{\pi}_3$ is a common representation for the completely positive maps T_1, T_2, and T_3 with corresponding dilations $(\tilde{V}_i, \hat{\pi}_1 \oplus \hat{\pi}_2 \oplus \hat{\pi}_3, \hat{\mathcal{K}}_1 \oplus \hat{\mathcal{K}}_2 \oplus \hat{\mathcal{K}}_3)$. Moreover, we see from Eq. (40) that $\tilde{V}_2$ only depends on the minimal dilations. In addition, we have

$$\tilde{V}_2^* \tilde{V}_1 = V_2^* V_1 \quad \text{and} \quad \tilde{V}_2^* \tilde{V}_3 = \check{V}_2^* \check{V}_3. \tag{43}$$

Now assume that $(\pi, V_i, \mathcal{K})$ and $(\check{\pi}, \check{V}_j, \check{\mathcal{K}})$ are chosen as in Proposition 6 such that

$$\|V_1 - V_2\| = \beta_\pi(T_1, T_2) = \beta(T_1, T_2), \tag{44}$$

$$\|\check{V}_2 - \check{V}_3\| = \beta_{\check{\pi}}(T_2, T_3) = \beta(T_2, T_3). \tag{45}$$

Hence, Eq. (43) and the triangle inequality for the operator norm imply that

$$\begin{aligned}
\beta(T_1, T_3) &\leqslant \|\tilde{V}_1 - \tilde{V}_3\| \\
&\leqslant \|\tilde{V}_1 - \tilde{V}_2\| + \|\tilde{V}_2 - \tilde{V}_3\| \\
&= \|V_1 - V_2\| + \|\check{V}_2 - \check{V}_3\| \\
&= \beta(T_1, T_2) + \beta(T_2, T_3),
\end{aligned} \tag{46}$$

concluding the proof. □

Appendix B. Bures' upper bound for positive functionals

The proof of the upper bound $\beta(T_1, T_2) \leqslant \sqrt{\|T_1 - T_2\|}$ for cp maps T_i that we present in Section 4 relies on the corresponding result for positive functionals. In his original paper [7] Bures assumed (normalized) states on von Neumann algebras. The generalization to arbitrary bounded positive functionals on C^*-algebras is straightforward. We nevertheless include it here for completeness and reference.

Proposition 11 *(Bures' bound for positive functionals). Let $\mathcal{A}$ be a C^*-algebra, and let $\omega_0, \omega_1 \in \mathcal{A}^*$ be positive functionals. We then have*

$$\beta(\omega_0, \omega_1) \leqslant \sqrt{\|\omega_0 - \omega_1\|}. \tag{47}$$

The following lemma will establish Proposition 11 under an additional dominance condition. This extra condition will then be removed with the help of Lemma 13, which proves the continuity of the Bures distance with respect to convex mixtures.

Lemma 12. *Let $\mathcal{A}$ be a C^*-algebra, and let $\omega_0, \omega_1 \in \mathcal{A}^*$ be two positive functionals such that $n\omega_0 \geqslant \omega_1$ for some $n \in \mathbb{N}$. Then Eq.* (47) *holds.*

Proof. We choose a common representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ such that the fibers $S(\omega_i, \pi)$ are non-empty, $i \in \{0, 1\}$. The functionals ω_i admit unique normal extensions to the von Neumann algebra $\mathcal{A}^\pi := \pi(\mathcal{A})''$, which we denote by ω_i^π. The dominance condition transfers, hence $n\omega_0^\pi - \omega_1^\pi \geqslant 0$ for some $n \in \mathbb{N}$. Sakai's Radon–Nikodym theorem [17,18] then allows us to find a positive operator $h \in \mathcal{A}^\pi$ such that $\omega_1^\pi(a) = \omega_0^\pi(hah)$ for all $a \in \mathcal{A}$. For $|\psi\rangle \in S(\omega_0, \pi)$ we have $h|\psi\rangle \in S(\omega_1, \pi)$, and hence

$$\beta^2(\omega_0, \omega_1) \leqslant \big\|(\mathbb{1} - h)|\psi\rangle\big\|^2 = \omega_0^\pi\big((\mathbb{1} - h)^2\big). \tag{48}$$

Let $h = \int \lambda p(d\lambda)$ denote the spectral decomposition of h, and set $p := \int_{\lambda=0}^1 p(d\lambda)$. We then find

$$\omega_0^\pi\big((\mathbb{1} - h)^2 p\big) \leqslant \omega_0^\pi\big(\big(\mathbb{1} - h^2\big)p\big) \quad \text{and} \tag{49}$$

$$\omega_0^\pi\big((\mathbb{1} - h)^2(\mathbb{1} - p)\big) \leqslant \omega_0^\pi\big(\big(\mathbb{1} - h^2\big)(p - \mathbb{1})\big). \tag{50}$$

