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On global solution to the Klein–Gordon–Hartree equation below energy space

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A B S T R A C T

In this paper, we consider the Cauchy problem for Klein–Gordon equation with a cubic convolution nonlinearity in $\mathbb{R}^3$. By making use of Bourgain's method in conjunction with a precise Strichartz estimate of S. Klainerman and D. Tataru, we establish the H^s $(s<1)$ global well-posedness of the Cauchy problem for the cubic convolution defocusing Klein–Gordon–Hartree equation. Before arriving at the previously discussed conclusion, we obtain global solution for this non-scaling equation with small initial data in $H^{s_0}\times H^{s_0-1}$ where $s_0=\frac{\gamma}{6}$ but not $\frac{\gamma}{2}-1$, for this equation that we consider is a subconformal equation in some sense. In doing so a number of nonlinear prior estimates are already established by using Bony's decomposition, flexibility of Klein–Gordon admissible pairs which are slightly different from that of wave equation and a commutator estimate. We establish this commutator estimate by exploiting cancellation property and utilizing Coifman and Meyer multilinear multiplier theorem. As far as we know, it seems that this is the first result on low regularity for this Klein–Gordon–Hartree equation.

1. Introduction

We study the following Cauchy problem for the Klein–Gordon-Hartree equation:

$$\begin{cases} \Box\phi+\phi+\left(|x|^{-\gamma}*|\phi|^2\right)\phi=0 & \text{in } \mathbb{R}\times\mathbb{R}^3, \\ \phi|_{t=0}=\phi_0, \qquad \partial_t\phi|_{t=0}=\phi_1. \end{cases} \tag{1.1}$$

Here $\phi(t,x)$ is a complex valued function defined in space time $\mathbb{R}^{1+3}$, and $\Box=\partial_{tt}-\Delta$.

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Recently the Cauchy problem (1.1) has been extensively studied in the case with initial data $(\phi_0, \phi_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. The well-posedness and the asymptotic behavior of solution to the Cauchy problem (1.1) have been studied by Menzala and Strauss [16,17] and Machihara, Nakanishi, and Ozawa [14]. On the other hand, the nonlinear Schrödinger equation and the Dirac equation with interaction term $(|x|^{-\gamma} * |\phi|^2)\phi$ has also been extensively studied, see [9,20,21,15]. Ginibre and Velo in [9] gave the scattering theory of Hartree equation for the energy subcritical case. For the energy critical case and mass critical, one can refer to [20,21] with radial initial data.

Many authors [3,7,10,18,27] have studied the local well-posedness (as well as global well-posedness) in fractional Sobolev spaces for the Cauchy problem of general semilinear wave or Schrödinger equations under minimal regularity assumptions on the initial data. For example, Tao [27] established the sharp local well-posedness of nonlinear wave equation. Kenig, Ponce, and Vega [10] had established the global well-posedness under the energy norm for the Cauchy problem of nonlinear wave equations with rough initial data (in particular, in $\dot{H}^s(\mathbb{R}^3)$, $\frac{3}{4} < s < 1$ for cubic wave equation). They used the Fourier truncation method discovered by Bourgain [4]. And also [18] extended Kenig–Ponce–Vega's result to the dimension $n \geqslant 4$. Recently, I. Gallagher and F. Planchon [7] presented a different proof of the result in [10] for $\frac{3}{4} < s < 1$. H. Bahouri and J.-Y. Chemin [2] proved global well-posedness for $s = \frac{3}{4}$ by using a nonlinear interpolation method and logarithmic estimates from S. Klainerman and D. Tataru [12]. We also find Roy [23] obtains the global well-posedness for rough initial data in $\dot{H}^s$, $\frac{13}{18} < s < 1$ by following the I-method [5] and scaling transformation. However, if one similarly deals with Klein–Gordon equation by using I-method, he or she may meet a problem caused by the lack of the scaling property. More studies and discussions on the low regularity of nonlinear wave or dispersive Schrödinger equations could be found in [4,28]. However, as far as we know, very few authors are engaged in studying the global well-posedness of the Cauchy problems (1.1) with less regular initial data. It is natural to ask whether a similar or better result holds for the problem (1.1).

This paper endeavors to find a global well-posedness solution to the Cauchy problem (1.1) with initial data $(\phi_0, \phi_1) \in H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ for some $s > \frac{\gamma}{4}$ with $\gamma \in (2, 3)$. Now we should remark some differences between (1.1) and cubic wave equation. If one views (1.1) as a wave equation by dropping the massive term and then makes some scaling analysis, we will find this nonlocal nonlinear term shares the scaling property of the nonlinearity $|u|^{\frac{4}{5-\gamma}} u$. One can check that $k := \frac{4}{5-\gamma} + 1 < 3$ when $2 < \gamma < \min\{n, 4\}$ with $n = 3$ and this result shows that the equation which we consider is in subconformal case. To obtain the global well-posedness theory, some previous literatures also show the subconformal equations are slightly different from the superconformal ones. For instance, Lindblad and Sogge [13,24] have shown the global existence and scattering theory for small data in a less regularity space for the superconformal case, while not for the subconformal case. Inspired by [7], we also split the initial data into low frequency part data in H^1 and high frequency part data in H^{s_0} with a suitable s_0. Since the problem (1.1) is global well-posed for large data in H^1 and small data in H^{s_0}, one may be tempted to follow a general principle of nonlinear interpolation and claim the problem (1.1) is global well-posed between them. Compared with the cubic wave equation, speaking of the Strichartz estimate, we believe that the global solution with high frequency data should exist in $H^{\frac{\gamma}{2}-1}$. It is well known that the Strichartz estimate is associated with scaling transform and it is scaling invariant. Unfortunately, the equation that we consider is a subconformal one, and its concentration effects take over scaling. Since the Strichartz estimate is applied to our subconformal equation, hence this brings about some loss to get a better result. In order to get a better result, one should establish an estimate which is conformal invariant. Fortunately, we can take $0 \leqslant \theta \leqslant 1$ as a parameter for the flexible admissible pairs (see Definition 2.3) to make the Strichartz estimate of Klein–Gordon more flexible than wave equation. This helps us to get a global solution with the high frequency data, at the cost of $0 \leqslant \theta = \frac{6}{\gamma} - 2 \leqslant 1$ which weakens the Strichartz estimate and causes $2 < \gamma < 3$. One can refer the detail in Section 3.

We point out that it is easy to have the result for $\frac{\gamma}{3} - \frac{1}{6} < s < 1$ by rough Hölder's inequality. But how to get our low bound $\frac{\gamma}{4} < s < 1$? A good way to think about this is via precise Strichartz estimate to obtain index s as low as possible. The nonlinearity including a formal negative derivative brings us some difficulties caused by the fact that the negative derivative acts on the low frequency part. And this leads us to restricts $s > \frac{\gamma}{4}$ rather than $s > \max\{\frac{1}{2}, \frac{\gamma}{2} - \frac{3}{4}\}$. At the end of this section, we also give some intuitive analysis to show our result is reasonable. As a limited case, our result recovers the result of [7,10] when γ tends to 3.

During the process of proving our key estimate Lemma 5.1, the nonlocal nonlinearity brings about some essential difficulties when we try to make use of the precise Strichartz estimate. Compared with the general semilinear nonlinearity, the convolution nonlinearity not only essentially represents a negative derivation in it but also has a difference construction of nonlinearity. These differences and difficulties prevent us from obtaining directly our expected result $s > \frac{\gamma}{4}$ by restricting the range of the parameter r. To overcome these difficulties, we firstly construct a commutator and establish this commutator estimate by exploiting cancellation property and utilizing Coifman and Meyer multilinear multiplier theorem and then go on our process through using precise Strichartz estimate.

Now we state our main result:

Theorem 1.1. *Let $\frac{\gamma}{4} < s < 1$ with $2 < \gamma < 3$. If $(\phi_0, \phi_1) \in H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$, then there exists a unique global solution ϕ of* (1.1) *such that $\phi \in C(\mathbb{R}^+; H^s(\mathbb{R}^3))$.*

We conclude this section by giving a sketch of the proof of Theorem 1.1 and one shall read more detailed information in the rest of this paper. Without loss of generality, we only consider ϕ as a real function for simplicity from now on. Since the problem (1.1) is global well-posed for large data in H^1 and small data in H^{s_0} with $s_0 = \frac{\gamma}{6}$, one may be tempted to follow a general principle of nonlinear interpolation and believe the problem (1.1) to be global well-posed between them, as well as the cubic defocusing wave equation [7]. To make sense of this heuristic, we proceed it in the following steps.

Step 1. The purpose of this step is to show the global well-posedness for the high frequency part. We split the initial data:

$$\phi_i = (\mathrm{I} - S_J)\phi_i + S_J\phi_i \stackrel{\text{def}}{=} v_i + u_i, \quad i = 0, 1$$

where I is the identity operator and S_J is the Littlewood–Paley operator, referring to Section 2. It is easy to see that

$$\|u_0\|_{H^1} \lesssim 2^{J(1-s)}\|\phi_0\|_{H^s}, \qquad \|u_1\|_{L^2} \lesssim \|\phi_1\|_{L^2}$$

and

$$\|v_0\|_{H^\beta} \lesssim 2^{J(\beta-s)}\|\phi_0\|_{H^s}, \qquad \|v_1\|_{H^{\beta-1}} \lesssim 2^{J(\beta-s)}\|\phi_1\|_{H^{s-1}} \quad \text{for all } \beta \leqslant s.$$

Thus it follows that

$$\mathcal{E}_{h,\sigma} \lesssim 2^{J(\sigma-s)}\mathcal{E}_s, \quad \text{for } \sigma \leqslant s, \tag{1.2}$$

$$\mathcal{E}_{\ell,1} \lesssim 2^{J(1-s)}\mathcal{E}_s, \quad \text{for } s \leqslant 1, \tag{1.3}$$

where

$$\mathcal{E}_s \stackrel{\text{def}}{=} \|\phi_0\|_{H^s} + \|\phi_1\|_{H^{s-1}}, \tag{1.4}$$

$$\mathcal{E}_{h,\sigma} \stackrel{\text{def}}{=} \|v_0\|_{H^\sigma} + \|v_1\|_{H^{\sigma-1}}, \tag{1.5}$$

$$\mathcal{E}_{\ell,\sigma} \stackrel{\text{def}}{=} \|u_0\|_{H^\sigma} + \|u_1\|_{H^{\sigma-1}}. \tag{1.6}$$

Choosing J large enough, one can achieve $\mathcal{E}_{h,s_0}$ small enough, in other words, initial data of the following problem

$$\begin{cases} \Box v + v + \left(|x|^{-\gamma} * v^2\right)v = 0 & \text{in } \mathbb{R} \times \mathbb{R}^3, \\ v|_{t=0} = v_0, \quad \partial_t v|_{t=0} = v_1 \end{cases} \tag{1.7}$$

is small enough in $H^{s_0}(\mathbb{R}^3) \times H^{s_0-1}(\mathbb{R}^3)$ where $s_0 < s$. Due to some technique difficulties and this equation is subconformal one, we are restricted to choose $s_0 = \frac{\gamma}{6}$ while not $\frac{\gamma}{2} - 1$ proposed by scaling analysis or $\frac{\gamma}{4} - \frac{1}{4}$ proposed by conformal analysis. We will get a global well-posed solution to the Cauchy problem (1.7), see Section 3 for details.

Step 2. In order to recover a solution to our problem (1.1), we solve a perturbed equation with large initial data in $H^1 \times L^2$,

$$\begin{cases} \Box u + u + \mathcal{I}\left(u^2\right)u + 2\mathcal{I}(uv)u + \mathcal{I}\left(v^2\right)u + \mathcal{I}(u^2)v + 2\mathcal{I}(uv)v = 0, \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1, \end{cases} \tag{1.8}$$

where the operator $\mathcal{I}$ is the operator $(-\Delta)^{\frac{\gamma-3}{2}}$. We will prove there exists a unique local solution to (1.8) in $C([0,T];H^1)$.

Step 3. To complete the proof of Theorem 1.1, the key is how to extend the local solution to a global solution. We should establish a priori bound on the energy of the local solution u. In fact, the energy estimate yields

$$\begin{aligned} &\frac{1}{2}\left(\left\|u(t)\right\|_{H^1}^2 + \left\|u_t(t)\right\|_{L^2}^2\right) + \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} |x-y|^{-\gamma} u^2(y,t)u^2(x,t)\,dy\,dx \\ &\quad\leqslant \frac{1}{2}\left(\|u_0\|_{H^1}^2 + \|u_1\|_{L^2}^2\right) + \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} |x-y|^{-\gamma} u_0^2(y)u_0^2(x)\,dy\,dx \\ &\qquad + \left|\int_0^t\int_{\mathbb{R}^3} \mathcal{I}\left(v^2\right)(x,\tau)u(x,\tau)\partial_\tau u(x,\tau)\,dx\,d\tau\right| \\ &\qquad + 2\left|\int_0^t\int_{\mathbb{R}^3} \mathcal{I}(uv)(x,\tau)v(x,\tau)\partial_\tau u(x,\tau)\,dx\,d\tau\right| \\ &\qquad + \left|\int_0^t\int_{\mathbb{R}^3} \mathcal{I}\left(u^2\right)(x,\tau)v(x,\tau)\partial_\tau u(x,\tau)\,dx\,d\tau\right| \\ &\qquad + 2\left|\int_0^t\int_{\mathbb{R}^3} \mathcal{I}(uv)(x,\tau)u(x,\tau)\partial_\tau u(x,\tau)\,dx\,d\tau\right|. \end{aligned}$$

Let $H_T(u) := \sup_{t<T} H(u)(t)$ where

$$H(u)(t) \stackrel{\text{def}}{=} \left(\frac{1}{2}\left\|u(t)\right\|_{H^1}^2 + \frac{1}{2}\left\|u_t(t)\right\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} |x-y|^{-\gamma} u^2(y,t)u^2(x,t)\,dy\,dx\right)$$

and then by making use of Hölder's inequality and Sobolev embedding, it follows that

$$\begin{aligned} H_T(u) &\lesssim H(u)(0) + H_T(u)\int_0^T \|v(\tau)\|^2_{L^{\frac{6}{4-\gamma}}}\,d\tau + H_T^{\frac{3}{2}}(u)\int_0^T \|v(\tau)\|_{L^{\frac{6}{7-2\gamma}}}\,d\tau \\ &\lesssim H(u)(0) + H_T(u)T^{\frac{7-\gamma}{6}}\|v\|^2_{X^\beta} + H_T^{\frac{3}{2}}(u)T^{\frac{5-\gamma}{3}}\|v\|_{X^\alpha} \\ &\lesssim 2^{2J(1-s)} + H_T(u)T^{\frac{7-\gamma}{6}}2^{2J(\beta-s)} + H_T^{\frac{3}{2}}(u)T^{\frac{5-\gamma}{3}}2^{J(\alpha-s)} \end{aligned}$$

where $\alpha = \frac{2\gamma-4}{3}$, $\beta = \frac{\gamma-1}{3}$ and the space X^α is defined in the coming section. What we want to do is to control $H_T(u)$ for arbitrarily large T. As long as $s > (\alpha+1)/2 = \frac{\gamma}{3} - \frac{1}{6}$, by choosing J large enough, bootstrap argument yields

$$H_T(u) \lesssim 2^{2J(1-s)}.$$

One can see that, if $s > \frac{\gamma}{3} - \frac{1}{6}$, the argument is trivial, since the above mentioned result can be deduced from some rough estimates such as the Hölder estimate. On the other hand, since the scaling suggests us that $X^{\frac{\gamma}{2}-1}$ is the lowest regularity space which v could belong to, it is tempting and reasonable to believe that the best result obtained by this method is $s > (\frac{\gamma}{2} - 1 + 1)/2 = \frac{\gamma}{4}$ instead of α by $\frac{\gamma}{2} - 1$. To obtain this optimal result $s > \frac{\gamma}{4}$, we adopt some more sophisticated tools such as precise Strichartz estimate, Bony's paraproduct estimates and twice Bony's decomposition. This result is achieved under an assumption of a core estimate which will be shown through the precise Strichartz estimate and a commutator estimate.

The paper is organized as follows: In the coming section, we recall some notations and recollect some well-known results on Besov spaces in conjunction with the Littlewood–Paley theory which will be used in the course of the proofs. Meanwhile, we also introduce the precise Strichartz estimate. Section 3 provides the global well-posedness of original equation evoking the high frequency part of initial data in H^{s_0}. In Section 4, we prove a local well-posedness of perturbed equation with the low frequency of the initial data in H^1 by the standard fixed point theorem. In Section 5, we give an energy estimate for the low frequency part provided an assumption the key estimate in Lemma 5.1. We extend the local well-posedness of the perturbed equation to globally well-posed by the bootstrap argument in Section 6. In the final section, we prove our essential and key lemma by the precise Strichartz estimate, commutator estimate and Coifman and Meyer multiplier theorem.

2. Preliminaries

In this section, we shall present some well-known facts on the Littlewood–Paley theory and introduce some notations, definitions and estimates which are needed in this paper. Let $\mathcal{S}(\mathbb{R}^3)$ be the Schwartz class of rapidly decreasing functions. Given $f \in \mathcal{S}(\mathbb{R}^3)$, its Fourier transform $\mathcal{F}f = \hat{f}$ is defined by

$$\hat{f}(\xi) = (2\pi)^{-\frac{3}{2}}\int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)\,dx, \qquad \mathcal{F}^{-1}f = \hat{f}(-\xi).$$

Choose two nonnegative radial functions $\chi, \varphi \in \mathcal{S}(\mathbb{R}^3)$ supported respectively in $\mathcal{B} = \{\xi \in \mathbb{R}^3,\ |\xi| \leqslant \frac{4}{3}\}$ and $\mathcal{C} = \{\xi \in \mathbb{R}^3,\ \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$ such that

$$\chi(\xi) + \sum_{j\geqslant 0} \varphi\big(2^{-j}\xi\big) = 1, \quad \xi \in \mathbb{R}^3,$$
$$\sum_{j\in\mathbb{Z}} \varphi\big(2^{-j}\xi\big) = 1, \quad \xi \in \mathbb{R}^3\backslash\{0\},$$

and

$$\operatorname{supp}\varphi\big(2^{-j}\cdot\big) \cap \operatorname{supp}\varphi\big(2^{-j'}\cdot\big) = \emptyset, \quad |j-j'| \geqslant 2,$$
$$\operatorname{supp}\chi(\cdot) \cap \operatorname{supp}\varphi\big(2^{-j}\cdot\big) = \emptyset, \quad j \geqslant 1.$$

Now we are in position to define the Littlewood–Paley operators S_j, $\dot{S}_j$, $\triangle_j$ and $\dot{\triangle}_j$ which are used to define Besov space

$$\triangle_j u \overset{\text{def}}{=} \begin{cases} 0, & j \leqslant -2, \\ \mathcal{F}^{-1}(\chi(\xi)\hat{u}(\xi)), & j = -1, \\ 2^{jn}\int_{\mathbb{R}^n} (\mathcal{F}^{-1}\varphi)(2^j y)u(x-y)\,dy, & j \geqslant 0, \end{cases}$$

$$S_j u \overset{\text{def}}{=} \sum_{j'\leqslant j-1} \triangle_{j'} u = 2^{jn} \int\limits_{\mathbb{R}^n} \big(\mathcal{F}^{-1}\chi\big)\big(2^j y\big)u(x-y)\,dy,$$

$$\dot{\triangle}_j u \overset{\text{def}}{=} 2^{jn} \int\limits_{\mathbb{R}^n} \big(\mathcal{F}^{-1}\varphi\big)\big(2^j y\big)u(x-y)\,dy, \quad j \in \mathbb{Z},$$

$$\dot{S}_j u \overset{\text{def}}{=} \sum_{j'\leqslant j-1} \dot{\triangle}_{j'} u.$$

One easily shows that $\dot{\triangle}_j = \dot{S}_{j+1} - \dot{S}_j$ for $j \in \mathbb{Z}$ and

$$\triangle_{-1} = S_0, \qquad \dot{\triangle}_j = \triangle_j, \quad j \geqslant 0.$$

Now we give the Littlewood–Paley's description of the Besov spaces.

Definition 2.1. Let $s \in \mathbb{R}$, $1 \leqslant p, q \leqslant \infty$. The homogeneous Besov space $\dot{B}^s_{p,q}$ is defined by

$$\dot{B}^s_{p,q} = \big\{ f \in \mathcal{Z}'\big(\mathbb{R}^3\big)\colon \|f\|_{\dot{B}^s_{p,q}} < \infty \big\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} (\sum_{j\in\mathbb{Z}} 2^{jsq}\|\dot{\triangle}_j f\|_p^q)^{\frac{1}{q}}, & \text{for} \quad q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\dot{\triangle}_j f\|_p, & \text{for } q = \infty, \end{cases}$$

and $\mathcal{Z}'(\mathbb{R}^3)$ can be identified by the quotient space $\mathcal{S}'/\mathcal{P}$ with the space $\mathcal{P}$ of polynomials.

Definition 2.2. Let $s \in \mathbb{R}$, $1 \leqslant p, q \leqslant \infty$. The inhomogeneous Besov space $B^s_{p,q}$ is defined by

$$B^s_{p,q} = \big\{ f \in \mathcal{S}'\big(\mathbb{R}^3\big)\colon \|f\|_{B^s_{p,q}} < \infty \big\},$$

where

$$\|f\|_{B^s_{p,q}} = \begin{cases} (\sum_{j\geqslant 0} 2^{jsq}\|\triangle_j f\|_p^q)^{\frac{1}{q}} + \|S_0(f)\|_p, & \text{for } q<\infty,\\ \sup_{j\geqslant 0} 2^{js}\|\triangle_j f\|_p + \|S_0(f)\|_p, & \text{for } q=\infty. \end{cases}$$

If $s>0$, then $B^s_{p,q} = L^p \cap \dot{B}^s_{p,q}$ and $\|f\|_{B^s_{p,q}} \approx \|f\|_p + \|f\|_{\dot{B}^s_{p,q}}$. We refer the reader to [1,22,29] for details.

In order to investigate the low regularity solution of the Cauchy problem (1.1), we require the use of the smoothing effect described by the Strichartz estimates and precise Strichartz estimates. For the purpose of conveniently making use of the Strichartz estimate, we introduce the admissible definition and the resolution space.

Definition 2.3. We shall say that a pair (q,r) is admissible, for $0\leqslant\theta\leqslant 1$, if

$$q,r\geqslant 2, \quad (q,r,\theta)\neq(2,\infty,0) \quad \text{and} \quad \frac{1}{q}+\frac{2+\theta}{2r}\leqslant\frac{2+\theta}{4}.$$

Remark 2.1. The above admissible pairs in Definition 2.3 is more flexible than wave admissible pairs, since θ can vary from 0 to 1. Obviously, an admissible pair in Definition 2.3 will become a wave admissible pair when $\theta=0$. When we consider the global existence for the high frequency part, we shall use $\theta=\frac{6}{\gamma}-2$ since the equation that we consider is a subconformal one.

The resolution space is defined in the following way based on the admissible definition:

$$X^\mu(I) := \bigcap_{0\leqslant\theta\leqslant 1} X^\mu_\theta(I)$$

where

$$X^\mu_\theta(I) := \bigg\{u\colon\ u\in\big(C\cap L^\infty\big)\big(I;H^\mu\big)\cap L^q\big(I;B^\sigma_{r,2}\big),\ (q,r) \text{ is admissible},\\ \frac{1}{q}=(3+\theta)\left(\frac{1}{2}-\frac{1}{r}\right)+\sigma-\mu\bigg\}.$$

We go on this section by recalling the classical Strichartz estimate and the precise Strichartz estimate. This kind of estimate goes back to Strichartz [26], and has been proved in its generality by Ginibre and Velo [8], and Keel and Tao [11]. The Strichartz estimates for the Klein–Gordon equation by using the above flexible admissible pairs can be found in [19].

Proposition 2.1. *Let u be a solution of*

$$\Box u + u = f \quad \text{in } \mathbb{R}\times\mathbb{R}^3 \quad \text{with} \quad u|_{t=0}=u_0, \quad \partial_t u|_{t=0}=u_1.$$

Then, for any admissible pairs (q_1,r_1) and (q_2,r_2), we have that

$$\begin{aligned} &\|\triangle_j u\|_{L^{q_1}(L^{r_1})} + 2^{-j}\|\partial_t\triangle_j u\|_{L^{q_1}(L^{r_1})}\\ &\quad\leqslant C2^{j(\frac{3+\theta}{2}-\frac{3+\theta}{r_1}-\frac{1}{q_1})}\big(\|\triangle_j u_0\|_{L^2}+2^{-j}\|\triangle_j u_1\|_{L^2}\big)\\ &\qquad + C2^{j[(3+\theta)(1-\frac{1}{r_1}-\frac{1}{r_2})-\frac{1}{q_1}-\frac{1}{q_2}-1]}\|\triangle_j f\|_{L^{q_2'}(L^{r_2'})}. \end{aligned} \tag{2.1}$$

We shall see that the classical Strichartz estimates are not enough to control some nonlinearities, and this leads us to resort to the following precise Strichartz estimates which were established by S. Klainerman and D. Tataru [12].

Proposition 2.2. *Let u be a solution of*

$$\Box u + u = 0 \quad \text{with} \quad u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1.$$

Assume that the supports of the Fourier transform of u_0 and u_1 are included in a ball $B(\xi_j, h2^j)$ with $|\xi_j| \in [2^{j-2}, 2^{j+2}]$ and $h < \frac{1}{8}$. Then we have that, for any admissible couple (q,r),

$$\|u\|_{L^q(L^r)} + 2^{-j}\|\partial_t u\|_{L^q(L^r)} \leqslant C2^{j(\frac{3}{2}-\frac{3}{r}-\frac{1}{q})} h^{\frac{1}{2}-\frac{1}{r}} \big(\|u_0\|_{L^2} + 2^{-j}\|u_1\|_{L^2}\big). \tag{2.2}$$

Let us recall the Hardy–Littlewood–Sobolev inequality [22,25] and a proposition of contraction which is generalization of Picard's theorem. We denote operator $\mathcal{I}$ by

$$\mathcal{I}u \overset{\text{def}}{=} (-\Delta)^{\frac{\gamma-3}{2}} u = |x|^{-\gamma} * u,$$

then

$$\|\mathcal{I}u\|_{L^q(\mathbb{R}^3)} \leqslant C_{p,q}\|u\|_{L^p(\mathbb{R}^3)} \tag{2.3}$$

for

$$0 < \gamma < 3, \quad 1 < p < q < \infty, \quad \text{and} \quad \frac{1}{q} = \frac{1}{p} - \frac{3-\gamma}{3}.$$

Proposition 2.3. *Let X be a Banach space and let $B : X \times X \times \cdots \times X \to X$ be an m-linear continuous operator $(m \geqslant 2)$ satisfying*

$$\big\|B(u_1, u_2, \ldots, u_m)\big\|_X \leqslant M\|u_1\|_X\|u_2\|_X \cdots \|u_m\|_X, \quad \forall u_1, u_2, \ldots, u_m \in X$$

for some constant $M > 0$. Let $\varepsilon > 0$ be such that $m(2\varepsilon)^{m-1}M < 1$. Then for every $y \in X$ with $\|y\|_X \leqslant \varepsilon$ the equation

$$u = y + B(u, u, \ldots, u) \tag{2.4}$$

has a unique solution $u \in X$ satisfying that $\|u\|_X \leqslant 2\varepsilon$. Moreover, the solution u continuously depends on y in the sense that, if $\|y_1\|_X \leqslant \varepsilon$ and $v = y_1 + B(v, v, \ldots, v)$, $\|v\|_X \leqslant 2\varepsilon$ then

$$\|u - v\|_X \leqslant \frac{1}{1 - m(2\varepsilon)^{m-1}M}\|y - y_1\|_X. \tag{2.5}$$

For the sake of convenience, we conclude this section by giving some notations. The solution ϕ to the Cauchy problem (1.1) is given by the following integral equation:

$$\phi(t,x) = \dot{K}(t)\phi_0 + K(t)\phi_1 + B(\phi, \phi, \phi) \overset{\text{def}}{=} \mathcal{T}\phi$$

where

$$K(t) := \frac{\sin(t\sqrt{I-\Delta})}{\sqrt{I-\Delta}},$$

$$B(u_1,u_2,u_3) := -\int_0^t K(t-\tau)\big(|x|^{-\gamma} * (u_1u_2)\big)u_3\,d\tau.$$

Throughout this article we shall denote by the letter C all universal constant and $\varepsilon > 0$ is an arbitrary small data. We shall sometimes replace an inequality of the type $f \leqslant Cg$ by $f \lesssim g$. Also, we shall denote by $(c_j)_{j\in\mathbb{Z}}$ any sequence of norm less than 1 in $\ell^2(\mathbb{Z})$.