Adding Eqs. (49) and (50), we see from Eq. (48) that

$$\begin{aligned} \beta^2(\omega_0, \omega_1) &\leqslant \omega_0^\pi\big(\big(\mathbb{1} - h^2\big)p\big) + \omega_0^\pi\big(\big(\mathbb{1} - h^2\big)(p - \mathbb{1})\big) \\ &= \omega_0^\pi\big(\big(\mathbb{1} - h^2\big)(2p - \mathbb{1})\big) \\ &= \big(\omega_0^\pi - \omega_1^\pi\big)(2p - \mathbb{1}) \\ &\leqslant \big\|\omega_0^\pi - \omega_1^\pi\big\| \\ &= \|\omega_0 - \omega_1\|, \end{aligned} \tag{51}$$

where we have used that $2p - \mathbb{1} = p - (\mathbb{1} - p)$ is a reflection, and hence $\|2p - \mathbb{1}\| = 1$. □

Lemma 13. *Let $\mathcal{A}$ be a C^*-algebra, and let $\omega_0, \omega_1 \in \mathcal{A}^*$ be two positive functionals. Then the inequality*

$$\big|\beta(\omega_0, \omega_1) - \beta\big((1 - s)\omega_0 + s\omega_1, \omega_1\big)\big| \leqslant \sqrt{s}\big(\sqrt{\|\omega_0\|} + \sqrt{\|\omega_1\|}\big) \tag{52}$$

holds for all $s \in [0, 1]$.

Proof. Again, the proof proceeds via a direct sum construction. For $|\psi_i\rangle \in S(\omega_i, \pi)$ we have $|\psi_0\rangle \oplus 0 \in S(\omega_0, \pi \oplus \pi)$ and $|\psi_0\rangle \oplus |\psi_1\rangle \in S(\omega_0 + \omega_1, \pi \oplus \pi)$, and thus

$$\beta(\omega_0, \omega_0 + \omega_1) \leqslant \big\| |\psi_0\rangle \oplus 0 - |\psi_0\rangle \oplus |\psi_1\rangle \big\| = \langle \psi_1 \mid \psi_1 \rangle = \sqrt{\|\omega_1\|}. \tag{53}$$

We know from Proposition 10 that the Bures distance is indeed a metric, and hence we can use the triangle inequality and then Eq. (53) to conclude that

$$\begin{aligned} \big|\beta(\omega_0, \omega_1) - \beta\big((1-s)\omega_0 + s\omega_1, \omega_1\big)\big| &\leqslant \beta\big(\omega_0, (1-s)\omega_0 + s\omega_1\big) \\ &\leqslant \beta\big(\omega_0, (1-s)\omega_0\big) + \beta\big((1-s)\omega_0, (1-s)\omega_0 + s\omega_1\big) \\ &\leqslant \sqrt{s}\big(\sqrt{\|\omega_0\|} + \sqrt{\|\omega_1\|}\big), \end{aligned} \tag{54}$$

just as suggested. □

We now have all the necessary tools at hand for the

Proof of Proposition 11. Given a parameter $s \in (0, 1]$, we define the convex mixture $\omega_s := (1-s)\omega_0 + s\omega_1$. Choosing a positive integer $n > s^{-1}$, we have $n\omega_s - \omega_1 > 0$, and hence $\beta(\omega_s, \omega_1) \leqslant \sqrt{\|\omega_s - \omega_1\|}$ follows from Lemma 12. We can then conclude from Lemma 13 that the estimate

$$\begin{aligned} \beta(\omega_0, \omega_1) &\leqslant \beta(\omega_s, \omega_1) + \sqrt{s}\big(\sqrt{\|\omega_0\|} + \sqrt{\|\omega_1\|}\big) \\ &\leqslant \sqrt{\|\omega_s - \omega_1\|} + \sqrt{s}\big(\sqrt{\|\omega_0\|} + \sqrt{\|\omega_1\|}\big) \\ &= \sqrt{1-s}\sqrt{\|\omega_0 - \omega_1\|} + \sqrt{s}\big(\sqrt{\|\omega_0\|} + \sqrt{\|\omega_1\|}\big) \end{aligned} \tag{55}$$

holds for all $s \in (0, 1]$. The limit $s \to 0$ yields the desired result. □

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