3. Global existence for the high frequency part

Let us consider the Cauchy problem with the high frequency data,

$$\begin{cases} \Box v + v + \big(|x|^{-\gamma} * v^2\big)v = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}^3, \\ v|_{t=0} = v_0, \qquad \partial_t v|_{t=0} = v_1, & x \in \mathbb{R}^3. \end{cases} \tag{3.1}$$

and then its integral formation becomes

$$\begin{aligned} v(t,x) &= \dot K(t)v_0(x) + K(t)v_1(x) - \int_0^t K(t-\tau)\big(|x|^{-\gamma} * v^2\big)v\,d\tau \\ &\overset{\text{def}}{=} \dot K(t)v_0(x) + K(t)v_1(x) + B(v,v,v). \end{aligned} \tag{3.2}$$

Our goal in this section is to prove the global well-posedness of (3.1) or (3.2). More precisely, we have the following proposition:

Proposition 3.1. *Let $s_0 = \frac{\gamma}{6}$ and suppose that $(v_0, v_1) \in H^\mu \times H^{\mu-1}$ for any $0 \leqslant \mu \leqslant 1$. There exists a constant $\varepsilon_0 > 0$ such that if*

$$\|v_0\|_{H^{s_0}} + \|v_1\|_{H^{s_0-1}} \leqslant \varepsilon_0,$$

then there exists a unique global solution v to (3.1) or (3.2) in $X^{s_0}(\mathbb{R}) \cap X^\mu(\mathbb{R})$. Moreover,

$$\|v\|_{X^\mu} \leqslant C_\mu\big(\|v_0\|_{H^\mu} + \|v_1\|_{H^{\mu-1}}\big).$$

Remark 3.1. We focus on $\mu = \frac{2\gamma-4}{3}$ and $\mu = \frac{\gamma-1}{3}$ in the coming section.

It is well known that the global existence theory for small initial data is a straightforward result of nonlinear estimate, thus how to obtain a suitable nonlinear estimate is essential. Before proving this proposition, we make some analysis on nonlinear estimate. As mentioned in the introduction, the nonlocal nonlinearity shares the scaling with a subconformal nonlinearity when $\gamma < 3$ and this may bring some troubles when we make a choice of a suitable resolution space X^{s_0}. Take $0 \leqslant \theta \leqslant 1$ as a parameter in the flexible admissible pairs (see Definition 2.3), and we make analysis on the relationship between θ and s_0. The Strichartz estimate, Hölder inequality and Hardy–Littlewood–Sobolev inequality imply that, for $\sigma \leqslant 0$,

$$\big\|B(v,v,v)\big\|_{X^{s_0}} \leqslant \big\|\big(|x|^{-\gamma} * |v|^2\big)v\big\|_{L^{q_1'}(B^{-\sigma}_{r_1',2})} \leqslant \|v\|_{L^{q_2}(B^{-\sigma}_{r_2,2})}\|v\|^2_{L^{q_3}(L^{r_3})},$$

satisfying

$$\frac{1}{q_1} = (3+\theta)\left(\frac{1}{2} - \frac{1}{r_1}\right) + \sigma + s_0 - 1,$$
$$\frac{1}{q_2} = (3+\theta)\left(\frac{1}{2} - \frac{1}{r_2}\right) - \sigma - s_0,$$

and

$$\frac{1}{q_3} = (3+\theta)\left(\frac{1}{2} - \frac{1}{r_3}\right) - s_0,$$
$$1 = \frac{1}{q_1} + \frac{1}{q_2} + \frac{2}{q_3},$$
$$2 = \frac{\gamma}{3} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{2}{r_3},$$

then

$$s_0 = \frac{\gamma}{2} - 1 + \frac{\gamma\theta}{6}.$$

We find the fact index s_0 is increasing when the parameter θ increases. It is tempting to choose $\theta = 0$ to get the smallest $s_0 = \frac{\gamma}{2} - 1$ proposed by scaling. However, in addition the admissible condition implies that

$$\frac{2}{q_1} \leqslant (2+\theta)\left(\frac{1}{2} - \frac{1}{r_1}\right),$$
$$\frac{2}{q_2} \leqslant (2+\theta)\left(\frac{1}{2} - \frac{1}{r_2}\right),$$
$$\frac{2}{q_3} \leqslant (2+\theta)\left(\frac{1}{2} - \frac{1}{r_3}\right).$$

Then a direction computation gives that

$$2\left(\frac{1}{q_1} + \frac{1}{q_2} + \frac{2}{q_3}\right) \leqslant (2+\theta)\left(2 - \frac{1}{r_1} - \frac{1}{r_2} - \frac{2}{r_3}\right)$$

which yields that

$$\frac{3}{\gamma} \leqslant 1 + \frac{\theta}{2}.$$

If we choose $\theta = 0$, then we are forced to $\gamma \geqslant 3$ which contradicts with our requirement $\gamma < 3$. But if we choose $\theta = \frac{6}{\gamma} - 2$ and then $s_0 = \frac{\gamma}{6}$ and we are allowed by $2 \leqslant \gamma \leqslant 3$.

Proof of Proposition 3.1. Thanks to Strichartz estimate, we have

$$\left\| B(v,v,v) \right\|_{X^\mu} \leqslant \left\| \left(|x|^{-\gamma} * |v|^2\right) v \right\|_{L^{q_1'}(B^{-\sigma}_{r_1',2})} \leqslant \|v\|_{L^{q_2}(B^{-\sigma}_{r_2,2})} \|v\|^2_{L^{q_3}(L^{r_3})},$$

where

$$\left(\frac{1}{q_1},\frac{1}{r_1}\right)=\left(\frac{3}{3+\gamma}(1-\mu-\sigma),\frac{1}{2}-\frac{\gamma}{3+\gamma}(1-\mu-\sigma)\right),$$

and

$$\left(\frac{1}{q_2},\frac{1}{r_2}\right)=\left(\frac{3(\mu+\sigma)}{3+\gamma},\frac{1}{2}-\frac{\gamma(\mu+\sigma)}{3+\gamma}\right),\qquad \left(\frac{1}{q_3},\frac{1}{r_3}\right)=\left(\frac{\gamma}{2(3+\gamma)},\frac{9+3\gamma-\gamma^2}{6(3+\gamma)}\right).$$

When $0\leqslant\mu\leqslant\frac{1}{2}+\frac{\gamma}{6}$, we choose $\sigma=0$; while $\frac{1}{2}+\frac{\gamma}{6}<\mu\leqslant 1$, we choose $\sigma=\frac{1}{2}+\frac{\gamma}{6}-\mu$. Thus,

$$\big\|B(v,v,v)\big\|_{X^\mu}\leqslant\|v\|_{X^\mu}\|v\|^2_{X^{s_0}}. \tag{3.3}$$

Combining this nonlinear estimate, Proposition 3.1 follows from a standard contraction argument and small initial data condition. □

4. Local existence for the low frequency part

In this part, we shall study the following perturbed problem in $\mathbb{R}\times\mathbb{R}^3$:

$$\begin{cases}\Box u+u+\mathcal{I}(u^2)u+2\mathcal{I}(uv)u+\mathcal{I}(v^2)u+\mathcal{I}(u^2)v+2\mathcal{I}(uv)v=0,\\ u|_{t=0}=u_0,\qquad \partial_t u|_{t=0}=u_1.\end{cases} \tag{4.1}$$

Proposition 4.1. *Let $\alpha=\frac{2\gamma-4}{3}$, $\beta=\frac{\gamma-1}{3}$ and assume that v be in $X^\alpha\cap X^\beta$ and $(u_0,u_1)\in H^1\times L^2$, then there exists a positive time T such that a unique solution u to* (4.1) *satisfying*

$$u\in C\big([0,T];H^1\big).$$

Proof of Proposition 4.1. In practice, solving (4.1) on $[0,T]$ is equivalent to solving the following integral equation

$$\begin{aligned}u&=\dot K(t)u_0+K(t)u_1+\int_0^t K(t-\tau)\big[\mathcal{I}\big(u^2\big)u+2\mathcal{I}(uv)u+\mathcal{I}\big(v^2\big)u+\mathcal{I}\big(u^2\big)v+2\mathcal{I}(uv)v\big]\,d\tau\\ &\triangleq\widetilde{\mathcal{T}}u.\end{aligned}$$

Using the Strichartz estimate, we have

$$\left\|\int_0^t K(t-\tau)\mathcal{I}\big(u^2\big)u\,d\tau\right\|_{L^\infty_T(H^1)}\lesssim\big\|\mathcal{I}\big(u^2\big)u\big\|_{L^1_T(L^2)}.$$

On one hand, we make use of Hölder's inequality and Hardy–Littlewood–Sobolev inequality to deduce that

$$\big\|\mathcal{I}\big(u^2\big)u\big\|_{L^1_T(L^2)}\leqslant C\big\|\mathcal{I}\big(u^2\big)\big\|_{L_T^{\frac{3}{2}}L^{\frac{9}{\gamma}}}\|u\|_{L^3_TL^{\frac{18}{9-2\gamma}}}\leqslant C\|u\|^3_{L^3_TL^{\frac{18}{9-2\gamma}}}\leqslant CT\|u\|^3_{L^\infty_TH^1}. \tag{4.2}$$

For the rest of terms, arguing similarly as above, it can be obtained that

$$\left\|\mathcal{I}(uv)u\right\|_{L^1_T(L^2)} \leqslant C\|u\|^2_{L^\infty_T L^6}\|v\|_{L^1_T L^{\frac{6}{7-2\gamma}}} \leqslant CT^{\frac{5-\gamma}{3}}\|u\|^2_{L^\infty_T H^1}\|v\|_{X^\alpha}, \tag{4.3}$$

$$\left\|\mathcal{I}(v^2)u\right\|_{L^1_T(L^2)} \leqslant C\|u\|_{L^\infty_T L^6}\|v\|^2_{L^2_T L^{\frac{6}{4-\gamma}}} \leqslant CT^{\frac{4-\gamma}{3}}\|u\|_{L^\infty_T H^1}\|v\|^2_{X^\beta}, \tag{4.4}$$

$$\left\|\mathcal{I}(u^2)v\right\|_{L^1_T(L^2)} \leqslant CT^{\frac{5-\gamma}{3}}\|u\|^2_{L^\infty_T H^1}\|v\|_{X^\alpha}, \tag{4.5}$$

$$\left\|\mathcal{I}(uv)v\right\|_{L^1_T(L^2)} \leqslant CT^{\frac{4-\gamma}{3}}\|u\|_{L^\infty_T H^1}\|v\|^2_{X^\beta}. \tag{4.6}$$

A combination of (4.2), (4.3)–(4.6) and the Strichartz estimate in Proposition 2.1 lead to the estimate

$$\begin{aligned}\|u\|_{L^\infty_T(H^1)} &\lesssim \|u_0\|_{H^1} + \|u_1\|_{L^2} + T\|u\|^3_{L^\infty_T(H^1)} + T^{\frac{5-\gamma}{3}}\|u\|^2_{L^\infty_T H^1}\|v\|_{X^\alpha} \\ &\quad + T^{\frac{4-\gamma}{3}}\|u\|_{L^\infty_T H^1}\|v\|^2_{X^\beta}.\end{aligned}$$

As long as choosing T is small enough, $\widetilde{\mathcal{T}}$ is a contraction mapping in ball $B(0, 2C\mathcal{E}_{\ell,1})$. By means of Picard's fixed point argument we have a unique solution u to (4.1) in $L^\infty([0,T]; H^1)$. Therefore, Proposition 4.1 is proved by the standard argument. □

5. Energy estimate for the low frequency part

In order to extend the local solution to a global solution, we shall prove a prior estimate for the Hamiltonian of u in this section. Let us recall the definition of Hamiltonian of u defined by

$$H(u)(t) \overset{\text{def}}{=} \left(\frac{1}{2}\left\|u(t)\right\|^2_{H^1} + \frac{1}{2}\left\|u_t(t)\right\|^2_{L^2} + \frac{1}{4}\int_{\mathbb{R}^3\times\mathbb{R}^3} |x-y|^{-\gamma}u^2(y,t)u^2(x,t)\,dy\,dx\right).$$

Similarly we give another notation of the energy of u, which is denoted by

$$E(u)(t) \overset{\text{def}}{=} \frac{1}{2}\left\|u(t)\right\|^2_{H^1} + \frac{1}{2}\left\|u_t(t)\right\|^2_{L^2}.$$

Let

$$H_T(u) \overset{\text{def}}{=} \sup_{t\leqslant T} H(u)(t), \qquad E_T(u) \overset{\text{def}}{=} \sup_{t\leqslant T} E(u)(t).$$

To extend the local existence to global existence, we have to do a number of nonlinear a priori estimates provided that $E_T(u) \leqslant 2CH(u)(0)$, see Proposition 5.1 and Lemma 5.1. As a direct consequence of the above assumption, we get an important relationship between $E(u)$ and $\mathcal{E}_s$ defined in the introduction

$$E_T(u) \lesssim 2^{2J(1-s)}\left(\mathcal{E}_s^2 + \mathcal{E}_s^4\right) \lesssim 2^{2J(1-s)}. \tag{5.1}$$

In fact, it follows from the Hardy–Littlewood–Sobolev inequality and the definition of u_0 that

$$\left\|\left(|x|^{-\gamma} * u_0^2\right)u_0^2\right\|_{L^1} \lesssim \|u_0\|^4_{\frac{12}{6-\gamma}} \leqslant \|S_0\phi_0\|^4_{\frac{12}{6-\gamma}} + \sum_{0\leqslant j\leqslant J}\|\triangle_j\phi_0\|^4_{\frac{12}{6-\gamma}}.$$

And then the right hand of the above inequality can controlled that as soon as $1 > s > \frac{\gamma}{4}$ by utilizing Bernstein inequality

$$\|S_0\phi_0\|_{L^2}^4 + \sum_{0\leqslant j\leqslant J} 2^{j4(\frac{\gamma}{4}-s)}2^{j4s}\|\triangle_j\phi_0\|_{L^2}^4 \lesssim 2^{2J(1-s)}\mathcal{E}_s^4.$$

From now on, we assume (5.1) to hold in our subsequence proof.

Proposition 5.1. *Assume that $(u_0, u_1) \in H^1 \times L^2$, then the following estimate holds for s_0, α, β defined in Proposition 3.1 and Proposition 4.1,*

$$\begin{aligned} H_T(u) &\lesssim H(u)(0) + T^{\frac{4-\gamma}{3}}2^{-2J(s-\beta)}E_T(u) + T^{\frac{5-\gamma}{3}}2^{-J(4s-\alpha-2s_0-1)}E_T(u) \\ &\quad + \big(T^{\frac{1}{2}+\frac{1}{r_1}}2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r_1})]} + T^{\frac{1}{2}+\frac{1}{r_2}}2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r_2})]} + T2^{-2J(s-\frac{1}{2})}\big)E_T(u) \end{aligned}$$

for $\max\{2, \frac{1}{3-\gamma}\} < r_1 < \frac{2}{3-\gamma}$ *and* $\frac{4}{\gamma-2} \leqslant r_2 < \infty$.

Proof. Multiplying (4.1) by $\partial_t u$ and integrating over x and t, we have

$$\begin{aligned} &\frac{1}{2}\big(\|u(t)\|_{H^1}^2 + \|u_t(t)\|_{L^2}^2\big) + \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} |x-y|^{-\gamma}u^2(y,t)u^2(x,t)\,dy\,dx \\ &\quad\leqslant \frac{1}{2}\big(\|u_0\|_{H^1}^2 + \|u_1\|_{L^2}^2\big) + \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} |x-y|^{-\gamma}u_0^2(y)u_0^2(x)\,dy\,dx \\ &\qquad + \left|\int_0^t\int_{\mathbb{R}^3} \mathcal{I}\big(v^2\big)(x,\tau)u(x,\tau)\partial_\tau u(x,\tau)\,dx\,d\tau\right| \\ &\qquad + 2\left|\int_0^t\int_{\mathbb{R}^3} \mathcal{I}(uv)(x,\tau)v(x,\tau)\partial_\tau u(x,\tau)\,dx\,d\tau\right| \\ &\qquad + \left|\int_0^t\int_{\mathbb{R}^3} \mathcal{I}\big(u^2\big)(x,\tau)v(x,\tau)\partial_\tau u(x,\tau)\,dx\,d\tau\right| \\ &\qquad + 2\left|\int_0^t\int_{\mathbb{R}^3} \mathcal{I}(uv)(x,\tau)u(x,\tau)\partial_\tau u(x,\tau)\,dx\,d\tau\right|. \end{aligned}$$

By taking the supremum over $t \leqslant T$, we have

$$\begin{aligned} H_T(u) &\lesssim H(u)(0) + \big\|\mathcal{I}\big(v^2\big)u\partial_t u\big\|_{L_T^1L^1} + \big\|\mathcal{I}(uv)v\partial_t u\big\|_{L_T^1L^1} \\ &\quad + \left|\int_0^T\int_{\mathbb{R}^3} \mathcal{I}\big(u^2\big)v\partial_t u\,dx\,dt\right| + \left|\int_0^T\int_{\mathbb{R}^3} \mathcal{I}(uv)u\partial_t u\,dx\,dt\right| \\ &\stackrel{\text{def}}{=} H(u)(0) + I + II + III + IV. \end{aligned} \tag{5.2}$$

The proof is broken down into the following several steps.

(i) Firstly, we estimate I and II. Making a similarly argument as (4.4) in the proof of Proposition 4.1, it can be obtained that

$$I \leqslant \left\| \mathcal{I}\left(v^2\right) u \right\|_{L^1_T L^2} \|u_t\|_{L^\infty_T L^2} \leqslant T^{\frac{4-\gamma}{3}} E_T(u) \|v\|^2_{X^\beta},$$

and then keeping in mind v has been estimated in Proposition 3.1, this together with (1.2) yields that

$$I \leqslant T^{\frac{4-\gamma}{3}} E_T(u) \mathcal{E}^2_{h,\beta} \leqslant T^{\frac{4-\gamma}{3}} E_T(u) 2^{-2J(s-\beta)} \mathcal{E}^2_s. \tag{5.3}$$

Arguing similarly, we easily get

$$II \leqslant T^{\frac{4-\gamma}{3}} E_T(u) 2^{-2J(s-\beta)} \mathcal{E}^2_s. \tag{5.4}$$

(ii) Secondly, we estimate the terms III and IV. As mentioned in the introduction, one can get the same type of estimate as above for the terms I and II, but that will lead to $s > \frac{\alpha}{2} + \frac{1}{2}$, which is worse than the exponent given in Theorem 1.1. To improve the lower bound on s, we have to utilize more precise estimate on III and IV.

We first split III and IV into two different pieces, respectively. One can write

$$v = v_F + B(v, v, v),$$

where v_F is its free part and the other one comes from nonlinear term. For the nonlinear part, it follows from (3.3) that

$$\left\| B(v, v, v) \right\|_{X^\alpha} \leqslant \|v\|_{X^\alpha} \|v\|^2_{X^{s_0}}.$$

This along with (4.5), one can see that

$$\begin{aligned} \left\| \mathcal{I}\left(u^2\right) B(v, v, v) u_t \right\|_{L^1_T L^1} &\leqslant \left\| \mathcal{I}\left(u^2\right) B(v, v, v) \right\|_{L^1_T L^2} \|u_t\|_{L^\infty_T L^2} \\ &\leqslant T^{\frac{5-\gamma}{3}} \|u\|^2_{L^\infty_T H^1} \left\| B(v, v, v) \right\|_{X^\alpha} \|u_t\|_{L^\infty_T L^2} \\ &\leqslant T^{\frac{5-\gamma}{3}} E_T(u)^{\frac{3}{2}} \|v\|_{X^\alpha} \|v\|^2_{X^{s_0}}. \end{aligned}$$

Moreover, we get by (1.2),

$$\left\| \mathcal{I}\left(u^2\right) B(v, v, v) u_t \right\|_{L^1_T L^1} \leqslant T^{\frac{5-\gamma}{3}} E_T^{\frac{3}{2}}(u) 2^{-J(3s-\alpha-2s_0)} \mathcal{E}^3_s. \tag{5.5}$$

By the same way as leading to (5.5), we easily infer that

$$\left\| \mathcal{I}\left(u B(v, v, v)\right) u u_t \right\|_{L^1_T L^1} \leqslant T^{\frac{5-\gamma}{3}} E_T^{\frac{3}{2}}(u) 2^{-J(3s-\alpha-2s_0)} \mathcal{E}^3_s. \tag{5.6}$$

Thus, it is sufficient to estimate these terms including free part v_F since (5.5) and (5.6). The following lemma gives estimates for the nonlinearity including free part v_F.

Lemma 5.1. *Let v_F be a solution of the free Klein–Gordon equation, and u be such that $E_T(u) \lesssim 2^{2J(1-s)}$. Then, for* $\max\{2, \frac{1}{3-\gamma}\} < r_1 < \frac{2}{3-\gamma}$ *and* $\frac{4}{\gamma-2} \leqslant r_2 < \infty$,

$$\left|\int_0^T \int_{\mathbb{R}^3} \mathcal{I}(u^2) v_F u_t \, dx\, dt\right| \lesssim \big(T^{\frac{1}{2}+\frac{1}{r_1}} 2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r_1})]} + T^{\frac{1}{2}+\frac{1}{r_2}} 2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r_2})]} + T 2^{-2J(s-\frac{1}{2})}\big) E_T(u), \tag{5.7}$$

$$\left|\int_0^T \int_{\mathbb{R}^3} \mathcal{I}(u v_F) u u_t \, dx\, dt\right| \lesssim T^{\frac{1}{2}+\frac{1}{r_2}} 2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r_2})]} E_T(u). \tag{5.8}$$

Hence these together with (5.5)–(5.6) yield that

$$\begin{aligned} III + IV \lesssim{}& T^{\frac{5-\gamma}{3}} E_T(u) 2^{-J[4s-\alpha-2s_0-1]} \mathcal{E}_s^4 \\ &+ \big(T^{\frac{1}{2}+\frac{1}{r_1}} 2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r_1})]} + T^{\frac{1}{2}+\frac{1}{r_2}} 2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r_2})]} + T 2^{-2J(s-\frac{1}{2})}\big) E_T(u). \end{aligned} \tag{5.9}$$

Therefore, we complete the proof of Proposition 5.1 provided that we had proved Lemma 5.1, whose proof is postponed in the last section. □

6. Proof of Theorem 1.1

Since the Cauchy problem (1.1) is split into Eq. (3.1) which is globally well-posed by choosing J enough to make $\mathcal{E}_{h,s_0} < \varepsilon_0$ and Eq. (4.1) which is locally well-posed (see Proposition 3.1 and Proposition 4.1), we have to show that the local solution to Eq. (4.1) can be extended globally.

Let us denote T_J^* the maximum time of existence in Proposition 4.1. Theorem 1.1 will be proved if

$$\lim_{J \to +\infty} T_J^* = +\infty.$$

Let us consider T_J the supremum of the $T < T_J^*$ such that

$$E_T(u) \leqslant 2C H(u)(0).$$

Thus, for any $T < T_J$, Proposition 5.1 gives us that

$$\begin{aligned} E_T(u) \leqslant H(u)(0)\big(&C + C_1 T^{\frac{4-\gamma}{3}} 2^{-2J(s-\beta)} \mathcal{E}_s^2 + C_2 T^{\frac{5-\gamma}{3}} 2^{-J(4s-\alpha-2s_0-1)} \mathcal{E}_s^4 \\ &+ C_3 T^{\frac{1}{2}+\frac{1}{r_1}} 2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r_1})]} \mathcal{E}_s^2 + C_4 T 2^{-2J(s-\frac{1}{2})} \mathcal{E}_s^2 \\ &+ C_5 T^{\frac{1}{2}+\frac{1}{r_2}} 2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r_2})]} \mathcal{E}_s^2\big). \end{aligned}$$

By the assumption of Theorem 1.1 $s > \frac{\gamma}{4}$, one easily verifies that

$$s > \max\left\{\beta, \frac{\alpha}{4} + \frac{s_0}{2} + \frac{1}{4}, \frac{1}{2}, \frac{\gamma}{2} - \frac{3}{4} + \frac{1}{2r_1}, \frac{\gamma}{2} - \frac{3}{4} + \frac{1}{2r_2}\right\}$$

if choosing r_1 sufficiently close to $\frac{2}{3-\gamma}$ and r_2 large enough. We infer that $T_J \geqslant \widetilde{T}_J$ if we choose $\widetilde{T}_J$ such that

$$\widetilde{T}_J \stackrel{\text{def}}{=} \min\left\{\left(\frac{2^{2J(s-\beta)}}{5C_1\mathcal{E}_s^2}\right)^{\frac{3}{4-\gamma}}, \left(\frac{2^{4J(s-\frac{1}{4}\alpha-\frac{s_0}{2}-\frac{1}{4})}}{5C_2\mathcal{E}_s^4}\right)^{\frac{3}{5-\gamma}}, \left(\frac{2^{2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r_1})]}}{5C_3\mathcal{E}_s^2}\right)^{\frac{2r_1}{r_1+2}},\right.$$
$$\left.\frac{2^{2J(s-\frac{1}{2})}}{5C_4\mathcal{E}_s^2}, \left(\frac{2^{2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r_2})]}}{5C_5\mathcal{E}_s^2}\right)^{\frac{2r_2}{r_2+2}}\right\}.$$

By the definition of T_J, we get $T_J^* \geqslant \widetilde{T}_J$. Obviously, $\widetilde{T}_J$ tends to infinity when J tend to infinity. This completes the proof of Theorem 1.1.

7. Proof of Lemma 5.1

In order to make conveniently use of the precise Strichartz estimate on which mostly the following proof relies, we begin this section by introducing a family of balls of center $(\xi_\nu^{j,k})_{\nu\in\Lambda_{j,k}}$ of radius 2^k and a function $\chi \in \mathcal{C}_c^\infty(B(0,1))$ such that for $j \geqslant 0$,

$$\forall \xi \in 2^j\mathcal{C}, \quad \sum_{\nu\in\Lambda_{j,k}} \chi\big(2^{-k}\big(\xi-\xi_\nu^{j,k}\big)\big) = 1 \quad \text{and} \quad C_0^{-1} \leqslant \sum_{\nu\in\Lambda_{j,k}} \chi^2\big(2^{-k}\big(\xi-\xi_\nu^{j,k}\big)\big) \leqslant C_0.$$

Let us define that, for some constant c,

$$\triangle_{j,k}^\nu a \stackrel{\text{def}}{=} \mathcal{F}^{-1}\big(\big(\varphi\big(2^{-j}\xi\big)\chi\big(2^{-k}\big(\xi-\xi_\nu^{j,k}\big)\big)\big)\hat{a}(\xi)\big),$$
$$\widetilde{\triangle_{j,k}^\nu} a \stackrel{\text{def}}{=} \mathcal{F}^{-1}\big(\big(\widetilde{\varphi}\big(2^{-j}\xi\big)\chi\big(c2^{-k}\big(\xi+\xi_\nu^{j,k}\big)\big)\big)\hat{a}(\xi)\big).$$

As the support of the Fourier transform of a product belongs to the sum of the support of each Fourier transform, we have

$$\triangle_j a = \sum_{\nu\in\Lambda_{j,k}} \triangle_{j,k}^\nu a, \qquad \triangle_j b = \sum_{\nu'\in\Lambda_{j,k}} \triangle_{j,k}^{\nu'} b.$$

In view of the fact that if $k \leqslant j-2$,

$$\triangle_k \sum_{\nu,\nu'\in\Lambda_{j,k}} \triangle_{j,k}^\nu a \triangle_{j,k}^{\nu'} b$$

is vanishing when $\xi_\nu^{j,k}$ is close to $\xi_{\nu'}^{j,k}$, without loss of generality, we can write

$$\triangle_k(\triangle_j a \triangle_j b) \approx \triangle_k \sum_{\nu\in\Lambda_{j,k}} \triangle_{j,k}^\nu a \widetilde{\triangle_{j,k}^\nu} b. \tag{7.1}$$

For the sake of convenience, we also fix the notation in this section that, for $0 \neq f(t,x) \in L_T^2 L^2$,

$$c_k = 2^{k\sigma}\big(\|\triangle_k v_0\|_{L^2} + 2^{-k}\|\triangle_k v_1\|_{L^2}\big)\mathcal{E}_{h,\sigma}^{-1}, \qquad \tilde{c}_k = \frac{\|\triangle_k f\|_{L_T^2 L_x^2}}{\|f\|_{L_T^2 L^2}}$$

with $\sigma = 1/2 + 1/r$ for $2 \leqslant r < \infty$.

Proof of Lemma 5.1. We first prove (5.7). In view of the fact that $\hat{v}_F$ only has high frequencies, Bony's decomposition implies that there exists constant N_0 such that

$$\mathcal{I}\big(u^2\big)v_F u_t = \sum_{j\geqslant J-N_0} S_{j+2}v_F\triangle_j\mathcal{I}\big(u^2\big)u_t + \sum_{j\geqslant J-N_0} S_{j-1}\mathcal{I}\big(u^2\big)\triangle_j v_F u_t. \tag{7.2}$$

Since the negative derivative $\mathcal{I}$ acts on the high frequency for the former term while on the low frequency for the latter one, the first term is much better than the second one. We shall estimate the first term by using merely the Hölder inequality, Bernstein inequality and classical Strichartz estimates. Firstly, we see that, for $2\leqslant r<\infty$,

$$\begin{aligned}\sum_{j\geqslant J-N_0}\big\|S_{j+2}v_F\triangle_j\mathcal{I}\big(u^2\big)\big\|_{L_x^2} &\lesssim \sum_{j\geqslant J-N_0}\sum_{j'\leqslant j+1}\|\triangle_{j'}v_F\|_{L_x^\infty}\big\|\triangle_j\mathcal{I}\big(u^2\big)\big\|_{L_x^2}\\ &\lesssim \sum_{j\geqslant J-N_0}\sum_{j'\leqslant j+1}2^{j'\frac{3}{r}}\|\triangle_{j'}v_F\|_{L^r}2^{j(\gamma-\frac{7}{2})}\|u\|^2_{L_T^\infty H^1}.\end{aligned}$$

The Bernstein inequality and (2.1) in Proposition 2.1 with $\frac{1}{p}+\frac{1}{r}=\frac{1}{2}$ for all $2\leqslant r<\infty$ imply that

$$\begin{aligned}&\Big\|\sum_{j\geqslant J-N_0}\big\|S_{j+2}v_F\triangle_j\mathcal{I}\big(u^2\big)\big\|_{L_x^2}\Big\|_{L_T^1}\\ &\quad\lesssim T^{1-\frac{1}{p}}\|u\|^2_{L_T^\infty H^1}\sum_{j\geqslant J-N_0}\sum_{j'\leqslant j+1}2^{j'\frac{3}{r}}\|\triangle_{j'}v_F\|_{L_T^pL^r}2^{j(\gamma-\frac{7}{2})}\\ &\quad\lesssim T^{1-\frac{1}{p}}\|u\|^2_{L_T^\infty H^1}\sum_{j\geqslant J-N_0}2^{j(\gamma-\frac{7}{2})}\sum_{j'\leqslant j+1}2^{j'(\frac{3}{2}-\frac{1}{p}-\sigma)}2^{j'\sigma}\big(\|\triangle_{j'}v_0\|_{L^2}+2^{-j'}\|\triangle_{j'}v_1\|_{L^2}\big).\end{aligned}$$

The right hand of the above inequality can be controlled by

$$T^{1-\frac{1}{p}}\|u\|^2_{L_T^\infty H^1}\sum_{j\geqslant J-N_0}2^{j(\gamma-\frac{7}{2})}\sum_{j'\leqslant j+1}2^{\frac{j'}{2}}c_{j'}\mathcal{E}_{h,\sigma}$$

and moreover it follows from (1.2), the definition of $\mathcal{E}_{h,\sigma}$ and Sobolev embedding that

$$\begin{aligned}\Big\|\sum_{j\geqslant J-N_0}\big\|S_{j+2}v_F\triangle_j\mathcal{I}\big(u^2\big)u_t\big\|_{L_x^1}\Big\|_{L_T^1} &\lesssim T^{1-\frac{1}{p}}\mathcal{E}_{h,\sigma}\|u\|^2_{L_T^\infty H^1}\sum_{j\geqslant J-N_0}2^{j(\gamma-3)}\|u_t\|_{L_T^\infty L^2}\\ &\lesssim T^{\frac{1}{2}+\frac{1}{r}}2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r})]}\mathcal{E}_s^2\|u\|^2_{L_T^\infty(H^1)},\end{aligned} \tag{7.3}$$

for $\frac{4}{\gamma-2}\leqslant r<\infty$.

Let us estimate the second term in (7.2) by the precise Strichartz estimates. Since this term contains that the negative derivative acts on the low frequency part $S_{j-1}(u^2)$, it leads to our new parameter $r<\frac{2}{3-\gamma}$ by some technique difficulties. Noting that Fourier–Plancherel formula and Hölder's inequality, we can see that

$$\begin{aligned}
&\sum_{j\geqslant J-N_0}\int_0^T\int_{\mathbb{R}^3} S_{j-1}\mathcal{I}\big(u^2\big)\triangle_j v_F u_t\,dx\,dt \\
&\lesssim \int \sum_{j\geqslant J-N_0}\sum_{-1\leqslant k\leqslant j-2} \triangle_k\mathcal{I}\big(u^2\big)\triangle_j v_F\triangle_j u_t\,dx\,dt \\
&\approx \sum_{k\geqslant -1}\int \triangle_k\mathcal{I}\big(u^2\big)\triangle_k \sum_{k\leqslant j-2,\,J-N_0\leqslant j}(\triangle_j v_F\triangle_j u_t)\,dx\,dt \\
&\lesssim \big\|u^2\big\|_{L^\infty_T(B^{\frac{1}{2}}_{2,1})}\int_0^T \sup_{k\geqslant -1} 2^{k(\gamma-\frac{7}{2})}\Big\|\triangle_k \sum_{k\leqslant j-2,\,J-N_0\leqslant j}(\triangle_j v_F\triangle_j u_t)\Big\|_{L^2}\,dt. \qquad (7.4)
\end{aligned}$$

On one hand, we have

$$\begin{aligned}
\int_0^T\Big\|\triangle_{-1}\sum_{J-N_0\leqslant j}(\triangle_j v_F\triangle_j u_t)\Big\|_{L^2}\,dt &\lesssim \sum_{j\geqslant J-N_0}\|\triangle_j v_F\triangle_j u_t\|_{L^1_T L^1} \\
&\leqslant T^{\frac{1}{2}}\sum_{j\geqslant J-N_0}\|\triangle_j v_F\|_{L^\infty_T L^2}\|\triangle_j u_t\|_{L^2_T L^2} \\
&\leqslant T^{\frac{1}{2}}\sum_{j\geqslant J-N_0} 2^{-sj}c_j\tilde{c}_j\mathcal{E}_{h,s}\|u_t\|_{L^2_{t,x}}.
\end{aligned}$$

If (7.4) is controlled by the term at $k=-1$, we can see that

$$\sum_{j\geqslant J-N_0}\big\|S_{j-1}\mathcal{I}\big(u^2\big)\triangle_j v_F\big\|_{L^2_T L^2}\lesssim T^{\frac{1}{2}}2^{-2J(s-\frac{1}{2})}\mathcal{E}_s^2\|u\|_{L^\infty_T(H^1)}. \qquad (7.5)$$

On the other hand, one denotes $g_k := \triangle_k\sum_{k\leqslant j-2}(\triangle_j v_F\triangle_j u_t)$ to estimate

$$\sum_{k\geqslant 0}2^{k(\gamma-\frac{7}{2}+\frac{3}{r})}\|g_k\|_{L^1_T L^{\frac{2r}{r+2}}}.$$

Let us write that

$$g_k=\sum_{k\leqslant j-2}\triangle_k\sum_{\nu\in\Lambda_{j,k}}\triangle^{\nu}_{j,k}v_F\triangle_j u_t.$$

As the support of the Fourier transform of a product is included in the sum of the support of each Fourier transform, we obtain

$$g_k=\sum_{k\leqslant j-2}\triangle_k\sum_{\nu\in\Lambda_{j,k}}\triangle^{\nu}_{j,k}v_F\widetilde{\triangle^{\nu}_{j,k}}u_t,$$

as well as in (7.1). Using the Hölder inequality, we get

$$\|g_k\|_{L^{\frac{2r}{r+2}}} \leqslant \sum_{k\leqslant j-2}\sum_{\nu\in\Lambda_{j,k}} \big\|\triangle_{j,k}^{\nu} v_F\big\|_{L^r}\big\|\widetilde{\triangle_{j,k}^{\nu}} u_t\big\|_{L^2}$$

and the Cauchy–Schwarz inequality and the L^2 quasi-orthogonality properties yield that

$$\begin{aligned}\|g_k\|_{L^{\frac{2r}{r+2}}} &\leqslant \sum_{k\leqslant j-2}\Big(\sum_{\nu\in\Lambda_{j,k}}\big\|\triangle_{j,k}^{\nu} v_F\big\|_{L^r}^2\Big)^{\frac12}\Big(\sum_{\nu\in\Lambda_{j,k}}\big\|\widetilde{\triangle_{j,k}^{\nu}} u_t\big\|_{L^2}^2\Big)^{\frac12}\\ &\leqslant \sum_{k\leqslant j-2}\Big(\sum_{\nu\in\Lambda_{j,k}}\big\|\triangle_{j,k}^{\nu} v_F\big\|_{L^r}^2\Big)^{\frac12}\|\triangle_j u_t\|_{L^2}. \end{aligned} \tag{7.6}$$

Precise Strichartz estimate implies that, for $\frac1p+\frac1r=\frac12$ with $2\leqslant r<\infty$,

$$\begin{aligned}\|g_k\|_{L^1_T(L^{\frac{2r}{r+2}})} \leqslant{}& T^{\frac12-\frac1p}\sum_{0\leqslant k\leqslant j-2} 2^{(k-j)(\frac12-\frac1r)}2^{j(\frac32-\frac3r-\frac1p)}\\ &\times\Big(\Big(\sum_{\nu\in\Lambda_{j,k}}\big\|\triangle_{j,k}^{\nu} v_0\big\|_{L^2}^2\Big)^{\frac12}+2^{-j}\Big(\sum_{\nu\in\Lambda_{j,k}}\big\|\triangle_{j,k}^{\nu} v_1\big\|_{L^2}^2\Big)^{\frac12}\Big)\|\triangle_j u_t\|_{L^2_{t,x}}\end{aligned}$$

and observe the quasi-orthogonality properties again, this can be dominated by

$$T^{\frac12-\frac1p}\sum_{0\leqslant k\leqslant j-2} 2^{(k-j)(\frac12-\frac1r)}2^{j(\frac32-\frac3r-\frac1p)}\big(\|\triangle_j v_0\|_{L^2}+2^{-j}\|\triangle_j v_1\|_{L^2}\big)\|\triangle_j u_t\|_{L^2_{t,x}}.$$

Keeping the definitions of $\mathcal{E}_{h,\sigma}$ and c_j in mind, one can see that

$$\|g_k\|_{L^1_T(L^{\frac{2r}{r+2}})} \lesssim T^{\frac12-\frac1p}2^{k(\frac12-\frac1r)}\sum_{k\leqslant j-2}2^{-\frac{2j}{r}}c_j\tilde c_j\mathcal{E}_{h,\sigma}E_T^{\frac12}(u)\lesssim T^{\frac12-\frac1p}2^{k(\frac12-\frac3r)}\mathcal{E}_{h,\sigma}E_T^{\frac12}(u).$$

Therefore, we get that

$$\sum_{k\geqslant 0}2^{k(\gamma-\frac72+\frac3r)}\|g_k\|_{L^1_T(L^{\frac{2r}{r+2}})}\lesssim T^{\frac12-\frac1p}\sum_{k\geqslant0}2^{k(\gamma-3)}\mathcal{E}_{h,\sigma}E_T^{\frac12}(u)$$

which implies nothing but

$$\left|\int_0^T\int_{\mathbb{R}^3}\sum_{j\geqslant J-N_0}S_{j-1}\mathcal{I}\big(u^2\big)\triangle_j v_F u_t\,dx\,dt\right|\lesssim T^{1-\frac1p}\mathcal{E}_{h,\sigma}E_T^{\frac32}(u). \tag{7.7}$$

Finally, we get that, for $\frac{4}{\gamma-2}\leqslant r<\infty$,

$$\left|\int_0^T\int_{\mathbb{R}^3}\sum_{j\geqslant J-N_0}S_{j-1}\mathcal{I}\big(u^2\big)\triangle_j v_F u_t\,dx\,dt\right|\lesssim T^{\frac12+\frac1r}2^{-2J[s-(\frac34+\frac1{2r})]}\mathcal{E}_s^2E_T(u).$$

However, although the r ranges $\frac{4}{\gamma-2} \leqslant r < \infty$, the above estimate still needs $s > \frac{3}{4}$ to continue our proof. If we only consider the high frequency $k \geqslant J$, the (7.7) can be modified by

$$\left|\int_0^T \int_{\mathbb{R}^3} \sum_{j \geqslant J-N_0} S_{j-1}\mathcal{I}\big(u^2\big)\triangle_j v_F u_t \, dx\, dt\right| \lesssim T^{1-\frac{1}{p}} 2^{J(\gamma-3)} \mathcal{E}_{h,\sigma} E_T^{\frac{3}{2}}(u) \tag{7.8}$$

and then we can obtain a better result

$$\left|\int_0^T \int_{\mathbb{R}^3} \sum_{j \geqslant J-N_0} S_{j-1}\mathcal{I}\big(u^2\big)\triangle_j v_F u_t \, dx\, dt\right| \lesssim T^{\frac{1}{2}+\frac{1}{r}} 2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r})]} \mathcal{E}_s^2 E_T(u),$$

which implies the bad influence comes from the low frequency part and this is consist of the effect of negative derivative acts on the low frequency. But if we choose $\tilde{\sigma} = \gamma - \frac{5}{2} + \frac{1}{r}$ instead of σ, we can improve (7.8), at cost of restricting r such that $\max\{2, \frac{1}{3-\gamma}\} < r < \frac{2}{3-\gamma}$ while not $2 \leqslant r < \infty$. Now we turn to details. It follows from similar argument that

$$\|g_k\|_{L_T^1(L^{\frac{2r}{r+2}})} \lesssim T^{\frac{1}{2}-\frac{1}{p}} 2^{k(\frac{1}{2}-\frac{1}{r})} \sum_{k \leqslant j-2} 2^{-j(\gamma-3+\frac{2}{r})} c_j \tilde{c}_j \mathcal{E}_{h,\tilde{\sigma}} E_T^{\frac{1}{2}}(u)$$

where $\tilde{\sigma} = \gamma - \frac{5}{2} + \frac{1}{r}$ with $\frac{1}{r} < 3 - \gamma < \frac{2}{r}$. We get

$$\sum_{k \geqslant 0} 2^{k(\gamma-\frac{7}{2}+\frac{3}{r})} \|g_k\|_{L_T^1(L^{\frac{2r}{r+2}})} \lesssim T^{\frac{1}{2}-\frac{1}{p}} \sum_{k \geqslant 0} \sum_{k \leqslant j-2} 2^{(k-j)(\gamma-3+\frac{2}{r})} c_j \tilde{c}_j \mathcal{E}_{h,\tilde{\sigma}} E_T^{\frac{1}{2}}(u)$$

which implies nothing but

$$\left|\int_0^T \int_{\mathbb{R}^3} \sum_{j \geqslant J-N_0} S_{j-1}\mathcal{I}\big(u^2\big)\triangle_j v_F u_t \, dx\, dt\right| \lesssim T^{1-\frac{1}{p}} \mathcal{E}_{h,\tilde{\sigma}} E_T^{\frac{3}{2}}(u)$$

by Young's inequality. Note that $\tilde{\sigma} \leqslant \frac{\gamma}{4} < s$ when r sufficiently closes to $\frac{2}{3-\gamma}$, therefore

$$\left|\int_0^T \int_{\mathbb{R}^3} \sum_{j \geqslant J-N_0} S_{j-1}\mathcal{I}\big(u^2\big)\triangle_j v_F u_t \, dx\, dt\right| \lesssim T^{\frac{1}{2}+\frac{1}{r}} 2^{-2J(s-\frac{\tilde{\sigma}+1}{2})} \mathcal{E}_s^2 E_T(u).$$

Combining this with (7.3) and (7.5), we complete the proof of (5.7) by obtaining

$$\begin{aligned}&\left|\int_0^T \int_{\mathbb{R}^3} \sum_{j \geqslant J-N_0} S_{j-1}\mathcal{I}\big(u^2\big)\triangle_j v_F u_t \, dx\, dt\right| \\ &\quad \lesssim \big(T^{\frac{1}{2}+\frac{1}{r_1}} 2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r_1})]} + T^{\frac{1}{2}+\frac{1}{r_2}} 2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r_2})]} + T 2^{-2J(s-\frac{1}{2})}\big) \mathcal{E}_s^2 E_T(u)\end{aligned}$$

with $\max\{2, \frac{1}{3-\gamma}\} < r_1 < \frac{2}{3-\gamma}$ and $\frac{4}{\gamma-2} \leqslant r_2 < \infty$.

We secondly prove (5.8) which is different from (5.7). To this end, we need to make Bony's decomposition more than once and establish a commutator estimate, which helps us to complete our proof. In view of the fact that $\hat{v}_F$ only has high frequencies again, it follows from Bony's decomposition that there exists N_0 such that

$$\mathcal{I}(uv_F)uu_t = \sum_{j\geqslant J-N_0} \mathcal{I}(S_{j+2}v_F\triangle_j u)uu_t + \sum_{j\geqslant J-N_0} \mathcal{I}(\triangle_j v_F S_{j-1}u)uu_t \overset{\text{def}}{=} I + II. \tag{7.9}$$

In order to estimate the term I, we split it into two pieces with $N_1 \gg N_0 > 0$,

$$\begin{aligned} I &= \sum_{j\geqslant J-N_0}\sum_{k} uu_t\triangle_k\mathcal{I}(S_{j+2}v_F\triangle_j u) \\ &= \sum_{j\geqslant J-N_0}\sum_{k\leqslant J-N_1} uu_t\triangle_k\mathcal{I}(S_{j+2}v_F\triangle_j u) + \sum_{j\geqslant J-N_0}\sum_{k\geqslant J-N_1} uu_t\triangle_k\mathcal{I}(S_{j+2}v_F\triangle_j u) \\ &\overset{\text{def}}{=} I_1 + I_2. \end{aligned}$$

The estimate of I_1 is broken down into the following two cases.

Case 1. $2 < \gamma \leqslant \frac{5}{2}$.

In this case, to our purpose, we obtain the following coarse estimate by Hölder's inequality

$$\begin{aligned} \|I_1\|_{L^1_x} &\lesssim \sum_{j\geqslant J-N_0}\sum_{k\leqslant J-N_1} \big\|\triangle_k\mathcal{I}(S_{j+2}v_F\triangle_j u)\big\|_{L^3}\|u\|_{L^\infty_T L^6}\|u_t\|_{L^\infty_T L^2} \\ &\lesssim \sum_{j\geqslant J-N_0}\sum_{k\leqslant J-N_1} 2^{k(\gamma-2)}\big\|\triangle_k(S_{j+2}v_F\triangle_j u)\big\|_{L^{\frac{3}{2}}} E_T(u) \\ &\lesssim \sum_{j\geqslant J-N_0}\sum_{k\leqslant J-N_1} 2^{k(\gamma-2)}\|S_{j+2}v_F\|_{L^6}\|\triangle_j u\|_{L^2} E_T(u) \\ &\lesssim \sum_{k\leqslant J-N_1} 2^{k(\gamma-2)} \sum_{j\geqslant J-N_0} 2^{-j}2^j\|\triangle_j u\|_{L^2}\sum_{j'\leqslant j}\|\triangle_{j'}v_F\|_{L^6}E_T(u). \end{aligned}$$

Choosing (p,r) such that $\frac{1}{p}+\frac{1}{r}=\frac{1}{2}$ with $2\leqslant r\leqslant 6$, the Strichartz estimate yields

$$\begin{aligned} \|I_1\|_{L^1_T L^1_x} &\lesssim T^{1-\frac{1}{p}} \sum_{k\leqslant J-N_1} 2^{k(\gamma-2)} \sum_{j\geqslant J-N_0} 2^{-j}\sum_{j'\leqslant j} 2^{j'(\frac{3}{r}-\frac{3}{6})}2^{j'(\frac{3}{2}-\frac{3}{r}-\frac{1}{p})} \\ &\quad\times \big(\|\triangle_{j'}v_0\|_{L^2} + 2^{-j'}\|\triangle_{j'}v_1\|_{L^2}\big)E_T^{\frac{3}{2}}(u). \end{aligned}$$

Arguing similarly as before it yields that

$$\begin{aligned} \|I_1\|_{L^1_T L^1_x} &\lesssim T^{\frac{1}{2}+\frac{1}{r}} \sum_{k\leqslant J-N_1} 2^{k(\gamma-2)} \sum_{j\geqslant J-N_0} 2^{-j}\sum_{j'\leqslant j} 2^{\frac{j'}{r}} c_{j'}\mathcal{E}_{h,1/2}E_T^{\frac{3}{2}}(u) \\ &\lesssim T^{\frac{1}{2}+\frac{1}{r}} 2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r})]}\mathcal{E}_s^2 E_T(u) \end{aligned}$$

with $2 \leqslant r \leqslant 6$. If choose $r = 6$, one can easily check that $\frac{\gamma}{4} > \frac{\gamma}{2} - \frac{3}{4} + \frac{1}{2r}$ when $2 < \gamma \leqslant \frac{5}{2}$. Although this result is enough for us to prove the main theorem, we want to improve the result for this term by loosen the upper bound of r from 6 to ∞ through the precise Strichartz estimate. Arguing similarly as before, we have

$$\|I_1\|_{L_x^1} \lesssim \sum_{j \geqslant J-N_0} \sum_{k \leqslant J-N_1} \left\| \triangle_k \mathcal{I}(S_{j+2} v_F \triangle_j u) \right\|_{L^3} \|u\|_{L_T^\infty L^6} \|u_t\|_{L_T^\infty L^2}$$

$$\lesssim \sum_{j \geqslant J-N_0} \sum_{k \leqslant J-N_1} 2^{k(\gamma-3)} 2^{k\frac{r+6}{2r}} \left\| \triangle_k (S_{j+2} v_F \triangle_j u) \right\|_{L^{\frac{2r}{r+2}}} E_T(u).$$

Since the Fourier transform of $S_{j-1} v_F \triangle_j u$ was supported in $2^j \mathcal{C}$ and $k \ll j$, $\triangle_k (S_{j-1} v_F \triangle_j u)$ vanishes which implies $\triangle_k (S_{j+2} v_F \triangle_j u) = \triangle_k (\widetilde{\triangle}_j v_F \triangle_j u)$. As the support of the Fourier transform of a product is included in the sum of the support of each Fourier transform, we also have

$$\triangle_k (\widetilde{\triangle}_j v_F \triangle_j u) = \triangle_k \left(\sum_{\nu, \nu' \in \Lambda_{j,k}} \triangle_{j,k}^{\nu} v_F \triangle_{j,k}^{\nu'} u \right) = \triangle_k \left(\sum_{\nu \in \Lambda_{j,k}} \triangle_{j,k}^{\nu} v_F \widetilde{\triangle_{j,k}^{\nu}} u \right).$$

Choosing (p, r) such that $\frac{1}{p} + \frac{1}{r} = \frac{1}{2}$ for $2 \leqslant r < \infty$, it follows from the Hölder inequality and L^2 quasi-orthogonality properties that

$$\left\| \triangle_k (S_{j+2} v_F \triangle_j u) \right\|_{L_T^1 (L^{\frac{2r}{r+2}})} \lesssim \left\| \sum_{\nu \in \Lambda_{j,k}} \left\| \triangle_{j,k}^{\nu} v_F \right\|_{L^r} \left\| \widetilde{\triangle_{j,k}^{\nu}} u \right\|_{L^2} \right\|_{L_T^1}$$

$$\lesssim T^{\frac{1}{2} - \frac{1}{p}} \left(\sum_{\nu \in \Lambda_{j,k}} \left\| \triangle_{j,k}^{\nu} v_F \right\|_{L^p L^r}^2 \right)^{\frac{1}{2}} \left\| \left(\sum_{\nu \in \Lambda_{j,k}} \left\| \triangle_{j,k}^{\nu} u \right\|_{L^2}^2 \right)^{\frac{1}{2}} \right\|_{L_T^2}$$

$$\lesssim T^{\frac{1}{2} - \frac{1}{p}} \left(\sum_{\nu \in \Lambda_{j,k}} \left\| \triangle_{j,k}^{\nu} v_F \right\|_{L^p L^r}^2 \right)^{\frac{1}{2}} \|\triangle_j u\|_{L_T^2 L^2}.$$

Then the precise Strichartz estimate yields that

$$\|I_1\|_{L_T^1 L_x^1} \lesssim T^{\frac{1}{2} - \frac{1}{p}} \sum_{k \leqslant J-N_1} 2^{k(\gamma-3)} 2^{k\frac{r+6}{2r}} \sum_{j \geqslant J-N_0} 2^{-j} 2^j \|\triangle_j u\|_{L_T^2 L^2} 2^{(k-j)(\frac{1}{2} - \frac{1}{r})} 2^{j(\frac{3}{2} - \frac{3}{r} - \frac{1}{p})}$$

$$\times \left(\left(\sum_{\nu \in \Lambda_{j,k}} \left\| \triangle_{j,k}^{\nu} v_0 \right\|_{L^2}^2 \right)^{\frac{1}{2}} + 2^{-j} \left(\sum_{\nu \in \Lambda_{j,k}} \left\| \triangle_{j,k}^{\nu} v_1 \right\|_{L^2}^2 \right)^{\frac{1}{2}} \right) E_T(u).$$

By the L^2-quasi-orthogonality properties, it gives that

$$\|I_1\|_{L_T^1 L_x^1} \lesssim T^{\frac{1}{2} - \frac{1}{p}} \sum_{k \leqslant J-N_1} 2^{k(\gamma-3)} 2^{k\frac{2r+4}{2r}} \sum_{j \geqslant J-N_0} 2^{-j} 2^j \|\triangle_j u\|_{L_T^2 L^2} 2^{j(\frac{1}{2} - \frac{1}{r})}$$

$$\times \left(\|\triangle_j v_0\|_{L^2} + 2^{-j} \|\triangle_j v_1\|_{L^2} \right) E_T(u).$$

Utilizing the technique as before yields that

$$\begin{aligned}\|I_1\|_{L^1_T L^1_x} &\lesssim T^{1-\frac{1}{p}} \sum_{k\leqslant J-N_1} 2^{k(\gamma-2+\frac{2}{r})} \sum_{j\geqslant J-N_0} 2^{-j(1+\frac{2}{r})} c_j \mathcal{E}_{h,\sigma} E_T^{\frac{3}{2}}(u)\\ &\lesssim T^{\frac{1}{2}+\frac{1}{r}} 2^{J(\gamma-3)} \mathcal{E}_{h,\sigma} E_T^{\frac{3}{2}}(u)\\ &\lesssim T^{\frac{1}{2}+\frac{1}{r}} 2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r})]} \mathcal{E}_s^2 E_T(u),\end{aligned}$$

with $\frac{4}{\gamma-2} \leqslant r < \infty$.

Case 2. $\frac{5}{2} < \gamma < 3$.

In the this case, the fact $\gamma - \frac{5}{2} > 0$ helps us to obtain the desirable result easily. Arguing similarly as before, we have

$$\begin{aligned}\|I_1\|_{L^1_x} &\lesssim \sum_{j\geqslant J-N_0} \sum_{k\leqslant J-N_1} \left\| \triangle_k \mathcal{I}(S_{j+2} v_F \triangle_j u) \right\|_{L^3} \|u\|_{L^\infty_T L^6} \|u_t\|_{L^\infty_T L^2}\\ &\lesssim \sum_{j\geqslant J-N_0} \sum_{k\leqslant J-N_1} 2^{k(\gamma-3)} 2^{3k(\frac{1}{2}-\frac{1}{3})} \left\| \triangle_k (S_{j+2} v_F \triangle_j u) \right\|_{L^2} E_T(u)\\ &\lesssim \sum_{j\geqslant J-N_0} \sum_{k\leqslant J-N_1} 2^{k(\gamma-\frac{5}{2})} \|S_{j+2} v_F\|_{L^\infty} \|\triangle_j u\|_{L^2} E_T(u).\end{aligned}$$

Choosing (p,r) such that $\frac{1}{p} + \frac{1}{r} = \frac{1}{2}$ with $2 \leqslant r < \infty$, the Strichartz estimate yields

$$\begin{aligned}\|I_1\|_{L^1_T L^1_x} &\lesssim T^{1-\frac{1}{p}} \sum_{k\leqslant J-N_1} 2^{k(\gamma-\frac{5}{2})} \sum_{j\geqslant J-N_0} 2^{-j} \sum_{j'\leqslant j} 2^{\frac{j'}{2}} c_{j'} \mathcal{E}_{h,\sigma} E_T^{\frac{3}{2}}(u)\\ &\lesssim T^{1-\frac{1}{p}} 2^{J(\gamma-\frac{5}{2})} \sum_{j\geqslant J-N_0} 2^{-\frac{j}{2}} \mathcal{E}_{h,\sigma} E_T^{\frac{3}{2}}(u)\\ &\lesssim T^{\frac{1}{2}+\frac{1}{r}} 2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r})]} \mathcal{E}_s^2 E_T^{\frac{3}{2}}(u).\end{aligned}$$

Combining these two cases, we have shown that

$$\|I_1\|_{L^1_T} \lesssim T^{\frac{1}{2}+\frac{1}{r}} 2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r})]} \mathcal{E}_s^2 E_T(u) \tag{7.10}$$

with $\frac{4}{\gamma-2} \leqslant r < \infty$. To control $\|I\|_{L^1_T L^1_x}$, it remains to estimate $\|I_2\|_{L^1_T L^1_x}$. Compared with $\|I_1\|_{L^1_T L^1_x}$, since the negative derivative acts on the high frequency, the upper bound of $\|I_2\|_{L^1_T L^1_x}$ is much easier to get. Here is the details:

$$\begin{aligned}\|I_2\|_{L^1_x} &\lesssim \sum_{j\geqslant J-N_0} \sum_{k\geqslant J-N_1} \left\| \triangle_k \mathcal{I}(S_{j+2} v_F \triangle_j u) \right\|_{L^3} \|u\|_{L^\infty_T L^6} \|u_t\|_{L^\infty_T L^2}\\ &\lesssim \sum_{j\geqslant J-N_0} \sum_{k\geqslant J-N_1} 2^{k(\gamma-3)} \|S_{j+2} v_F\|_{L^\infty} \|\triangle_j u\|_{L^3} E_T(u).\end{aligned}$$

Choosing (p,r) such that $\frac{1}{p}+\frac{1}{r}=\frac{1}{2}$ with $2\leqslant r<\infty$ again, the Strichartz estimate yields

$$\|I_2\|_{L^1_T L^1_x} \lesssim T^{1-\frac{1}{p}} \sum_{k\geqslant J-N_1} 2^{k(\gamma-3)} \sum_{j\geqslant J-N_0} 2^{-\frac{j}{2}} \sum_{j'\leqslant j} 2^{\frac{j'}{2}} c_{j'} \mathcal{E}_{h,\sigma} E_T^{\frac{3}{2}}(u)$$
$$\lesssim T^{\frac{1}{2}+\frac{1}{r}} 2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r})]} \mathcal{E}_s^2 E_T(u).$$

Combining this with (7.10), we obtain that

$$\|I\|_{L^1_T L^1_x} \lesssim T^{\frac{1}{2}+\frac{1}{r}} 2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r})]} \mathcal{E}_s^2 E_T(u) \tag{7.11}$$

for $\frac{4}{\gamma-2}\leqslant r<\infty$.

To complete the proof of Lemma 5.1, it remains to estimate *II*. One can proceed this as above by Hölder's inequality to estimate

$$\Big\| \sum_{j\geqslant J-N_0} 2^{j(\gamma-3)} \|\triangle_j v_F S_{j-1}u\|_{L^3} \Big\|_{L^1_T} E_T(u). \tag{7.12}$$

Resorting to the Hölder inequality and the classical Strichartz estimate, one can obtain that

$$\|II\|_{L^1_T L^1} \lesssim T^{\frac{1}{2}+\frac{1}{r}} 2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r})]} \mathcal{E}_s^2 E_T(u)$$

with $2\leqslant r\leqslant 6$. One also can try to improve the result by using the precise Strichartz estimate as before, but it fails and merely obtain that

$$\|II\|_{L^1_T L^1} \lesssim T^{\frac{1}{2}+\frac{1}{r}} 2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r})]} \mathcal{E}_s^2 E_T(u)$$

with $2\leqslant r\leqslant 4$.

One can easily check that the result is worse than the desirable result because of the restriction of r. Compared with the second term in (7.2), the negative derivative acts on the high frequency part so that it is tempting to obtain a better result than that of (7.2). But $\triangle_j v_F$ is bound with $S_{j-1}u$ by the operator $\mathcal{I}$, and this structure prevents us from using efficiently the precise Strichartz estimate. If one first resort to the Hölder inequality, as shown in (7.12), he or she merely obtains a loss result because of the range restriction of r. To go around this difficulty, we first establish a commutator estimate through exploiting cancellation property. Now we turn to details. Our task is to estimate

$$\Big|\int_0^T \int_{\mathbb{R}^3} \sum_{j\geqslant J-N_0} \mathcal{I}(\triangle_j v_F S_{j-1}u) u u_t \, dx\, dt\Big|.$$

In order to drag the $S_{j-1}u$ out of the operator $\mathcal{I}$, we construct $u\mathcal{I}(\triangle_j v_F)S_{j-1}u$ and the triangle inequality yields that

$$\Big|\int_0^T \int_{\mathbb{R}^3} \sum_{j\geqslant J-N_0} \mathcal{I}(\triangle_j v_F S_{j-1}u) u u_t \, dx\, dt\Big| \leqslant \sum_{j\geqslant J-N_0} \big\|\big(\mathcal{I}(\triangle_j v_F S_{j-1}u) - \mathcal{I}(\triangle_j v_F) S_{j-1}u\big) u u_t\big\|_{L^1_T L^1_x}$$
$$+ \Big|\int_0^T \int_{\mathbb{R}^3} \sum_{j\geqslant J-N_0} \mathcal{I}(\triangle_j v_F) S_{j-1}u u u_t \, dx\, dt\Big|.$$

We benefit from the cancellation when we deal with the first term. Since both the Fourier transformation of $\mathcal{I}(\triangle_j v_F S_{j-1}u)$ and $\mathcal{I}(\triangle_j v_F)S_{j-1}u$ are supported in a ring sized 2^j, the Hölder inequality and the Bernstein inequality lead to that

$$\big\|\big(\mathcal{I}(\triangle_j v_F S_{j-1}u)-\mathcal{I}(\triangle_j v_F)S_{j-1}u\big)u\big\|_{L^2_x}\leqslant 2^{\frac{j}{2}}\big\|\mathcal{I}(\triangle_j v_F S_{j-1}u)-\mathcal{I}(\triangle_j v_F)S_{j-1}u\big\|_{L^2_x}\|u\|_{L^6}.$$

Before estimating its right hand, we recall the Coifman and Meyer multiplier theorem. Consider an infinitely differentiable symbol $m:\mathbb{R}^{nk}\mapsto\mathbb{C}$ so that for all $\alpha\in\mathbb{N}^{nk}$ and all $\xi=(\xi_1,\xi_2,\ldots,\xi_k)\in\mathbb{R}^{nk}$, there is a constant $c(\alpha)$ such that

$$\big|\partial_\xi^\alpha m(\xi)\big|\leqslant c(\alpha)\big(1+|\xi|\big)^{-|\alpha|}. \tag{7.13}$$

Define the multilinear operator T by

$$\big[T(f_1,\ldots,f_k)\big](x)=\int\limits_{\mathbb{R}^{nk}} e^{ix\cdot(\xi_1+\cdots+\xi_k)}m(\xi_1,\ldots,\xi_k)\hat f_1(\xi_1),\ldots,\hat f_k(\xi_k)\,d\xi_1\cdots d\xi_k, \tag{7.14}$$

or

$$\mathcal{F}\big[T(f_1,\ldots,f_k)\big](\xi)=\int\limits_{\xi=\xi_1+\cdots+\xi_k} m(\xi_1,\ldots,\xi_k)\hat f_1(\xi_1),\ldots,\hat f_k(\xi_k)\,d\xi_1\cdots d\xi_{k-1}. \tag{7.15}$$

Proposition 7.1. *(See [6], p. 179.) Suppose $p_j\in(1,\infty)$, $j=1,\ldots k$, are such that $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_k}\leqslant 1$. Assume $m(\xi_1,\ldots,\xi_k)$ a smooth symbol as in* (7.13)*. Then there is a constant $C=C(p_i,n,k,c(\alpha))$ so that for all Schwarz class functions $f_1,\ldots,f_k$,*

$$\big\|\big[T(f_1,\ldots,f_k)\big](x)\big\|_{L^p(\mathbb{R}^n)}\leqslant C\|f_1\|_{L^{p_1}(\mathbb{R}^n)}\cdots\|f_k\|_{L^{p_k}(\mathbb{R}^n)}. \tag{7.16}$$

Since the operator $\mathcal{I}$ is a convolution operator with kernel $|x|^{-\gamma}$ in $\mathbb{R}^3$, we can write that

$$\begin{aligned}&\mathcal{F}\big[\mathcal{I}(\triangle_j v_F S_{j-1}u)-\mathcal{I}(\triangle_j v_F)S_{j-1}u\big](\xi)\\&\quad=\int\limits_{\xi=\xi_1+\xi_2}\big(|\xi_1+\xi_2|^{\gamma-3}-|\xi_1|^{\gamma-3}\big)\widehat{\triangle_j v_F}(\xi_1)\widehat{S_{j-1}u}(\xi_2)\,d\xi_2.\end{aligned}$$

By the mean value theorem, the right hand of the above formula becomes that

$$\int\limits_{\xi=\xi_1+\xi_2}|\xi_1+\lambda\xi_2|^{\gamma-4}\frac{(\xi_1+\lambda\xi_2)\cdot\xi_2}{|\xi_1+\lambda\xi_2|}\widehat{\triangle_j v_F}(\xi_1)\widehat{S_{j-1}u}(\xi_2)\,d\xi_2,$$

for a certain $\lambda\in[0,1]$. Moreover, we rewrite it as follows:

$$\int\limits_{\xi=\xi_1+\xi_2}m(\xi_1,\xi_2)\hat f_1(\xi_1)\hat f_2(\xi_2)\,d\xi_2,$$

with

$$m(\xi_1,\xi_2)=(\xi_1+\lambda\xi_2)|\xi_1+\lambda\xi_2|^{\gamma-5}|\xi_1|^{4-\gamma},\qquad f_1=|\nabla|^{\gamma-4}\triangle_j v_F,\qquad f_2=\nabla S_{j-1}u.$$

Observe that $|\xi_1| \geqslant 2^{j-1}$ and $2^{j-2} \geqslant |\xi_2|$, we have that $|\xi_1 + \lambda\xi_2| \sim |\xi_1| \geqslant 2^{J-N_0}$. Hence, we can check that the symbol $m(\xi_1, \xi_2)$ satisfies the estimate (7.13). Finally, it follows from Proposition 7.1 that

$$\left\| \mathcal{I}(\triangle_j v_F S_{j-1}u) - \mathcal{I}(\triangle_j v_F) S_{j-1} u \right\|_{L_x^2} \lesssim \|f_1\|_{L_x^r} \|f_2\|_{L_x^{\frac{2r}{r-2}}}$$

with $2 < r < \infty$. After making use of the Bernstein inequality, the right hand can be controlled by

$$2^{j(\gamma-4+\frac{3}{r})} \|\triangle_j v_F\|_{L_x^r} \|\nabla u\|_{L_x^2}.$$

Keeping in mind $j \geqslant J - N_0$ and recalling the definition of $\mathcal{E}_{h,\sigma}$, the Strichartz estimate and a direct calculation of summing in j show that

$$T^{1-\frac{1}{p}} \sum_{j \geqslant J-N_0} 2^{\frac{j}{2}} 2^{j(\gamma-4+\frac{3}{r})} \|\triangle_j v_F\|_{L_T^p L_x^r} \leqslant T^{1-\frac{1}{p}} \sum_{j \geqslant J-N_0} 2^{j(\gamma-3+\frac{1}{r})} 2^{j(\frac{1}{2}-s)} \mathcal{E}_{h,s}$$

with $\frac{1}{p} + \frac{1}{r} = \frac{1}{2}$ and $2 < r < \infty$. Choosing r such that $\max\{2, \frac{1}{3-\gamma}\} \leqslant r < \infty$, we have that

$$\begin{aligned} &\sum_{j \geqslant J-N_0} \left\| \big(\mathcal{I}(\triangle_j v_F S_{j-1}u) - \mathcal{I}(\triangle_j v_F) \;\; S_{j-1}u\big) u u_t \right\|_{L_T^1 L_x^1} \\ &\quad \lesssim T^{\frac{1}{2}+\frac{1}{r}} 2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r})]} \mathcal{E}_s^2 E_T(u). \end{aligned} \tag{7.17}$$

Now the rest of the paper devotes to estimate this term

$$\left| \int_0^T \int_{\mathbb{R}^3} \sum_{j \geqslant J-N_0} \mathcal{I}(\triangle_j v_F) S_{j-1} u u u_t \, dx \, dt \right|.$$

In order to use precise Strichartz estimate, we need to decompose this term by Bony's para-product decomposition again,

$$\begin{aligned} \mathcal{I}(\triangle_j v_F) S_{j-1} u u u_t &= \sum_k \big\{ S_{k-1}(u S_{j-1}u) \triangle_k \mathcal{I}(\triangle_j v_F) u_t + \triangle_k (u S_{j-1} u) S_{k+2} \mathcal{I}(\triangle_j v_F) \big\} \\ &= II_1 + II_2. \end{aligned}$$

After decomposing this, the term II_1 is similar to the second term in the (7.2) and the negative derivative acts on the high frequency $\triangle_j v_F$ leading to a better result than the second term in the (7.2). Thanks to Fourier–Plancherel formula and Hölder inequality, we obtain

$$\begin{aligned} \sum_{j \geqslant J-N_0} \int_0^T \int_{\mathbb{R}^3} II_1 \, dx \, dt &\approx \sum_{j \geqslant J-N_0} \sum_k \int S_{k-1}(u S_{j-1}u) \triangle_k \mathcal{I}(\triangle_j v_F) \triangle_k u_t \, dx \, dt \\ &\approx \sum_{j \geqslant J-N_0} \sum_k \int \sum_{k' \leqslant k-2} \triangle_{k'}(u S_{j-1}u) \triangle_k \mathcal{I}(\triangle_j v_F) \triangle_k u_t \, dx \, dt \\ &\lesssim \sum_{j \geqslant J-N_0} \sum_{k'} \int \triangle_{k'}(u S_{j-1}u) \triangle_{k'} \sum_{k' \leqslant k-2} \big(\triangle_k \mathcal{I}(\triangle_j v_F) \triangle_k u_t \big) \, dx \, dt \end{aligned}$$

$$\lesssim \sum_{j\geqslant J-N_0} \|uS_{j-1}u\|_{L^\infty \dot B^{\frac12}_{2,2}} \int_0^T \Big\| 2^{-\frac{k'}{2}} \Big\| \triangle_{k'} \sum_{k'\leqslant k-2} \big(\triangle_k \mathcal{I}(\triangle_j v_F)\triangle_k u_t\big)\Big\|_{L^2}\Big\|_{\ell^2}\,dt$$

$$\lesssim \sum_{j\geqslant J-N_0} \|u\|^2_{L^\infty H^1} \int_0^T \Big\| 2^{-\frac{k'}{2}} \Big\| \triangle_{k'} \sum_{k'\leqslant k-2} \big(\triangle_k \mathcal{I}(\triangle_j v_F)\triangle_k u_t\big)\Big\|_{L^2}\Big\|_{\ell^2}\,dt.$$

On the other hand, one denotes

$$g_{k',j} = \triangle_{k'} \sum_{k'\leqslant k-2} \big(\triangle_k \mathcal{I}(\triangle_j v_F)\triangle_k u_t\big),$$

to estimate

$$\sum_{k'} 2^{k'(-\frac12+\frac3r)} \|g_{k',j}\|_{L^1_T L^{\frac{2r}{r+2}}}.$$

Let us write that

$$g_{k',j} = \sum_{k'\leqslant k-2} \triangle_{k'} \bigg(\sum_{\nu\in\Lambda_{k,k'}} \triangle^{\nu}_{k,k'} \mathcal{I}(\triangle_j v_F)\triangle_k u_t\bigg).$$

As the support of the Fourier transform of a product is included in the sum of the support of each Fourier transform, we obtain

$$g_{k',j} = \sum_{k'\leqslant k-2} \triangle_{k'} \bigg(\sum_{\nu\in\Lambda_{k,k'}} \triangle^{\nu}_{k,k'} \mathcal{I}(\triangle_j v_F)\widetilde{\triangle^{\nu}_{k,k'}} u_t\bigg).$$

Using the Hölder inequality, we get

$$\begin{aligned}
\|g_{k',j}\|_{L^{\frac{2r}{r+2}}} &\leqslant \sum_{k'\leqslant k-2} \sum_{\nu\in\Lambda_{k,k'}} \big\|\triangle^{\nu}_{k,k'}\mathcal{I}(\triangle_j v_F)\big\|_{L^r} \big\|\widetilde{\triangle^{\nu}_{k,k'}} u_t\big\|_{L^2}\\
&\leqslant 2^{j(\gamma-3)} \sum_{k'\leqslant k-2} \bigg(\sum_{\nu\in\Lambda_{k,k'}} \big\|\triangle^{\nu}_{k,k'} v_F\big\|^2_{L^r}\bigg)^{\frac12} \bigg(\sum_{\nu\in\Lambda_{k,k'}} \big\|\triangle^{\nu}_{k,k'} u_t\big\|^2_{L^2}\bigg)^{\frac12}\\
&\leqslant 2^{j(\gamma-3)} \sum_{k'\leqslant k-2} \bigg(\sum_{\nu\in\Lambda_{k,k'}} \big\|\triangle^{\nu}_{k,k'} v_F\big\|^2_{L^r}\bigg)^{\frac12} \|\triangle_k u_t\|_{L^2}
\end{aligned}$$

the use of quasi-orthogonality properties is made in the last inequality.

Precise Strichartz estimate and the quasi-orthogonality properties imply that

$$\begin{aligned}
\|g_{k',j}\|_{L^1_T(L^{\frac{2r}{r+2}})} &\leqslant T^{\frac12-\frac1p} 2^{j(\gamma-3)} \sum_{k'\leqslant k-2} 2^{(k'-k)(\frac12-\frac1r)} 2^{k(\frac32-\frac3r-\frac1p)}\\
&\quad\times \bigg(\bigg(\sum_{\nu\in\Lambda_{k,k'}} \big\|\triangle^{\nu}_{k,k'} v_0\big\|^2_{L^2}\bigg)^{\frac12} + 2^{-k}\bigg(\sum_{\nu\in\Lambda_{k,k'}} \big\|\triangle^{\nu}_{k,k'} v_1\big\|^2_{L^2}\bigg)^{\frac12}\bigg) \|\triangle_k u_t\|_{L^2_T L^2_x}
\end{aligned}$$

$$\lesssim T^{\frac{1}{2}-\frac{1}{p}} 2^{j(\gamma-3)} \sum_{k' \leqslant k-2} 2^{(k'-k)(\frac{1}{2}-\frac{1}{r})} 2^{k(\frac{3}{2}-\frac{3}{r}-\frac{1}{p})}$$

$$\times \left(\|\triangle_k v_0\|_{L^2} + 2^{-k} \|\triangle_k v_1\|_{L^2} \right) \|\triangle_k u_t\|_{L_T^2 L_x^2}$$

with $\frac{1}{p} + \frac{1}{r} = \frac{1}{2}$ for $2 \leqslant r < \infty$. Therefore

$$\sum_{k'} 2^{k'(-\frac{1}{2}+\frac{3}{r})} \|g_{k',j}\|_{L_T^1(L^{\frac{2r}{r+2}})} \lesssim T^{\frac{1}{2}-\frac{1}{p}} 2^{j(\gamma-3)} \sum_{k'} \sum_{k' \leqslant k-2} 2^{(k'-k)\frac{2}{r}} c_k \tilde{c}_k \mathcal{E}_{h,\sigma} E_T^{\frac{1}{2}}(u).$$

A direct computation shows that

$$\sum_{k'} 2^{-\frac{k'}{2}} \|g_{k',j}\|_{L_T^1 L^2} \lesssim \sum_{k'} 2^{k'(-\frac{1}{2}+\frac{3}{r})} \|g_{k',j}\|_{L_T^1(L^{\frac{2r}{r+2}})} \lesssim T^{\frac{1}{2}-\frac{1}{p}} 2^{j(\gamma-3)} \mathcal{E}_{h,\sigma} E_T^{\frac{1}{2}}(u).$$

Hence, we have that

$$\left| \sum_{j \geqslant J-N_0} \int_0^T \int_{\mathbb{R}^3} II_1 \, dx \, dt \right| \lesssim 2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r})]} T^{\frac{1}{2}+\frac{1}{r}} \mathcal{E}_s^2 E_T(u) \tag{7.18}$$

with $\frac{4}{\gamma-2} \leqslant r < \infty$. Finally, we conclude this section by giving the estimate of II_2,

$$\begin{aligned}
&\left| \sum_{j \geqslant J-N_0} \int_0^T \int_{\mathbb{R}^3} II_2 \, dx \, dt \right| \\
&\lesssim T^{\frac{1}{2}} \sum_{j \geqslant J-N_0} \sum_k \left\| \triangle_k(u S_{j-1} u) S_{k+1} \mathcal{I}(\triangle_j v_F) \right\|_{L_T^2 L^2} \|u_t\|_{L_T^\infty L^2} \\
&\lesssim T^{1-\frac{1}{p}} \sum_{j \geqslant J-N_0} 2^{j(\gamma-3)} \sum_k \sum_{k' \leqslant k} \left\| \triangle_k(u S_{j-1} u) \right\|_{L_T^\infty L^2} 2^{k'\frac{3}{r}} \|\triangle_{k'} \triangle_j v_F\|_{L_T^p L^r} E_T^{\frac{1}{2}}(u) \\
&\lesssim T^{1-\frac{1}{p}} \sum_{j \geqslant J-N_0} 2^{j(\gamma-3)} \sum_k \sum_{k' \leqslant k} \left\| \triangle_k(u S_{j-1} u) \right\|_{L_T^\infty L^2} 2^{\frac{k'}{2}} c_{k'} \mathcal{E}_{h,\sigma} E_T^{\frac{1}{2}}(u) \\
&\lesssim T^{1-\frac{1}{p}} \sum_{j \geqslant J-N_0} 2^{j(\gamma-3)} \sum_{k'} c_{k'} \sum_{k' \leqslant k} 2^{\frac{k}{2}} \left\| \triangle_k(u S_{j-1} u) \right\|_{L_T^\infty L^2} 2^{(k'-k)\frac{1}{2}} \mathcal{E}_{h,\sigma} E_T^{\frac{1}{2}}(u) \\
&\lesssim T^{1-\frac{1}{p}} \sum_{j \geqslant J-N_0} 2^{j(\gamma-3)} \|c_{k'}\|_{\ell^2(\mathbb{Z})} \left\| 2^{\frac{k}{2}} \left\| \triangle_k(u S_{j-1} u) \right\|_{L_T^\infty L^2} \right\|_{\ell^2(\mathbb{Z})} \left\| 2^{-\frac{k}{2}} \right\|_{\ell^2(\mathbb{N})} \mathcal{E}_{h,\sigma} E_T^{\frac{1}{2}}(u) \\
&\lesssim T^{1-\frac{1}{p}} 2^{J(\gamma-3)} E_T^{\frac{3}{2}}(u) \mathcal{E}_{h,\sigma} \\
&\lesssim 2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r})]} T^{\frac{1}{2}+\frac{1}{r}} \mathcal{E}_s^2 E_T(u).
\end{aligned} \tag{7.19}$$

Collecting (7.18) and (7.19), we have proved that

$$\left|\int_0^T\int_{\mathbb{R}^3}\sum_{j\geqslant J-N_0}\mathcal{I}(\triangle_j v_F)S_{j-1}uuu_t\,dx\,dt\right|\lesssim T^{\frac{1}{2}+\frac{1}{r}}2^{-2J[s-(\frac{\gamma}{2}-\frac{3}{4}+\frac{1}{2r})]}\mathcal{E}_s^2E_T(u), \tag{7.20}$$

with $\frac{4}{\gamma-2}\leqslant r<\infty$. Finally, we complete the proof of (5.8) by (7.11) and (7.20), hence it ends the proof of Lemma 5.1. □

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